A SET OF POSTULATES FOR ORDINARY COMPLEX ALGEBRA*

BY

EDWARD V. HUNTINGTON

Introduction.

The well-known algebra which forms one of the main branches of elementary mathematics is a body of propositions expressible in terms of five fundamental concepts—(the class of complex numbers, with the operations of addition and of multiplication, and the subclass of real numbers, with the relation of order)—and deducible from a small number of fundamental propositions, or hypotheses.

The object of the present paper is to analyze these fundamental propositions, as far as may be, into their simplest component statements, and to present a list which shall not only be free from redundancies, and sufficient to determine the algebra uniquely, but shall also bring out clearly the relative importance of the several fundamental concepts in the logical structure of the algebra.

A more precise statement of the problem is the following: we consider two undefined classes, K and C; two undefined operations, which we may denote by $\oplus$ and $\odot$; and an undefined relation, which we may denote by $\circledcirc$; and we impose upon these five (undefined) fundamental concepts certain arbitrary conditions, or postulates, to serve as the fundamental propositions of an abstract deductive theory (the other propositions of the theory being all the propositions which are deducible from the fundamental propositions by purely logical processes); the problem then is, to choose these fundamental propositions so that all the theorems of algebra, regarded as formal or abstract propositions, shall be deducible from them—the class K and C corresponding to the classes of complex and real numbers, respectively, and the symbols $\oplus$, $\odot$, and $\circledcirc$ to the ordinary $+$, $\times$, and $<$.

Furthermore, the set of postulates, to be satisfactory, must determine the algebra *uniquely;* in other words, the set of postulates adopted must be such that any two systems $(K, C, \oplus, \odot, \circledcirc)$ which satisfy them all shall be *simply isomorphic* with respect to the fundamental concepts—that is, shall be capable of being brought into one-to-one correspondence in such a way that correspond-

* Presented to the Society December 30, 1904. Received for publication January 28, 1905.

ing operations performed on corresponding elements shall lead to corresponding results (see the detailed theorem in § 8).*

The ordinary algebra will appear, then, as one among many *equivalent* concrete interpretations of the abstract theory — all the possible interpretations being called *equivalent, with respect to the fundamental concepts* K, C, $\oplus$, $\odot$, *and* $\circledcirc$, because they are not distinguishable by any properties which can be stated in terms of these symbols.† (Illustrations of such equivalent, or abstractly identical, systems will be given below in § 10.)

Finally, for the sake of elegance, the postulates should be *independent;* that is, no one of them should be deducible from the rest.

A problem of this kind can be solved, no doubt, in a great variety of ways; in fact, one has considerable freedom, not only in the choice of the postulates themselves, but also in the choice of the fundamental concepts in terms of which the postulates are stated. For example, in the present paper, the class C might easily be defined in terms of K, $\oplus$, $\odot$, and $\circledcirc$, instead of being introduced as a fundamental concept.‡ Or again, it would be possible, I believe, to develop the whole algebra on the basis of a single fundamental operation, in terms of which C, $\oplus$, $\odot$, and $\circledcirc$ could all be defined.§

For the present purpose, however, I have not attempted to reduce the number of fundamental concepts to a minimum, but have sought to give a set of postulates which shall conform as closely as possible to familiar forms of presentation. With this end in view, I have omitted the circles from the symbols $\oplus$, $\odot$ and $\circledcirc$, wherever it is possible to do so without confusion with the ordinary $+$, $\times$, and $<$ of arithmetic; it must be constantly borne in mind, however, that the

---

* A set of postulates having this property has been called a *categorical* set, as distinguished from a *disjunctive* set; see O. VEBLEN, Transactions, vol. 5 (1904), p. 346. The notion of equivalence, which had long been familiar in the case of two isomorphic groups, became in the hands of G. CANTOR the fundamental notion of his theory of classes (Mengenlehre, théorie des ensembles).

† In the case of any categorical set of postulates one is tempted to assert the theorem that if any proposition can be stated in terms of the fundamental concepts, either it is itself deducible from the postulates, or else its contradictory is so deducible; it must be admitted, however, that our mastery of the processes of logical deduction is not yet, and possibly never can be, sufficiently complete to justify this assertion. My statement in the last footnote on page 17 of the present volume of the Transactions must therefore be taken, as Mr. H. N. DAVIS first pointed out to me, with some qualification. Compare in this connection remarks by D. HILBERT in his address on the problems of mathematics at the Paris congress of 1900, translated in the Bulletin of the American Mathematical Society, ser. 2, vol. 8 (1901–02), especially pp. 444–445; also his "Axiom of Completeness" for real numbers in the Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 8 (1899), p. 183.

‡ To do this, we have merely to demand the existence of a class C having all the properties mentioned in the postulates of groups II–V, below. The additional postulate required for this purpose would be thus in a high degree a compound statement.

§ The operation I have in mind is the operation of taking "the absolute value of the difference of."

symbols $+$, $\times$, and $<$, when used in this more general sense, are simply arbitrary signs, having no properties not expressly stated in the postulates.

Furthermore, in selecting the postulates, I have chosen only such statements as do not involve the assumption of the existence of any kind of numbers.

It should be said, in conclusion, that no attempt is here made to give a metaphysical analysis of the concepts *class*, *operation*, and *relation*, on which the algebra is based, or of the *laws of deductive logic* by which its propositions are deduced — the discussion of these more fundamental notions, here assumed as familiar, being matter for the trained student of philosophy. I hope, however, that a paper like the present may be indirectly of service on the philosophical side of the subject, by enabling one to formulate very precisely the problems involved in the question: *What is algebra?*

I have made use, in the course of the work, of results obtained in my previous paper on real algebra, which will be found in the present volume of the Transactions.* To this paper, and to the *Theoretische Arithmetik* of STOLZ and GMEINER (1901–), as well as the *Principles of Mathematics*, by BERTRAND RUSSELL (vol. 1, 1903), the reader is referred for bibliographical references. Compare also the addresses by JOSIAH ROYCE † and MAXIME BÔCHER ‡ at the Congress of Arts and Science, St. Louis, September, 1904, and a series of articles by LOUIS COUTURAT in a French review.§

PART I. THE POSTULATES FOR COMPLEX ALGEBRA, AND DEDUCTIONS FROM THEM.

We consider *two classes*, K and C, *two operations*, $+$ and $\times$, and a *relation*, $<$, subject to the conditions prescribed in the twenty-eight postulates of this section. ‖ The symbols $a + b$ and $a \times b$ (or ab) may be read as the "sum" and the "product" of the elements a and b, by analogy with the ordinary language of arithmetic, while the relation $a < b$, or $b > a$, may be read "a pre-

* E. V. HUNTINGTON, *A set of postulates for real algebra, comprising postulates for a one-dimensional continuum and for the theory of groups*, Transactions, vol. 6 (1905), pp. 17–41. — The statement on page 21, that the postulates $R1$–$R8$ form a categorical set, is clearly erroneous, and is corrected in § 4 below. Since this statement was merely parenthetical, the correction does not affect the rest of the paper. — In postulate $R6$, on pages 20 and 32, the element x in 2°) must be noted as "different from X," in order to make the proof of independence for $R3$ conclusive.

† J. ROYCE, *The sciences of the ideal*, published in Science for October 7, 1904 (new series, vol 20, pp. 449–462).

‡ M. BÔCHER, *The fundamental conceptions and methods of mathematics*, published in the Bulletin of the American Mathematical Society, vol. 11 (1904–05), pp. 115–135.

§ L. COUTURAT, *Les principes des mathématiques*, in the Revue de Métaphysique et de Morale, vol. 12, 1904.

‖ A summary of these postulates will be given below, in § 9.

cedes b," or "b follows a."* As already noted, however, these interpretations of the symbols are by no means the only possible ones (compare the examples below, in § 10).

The postulates are arranged in five groups, numbered I–V.

§ 1. *The class K with regard to $+$ and $\times$.*

The following thirteen postulates, I:1–13, make the class K a *field*† with respect to the operations $+$ and $\times$.

The first seven postulates state the general laws of operation in the field, and are to be understood as holding only in so far as the elements, sums, and products involved are elements of K. The remaining six postulates give the requisite "existence-theorems."

POSTULATE I,1. $a + b = b + a$. (Commutative law for addition.)

POSTULATE I,2. $(a + b) + c = a + (b + c)$. (Associative law for addition.)

POSTULATE I,3. *If* $a + b = a + b'$, *then* $b = b'$.

This may be called the (left-hand) law of cancelation for addition.

POSTULATE I,4. $ab = ba$. (Commutative law for multiplication.)

POSTULATE I,5. $(ab)c = a(bc)$. (Associative law for multiplication.)

POSTULATE I,6. *If* $ab = ab'$, *and* $a + a \neq a$, *then* $b = b'$.

This may be called the (left-hand) law of cancelation for multiplication.

POSTULATE I,7. $a(b + c) = (ab) + (ac)$.

This is the (left-hand) distributive law for multiplication with respect to addition.

POSTULATE I,8. *If a and b are elements of K, then their "sum," $a + b$, is an element of K*; that is, there is an element s, uniquely determined by a and b, such that $a + b = s$.

POSTULATE I,9. *There is an element x in K, such that $x + x = x$.*

From postulates I:1,2,3,8, it follows that there cannot be more than one element z such that $z + z = z$, and by postulate I,9 there must be at least one such element. This unique element z is called the *zero-element* of the system, and will be denoted by 0, whenever there is no danger of confusion with the ordinary 0 of arithmetic.

From the same postulates we have: $0 + a = a + 0 = a$, for every element a. (For proof of this and the preceding statement see p. 23.)

* The expressions "before" and "after," etc., are preferable to the expressions "greater" and "less," etc., in this connection, since the notion of *size* is not involved in the notion of *order*.

† Cf. E. V. HUNTINGTON, *Note on the definitions of abstract groups and fields*, with bibliography, in the present number of the Transactions.

POSTULATE I,10. *If there is a unique zero-element* 0, *then for every element* a *there is an element* a' *such that* $a + a' = 0$.

From postulates I:1,3,10, it follows that a' is uniquely determined by a, and that $a + a' = a' + a = 0$. This element a' is called the *negative of* a, and is denoted by $-a$, so that

$$a + (-a) = (-a) + a = 0.$$

Furthermore, every two elements a and b determine uniquely an element x such that $a + x = b$; this element x, namely $x = (-a) + b$, is called their *difference*, b *minus* a, and is denoted by $b - a$, so that

$$a + (b - a) = b.$$

(The proof depends on postulates I:1,2,3,8,9,10.) The usual properties of subtraction follow readily from this definition.*

POSTULATE I,11. *If* a *and* b *are elements of* K, *then their "product,"* ab, *is an element of* K; that is, there is an element p, uniquely determined by a and b, such that $ab = p$.

POSTULATE I,12. *If there is a unique zero-element* 0 *in* K, *then there is an element* y *in* K *different from* 0, *and such that* $yy = y$.

From postulates I:4,5,6,11, we see that there cannot be more than one element u, different from 0, such that $uu = u$, and by postulate I,12 there must be at least one such element. This unique element u is called the unit-element of the system, and will be denoted by 1 whenever there is no danger of confusion with the ordinary 1 of arithmetic.

From the same postulates we have: $1 \times a = a \times 1 = a$, for every element a. (Proof: if $uu = u$, then $u(ua) = (uu)a = ua$, whence $ua = a$, by I,6.)

POSUTLATE I,13. *If there is a unique zero-element* 0, *and a unique unit-element* 1, *different from* 0, *then for every element* a, *provided* a *is not* 0, *there is an element* a'' *such that* $aa'' = 1$.

From postulates I:4,6,13, it follows that a'' is uniquely determined by a, and that $aa'' = a''a = 1$ (where $a \neq 0$). This element a'' is called the *reciprocal of* a, and is denoted by $\frac{1}{a}$, or $1/a$, so that

$$a(1/a) = (1/a)a = 1 \text{ if } a \neq 0.$$

Furthermore, every two elements a and b, provided $a \neq 0$, determine uniquely an element y such that $ay = b$; this element y, namely $y = (1/a)b$, is called

* It would probably be preferable, however, on both pedagogical and logical grounds, not to introduce the operation of subtraction at all, but to regard $b - a$ always as the sum of the two elements b and $-a$.

their *quotient*, *b divided by a*, and is denoted by $\frac{b}{a}$, or b/a, so that

$$a(b/a) = (b/a)a = b \text{ if } a \neq 0.$$

(The proof depends on postulates I:4,5,6,11,12,13.) The usual properties of division follow readily from this definition.*

Now from the distributive law (postulate I,7), which we have not yet utilized, we see that

$$0 \times a = a \times 0 = 0,$$

for every element a. Then if a is different from 0, its reciprocal, $1/a$, will also be different from 0, since otherwise we should have $1 = 0$. Hence the important theorem that *if a and b are both different from* 0, *their product is also different from* 0; or in other words, a product ab cannot be zero unless at least one of its factors is zero. [Proof: if $ab = 0$ and $a \neq 0$, then $b = 1 \times b = (1/a)a \times b = (1/a) \times ab = (1/a) \times 0 = 0$.]

These theorems are sufficient to show that any system $(K, +, \times)$ which satisfies the postulates I:1–13 has all the properties of a *field* with respect to the operations $+$ and $\times$. To summarize:

The elements of K form an abelian group with respect to $+$; *the elements excluding the zero of that group form an abelian group with respect to* $\times$; *for every element* a, $a \times 0 = 0 \times a = 0$; *and the operation* $\times$ *is distributive with respect to the operation* $+$.

We now consider the multiples, submultiples, and rational fractions of any element of the field.

Assuming the system of the ordinal, or natural, numbers, the characteristic properties of which are summarized in PEANO's five postulates (see page 27), we define the *mth multiple*, ma, *of any elememt* a by the usual recurrent formulæ

$$1a = a,\ 2a = 1a + a,\ 3a = 2a + a,\ \cdots,\ (k+1)a = ka + a,$$

where $1, 2, 3, \cdots, k, \cdots, m$ denote ordinal numbers, and $k + 1$ the arithmetical sum of k and 1.

The *mth submultiple of* a, denoted by a/m, is then defined by the equation $a/m = a/(mu)$,† from which we have $m(a/m) = a$.

From these definitions it follows at once that $p(a/q) = (pa)/q$, so that we may denote either member of this equation by pa/q.

* It would probably be preferable, however, on both pedagogical and logical grounds, not to introduce the operation of division at all, but to regard b/a always as the product of the two elements b and $1/a$.

† The unit-element of the field is here denoted by u, instead of by 1, to avoid confusion with the number 1.

Any element of the form pa/q (where p and q are ordinal numbers) is called a *rational fraction of* a. The following theorems on rational fractions in which m, n, p, q denote ordinal numbers, are readily deduced from the properties of a field:

$$(a)\quad \frac{mpa}{mq} = \frac{pa}{q}; \qquad (b)\quad \frac{pa}{q} + \frac{ma}{n} = \frac{(np+mq)a}{qn}; \qquad (c)\quad \frac{pa}{q} \times \frac{ma}{n} = \frac{pma}{qn}.$$

It should be noticed that we cannot yet infer from $ma = na$ that $m = n$, since, as far as postulates I:1–13 are concerned, the field may be finite, that is, contain only a finite number of elements.* (See below, under postulate IV, 2.)

§ 2. *The subclass C, with regard to $+$ and $\times$.*

The following seven postulates, II:1–7, make C a subclass in K, which shall be, like K, a field with respect to the operations $+$ and $\times$.

POSTULATE II,1. *If a is an element of C, then a is an element of K.*

POSTULATE II,2. *The class C contains at least one element.*

POSTULATE II,3. *If a is an element of C, then there is an element b in C, such that $a \neq b$.*

These three postulates tell us that C is a subclass in K, containing at least two elements.

POSTULATE II,4. *If a and b are elements of C, then their sum, $a + b$, if it exists at all in K, is an element of C.*

POSTULATE II,5. *If a is an element of C, then its negative, $-a$, if it exists at all in K, is an element of C.*

Hence the zero element, 0, of K is an element of C.

POSTULATE II,6. *If a and b are elements of C, then their product, ab, if it exists at all in K, is an element of C.*

POSTULATE II,7. *If a is an element of C, then its reciprocal, $1/a$, if it exists at all in K, is an element of C.*

Hence the unit element, 1, of K is an element of C.

Now by virtue of postulate II,1, all the general laws of operation, I:1–7, hold as well for the subclass C as for the whole class K; hence the postulates of I and II make the subclass C, like K, a *field* with respect to $+$ and $\times$.

It should be noticed that the subclass C may be identical with K, as far as the postulates of I and II are concerned. (See § 6.)

§ 3. *The subclass C, with regard to $<$.*

In the following five postulates, III:1–5, we impose further restrictions on the subclass C, which, together with the postulates II:2–3, make the class C a *one-*

* As is well known, the number of elements in a finite field must be a prime number, or a power of a prime.

dimensional continuum (in the sense defined by DEDEKIND) with respect to the relation $<$. (On the use of the word continuum, see remarks in §4.) The fifth postulate, III,5, however, proves to be a consequence of the other four, when the properties of the field (I and II), and the postulate IV:1–2 are utilized; it should therefore not be included in the final list.

POSTULATE III,1. *If a and b are elements of C, and $a \neq b$, then either $a < b$ or else $a > b$.*

POSTULATE III,2. *If $a < b$, then $a \neq b$.*

That is, the relation $<$ is *non-reflexive* throughout C.

POSTULATE III,3. *If a, b and c are elements of C, and if $a < b$ and $b < c$, then $a < c$.*

That is, the relation $<$ is *transitive* throughout C.

From III:2–3 we see that the relations $a < b$ and $a > b$ cannot both be true;* that is, the relation $<$ is *non-symmetric* for every pair of elements in C. Hence, *if a and b are elements of C, we must have either*

$$a = b, \text{ or } a < b, \text{ or } a > b;$$

and these three relations are mutually exclusive.

POSTULATE III,4. *If Γ is a non-empty subclass in C, and if there is an element b in C such that every element of Γ precedes b, then there is an element X in C having the following two properties with regard to the subclass Γ:*

1°) *every element of Γ precedes X or is equal to X; while*

2°) *if x' is any element of C which precedes X, then there is at least one element of Γ which follows x'.*

This is the postulate of *continuity* in the form essentially due to DEDEKIND.†

The element X, which is readily seen to be uniquely determined by the subclass Γ, is called the "upper limit" of Γ, or sometimes its "least upper bound." From III:1–4 can be deduced, as on page 20, the following theorem, which is DEDEKIND's original form of the postulate of continuity:

If the elements of C are divided into two non-empty classes, Γ and Γ', such that every element of C belongs either to Γ or to Γ', and if every element of Γ precedes every element of Γ', then there is an element X in C which has the following properties:

1°) *every element of C which precedes X belongs to Γ; and*

2°) *every element of C which follows X belongs to Γ'.*

It should be noticed in regard to this postulate of continuity, so-called, that all the postulates III:1–4 might be satisfied by a *discrete* assemblage; in order to introduce the property of *density*, we add the following postulate, III,5:

* In my previous paper (§1) this proposition, at least for the case $a \neq b$, was used as a postulate, the other postulates being "weakened" to allow of its independence. It has not seemed worth while to carry the analysis so far in the present article.

† Loc. cit., below.

Postulate III,5. If a and b are elements of C, and $a < b$, then there is at least one element x in C, such that $a < x$ and $x < b$. Such elements x are said to lie *between* a and b.

These five postulates, III:1–5, together with postulates II:2–3 (which merely tell us that the class C contains at least two elements), make the class C a *one-dimensional continuum*, in DEDEKIND's sense, with respect to the relation $<$ (see § 1 in my previous paper).

The postulate III,5 is not printed in italics, because, as already stated, when we utilize the postulates of I, II and IV, as well as those of III, it can be shown to be redundant. The proof will be given in § 5.

§ 4. *Note on the use of the word "continuum."*

The term *one-dimensional* (or *linear*) *continuum* with respect to the relation of order ($<$), as used in this and my previous paper, must be understood not in the exact sense defined by CANTOR* in 1895, but in the older and wider meaning in which the word was used by DEDEKIND† and WEBER.‡

DEDEKIND would call any system (K, $<$) which has at least two elements and satisfies the postulates III:1–5, a continuum.

CANTOR restricts the term to systems which possess not only all these properties, but also two further properties, namely: 1) the class K shall have a first and a last element; and 2) the class K *shall contain a denumerable subclass*, R, *such that between every two distinct elements of* K *there shall be at least one element of* R.§ (A denumerable class means an infinite class which can be placed in one-to-one correspondence with the class of natural numbers.) || He

---

* G. CANTOR, *Beiträge zur Begrundung der transfiniten Mengenlehre*, Mathematische Annalen, vol. 46 (1895), p. 510; French translation by F. MAROTTE, *Sur les fondements de la théorie des ensembles transfinis*, 1899, p. 42. Cf. B. RUSSELL, loc. cit., chap. 36, and L. COUTURAT, loc. cit., pp. 664–675.

† R. DEDEKIND, *Stetigkeit und irrationalen Zahlen*, 1872; English translation by BEMAN.

‡ H. WEBER, *Algebra*, vol. 1 (1898), pp. 4–12.

§ Compare also A. N. WHITEHEAD, in a footnote on p. 299 of B. RUSSELL's *Principles of Mathematics*, vol. 1 (1903).

|| The term *abzählbar*, introduced by G. CANTOR in 1879 [Mathematische Annalen, vol. 15 (1879), p. 4; vol. 20 (1882), p. 116; vol. 21 (1883), p. 550], and uniformly translated in French by *dénombrable* [Acta Mathematica, vol. 2 (1883), p. 365; Formulaire de Mathématiques, vol. 4 (1903), p. 130], has been variously rendered in English. The Century Dictionary gave *enumerable* in 1889; but the editor in charge of the mathematical vocabulary, Mr. C. S. PEIRCE, has since used *numerable*, and finally *denumerable*, in his own writings [see Monist, vol. 2, p. 541, 1892; and especially vol. 7, p. 214, 1897]. *Enumerable* is also the form proposed by Professor OSGOOD [American Journal of Mathematics, vol. 19, p. 161, 1897], and by Professor BEMAN [Bulletin of the American Mathematical Society, vol. 3, p. 174, 1897]. At least one writer, W. H. YOUNG, uses *countable* [Proceedings of the London Mathematical Society, vol. 34, p. 285, 1902, and later papers]; while the Encyclopedia Brittanica prefers *numerable* (article "Number," vol. 31, p. 282, 1902; G. B. MAT-

then shows* that any two systems which satisfy this definition of the linear continuum are *ordinally similar*, and equivalent to the class of real numbers from 0 to 1 inclusive, as far as the relation $\lesssim$ is concerned. His postulates therefore form a *categorical set*, with regard to the relation $<$.

DEDEKIND's definition, on the other hand, is not categorical, but admits various non-equivalent types of order. This possibility is due not merely to the ambiguity in regard to the presence or absence of end points, but rather to the omission of CANTOR's second requirement, concerning the existence of a denumerable subclass within the system. This remarkable postulate is independent of all the other properties, as may be shown by the following example: †

Let $K =$ the class of all complex numbers (α, β) in which α and β are real numbers such that $0 \leqq \alpha \leqq 1$ and $0 \leqq \beta \leqq 1$; with $(\alpha_1, \beta_1) \odot (\alpha_2, \beta_2)$ whenever $\alpha_1 < \alpha_2$; and if $\alpha_1 = \alpha_2$, then whenever $\beta_1 < \beta_2$.

This system (which can be stated, if one prefers, in geometric terms) satisfies all of DEDEKIND's postulates, and has a first element, $(0, 0)$, and a last element, $(1, 1)$; but it is not a continuum under CANTOR's definition, since it is not possible to find within it a denumerable subclass of the kind required.‡

CANTOR's definition is thus clearly superior to DEDEKIND's for the purposes of a purely ordinal discussion of the continuum; if, however, as in algebra, the elements of the system considered are not only connected by the relation of order, but are also combined by the operation of addition,§ then DEDEKIND's definition may be, as it is in this paper, amply sufficient. It is therefore perhaps not undesirable that both definitions should remain in current use.

A further analysis of CANTOR's last postulate, with a general discussion of geometric continuity, will be found in a paper by O. VEBLEN in the present number of the Transactions.

§ 5. *Further postulates for subclass C.*

The following two postulates, IV:1–2, serve to connect the operations $+$ and $\times$ and the relation $<$, within the subclass C, and make it equivalent to the class of all real numbers with respect to $+$, $\times$, and $<$.

THEWS]. The form *denumerable* is the form adopted by RUSSELL and WHITEHEAD in their *Principles of Mathematics*, and seems to me preferable to the other terms because it avoids a suggestion of finitude (cf. PEIRCE, loc. cit., 1897). I am also following RUSSELL and WHITEHEAD, and many Italian writers, in translating *Menge* (Mannigfaltigkeit, ensemble, insieme, aggregato) by *class* instead of by manifold, mass, set, ensemble, collection, assemblage, or aggregate.

* Loc. cit.

† Cf. H. WEBER, loc. cit., p. 11.

‡ For, there would have to be elements of the subclass for every value of α between 0 and 1, and these values of α form an infinite class which is not denumerable.

§ WEBER, loc. cit., speaks of such systems as systems which admit *measurement* (messbare Mannigfaltigkeiten).

POSTULATE IV,1. *Within the class C, if $x < y$, then $a + x < a + y$, whenever* $a + x \neq a + y$.*

From this postulate we have at once: if $a > 0$ and $b > 0$, then $a + b > a$; and if $a < 0$ and $b < 0$, then $a + b < a$ (compare postulates $RA1$ and $RA2$ in my previous paper, page 25). Hence, if we define *positive and negative elements* as those elements of C which are > 0 and < 0 respectively, we can show that if $a < b$ then there is a positive x such that $a + x = b$, and conversely, if x is positive in $a + x = b$, then $a < b$. [Proof as for theorem 14 on page 26.]

POSTULATE IV,2. *Within the class C, if $a > 0$ and $b > 0$, then $ab > 0$.*

By the aid of this postulate it is easy to show that the product, ab, of two elements of C will be positive when a and b are both positive or both negative; and negative if a and b are of opposite signs (compare theorem 33 on page 40). Hence the unit-element 1, or u, with all its multiples and submultiples, will be positive;† and therefore all the multiples and submultiples of any positive element a will also be positive.

We are now in a position to prove the theorem of *density*, which was stated provisionally in § 3 as postulate III,5. The proof, which involves all the postulates I–IV, is as follows: If $a < b$, $b - a$ will be positive, and if we take $c = (b - a)/2$, c will also be positive; if then we take $x = a + c$ (from which follows $x + c = b$), we find at once that $a < x$ and $x < b$.

It thus appears that all the postulates of the set for real algebra given in the appendix of my previous paper (page 39) are satisfied by the class C; hence, in any system $(K, C, +, \times, <)$ which satisfies the twenty-six postulates of groups I–IV, the subclass C will be equivalent to the class of all real numbers with respect to $+$, $\times$, and $<$.

§ 6. *Note on real algebra.*

As we have just seen, any class C which is a field with respect to $+$ and $\times$, and possesses DEDEKIND's property of continuity with respect to $<$, and satisfies the postulates IV:1–2, will be equivalent to the class of all real numbers with respect to $+$, $\times$, and $<$.

Hence the twenty-six postulates of groups I–IV, with an additional postulate demanding that the class K shall contain no elements which do not belong to C, would form a categorical set of postulates for real algebra.

---

* This proviso makes possible the proof of the independence of postulate I,3, which would otherwise be deducible from IV,1, with the aid of III,2.

† For, if we suppose $u < 0$, then $-u > 0$, whence $u \times (-u) < 0$, or $-u < 0$, and therefore $u > 0$. The multiples of u will form an ascending sequence: $u < 2u < 3u < \cdots$; whence: if $mu = nu$, then $m = n$; or more generally, if $ma = na$, and $a \neq 0$, then $m = n$, where m and n are ordinal numbers.

On the other hand, these same twenty-six postulates, with the addition of the two postulates of group V, below, form a categorical set for complex algebra.

We may say, then, that the postulates of groups I–IV contain the properties common to both real and complex algebra.

§ 7. *Postulates peculiar to complex algebra.*

The two following postulates, V:1–2, which complete the list of postulates for complex algebra, concern the existence of elements of K which do not belong to the subclass C.

POSTULATE V,1. *If K is a field with respect to $+$ and $\times$, then there is an element j in K such that $jj = -1$ (where -1 is the negative of the unit-element of the field).*

If i is one element which has this property (namely, $ii = -1$), then $-i$ will have the same property; and this will be true of no other element besides these two. That is, the element j whose existence is postulated in V.1 is not uniquely determined, but may have two values, one the negative of the other. Which of these values we shall denote by i and which by $-i$ is a matter of arbitrary choice.

Neither i nor $-i$ can be an element of C, since the square of every element of C is positive or zero; hence, if x and y are elements of C, the product iy and the sum $x + iy$ will be elements of K which do not belong to C unless $y = 0$. This gives us the important theorem that if $x + iy = x' + iy'$, where x, y, x', y', belong to C, then $x = x'$ and $y = y'$.*

The next, and last, postulate demands that the class K shall contain no further elements, that is, no elements not expressible in the form $x + iy$, where x and y are elements of C.

POSTULATE V,2. *If K and also C are fields with respect to $+$ and $\times$, and if there is an element i such that $ii = -1$ (see postulate* V,1), *then for every element a in K there are elements x and y in C such that $x + iy = a$.*

We are now in a position to establish the categorical character of the whole set of postulates.

§ 8. *Proof of the equivalence of all systems that satisfy the postulates* I–V.

The following theorem for complex algebra is analogous to the theorem numbered 37 in my paper on real algebra (see page 40), and is proved in a similar way.

If $(K, C, +, \times, <)$ and $(K', C', +, \times, <)$ are any two systems satisfying the twenty-eight postulates of groups I–V, then these systems are EQUIV-

* Thus, from $x + iy = x' + iy'$ follows $x - x' = i(y' - y)$; hence $y' - y = 0$, and therefore $x - x' = 0$.

ALENT, *or* ABSTRACTLY IDENTICAL, *with respect to* K, C, $+$, $\times$, *and* $<$; *that is, the classes* K *and* K' *can be brought into one-to-one correspondence in such a way that if* a', b', *etc., are the elements of* K' *which correspond to the elements* a, b, *etc., in* K, *then we shall have :*

1°) *the subclass* C *corresponds to the subclass* C' *;*

2°) *if* $a + b = c$, *then* $a' + b' = c'$, *and conversely ;*

3°) *if* $ab = c$, *then* $a'b' = c'$, *and conversely ;*

4°) *within the subclasses* C *and* C', *if* $a < b$, *then* $a' < b'$, *and conversely.*

In other words, the twenty-eight postulates form a categorical set.*

The proof of the theorem consists simply in bringing the elements of C in K into correspondence with the elements of C' in K', as in the proof of the analogous theorem for real algebra, and then making the element i in K correspond to the element i' in K'. Then every element $x + iy$ in K will correspond to an element $x' + iy'$ in K', where x' and y' are the elements in C' which correspond to the elements x and y in C.

It should be noticed that this correspondence between two equivalent systems can be set up in two, and only two, ways; the only ambiguity resulting from the arbitrary choice of the element i (see postulate V,2)†.

PART II. SUMMARY OF THE POSTULATES, AND PROOF OF THEIR CONSISTENCY AND INDEPENDENCE.

In this part, for convenience of reference, I give a list of the twenty-eight postulates of groups I–V, using the general symbols $\oplus$, $\odot$, $\circledcirc$, z, and u, instead of the symbols $+$, $\times$, $<$, 0, and 1; these latter symbols are used, in this part, only in their ordinary arithmetic meanings.

Next, I give some examples of systems $(K, C, \oplus, \odot, \circledcirc)$ which satisfy all the twenty-eight postulates. These systems, as has just been shown, are all equivalent, or *abstractly identical*, with respect to $\oplus$, $\odot$, and $\circledcirc$; the existence of any one of them proves the *consistency* of the postulates.

Finally, I give, for each of the twenty-eight postulates, an example of a system $(K, C, \oplus, \odot, \circledcirc)$ which satisfies all the other postulates, but not the one in question, thus establishing in the usual way the *independence* of all the postulates.

In constructing these systems, the existence of the ordinary systems of real and complex numbers, as derived by the "genetic" method from the system of the natural numbers, is assumed, as well as the usual geometric representations of these numbers by points of the plane or the sphere.

---

* See the first two footnotes in the introduction, above.

† Speaking in geometric terms, the complex plane has two *aspects:* the first plane may be rotated through 180° about the axis of reals before being applied to the second plane.

§ 9. *List of the postulates.*

Before giving the postulates themselves, it will be convenient to repeat the more important definitions which have been used in the course of the work. All these definitions are simply abbreviations, introduced in order to avoid tedious circumlocution.

DEFINITION 1. If there is a uniquely determined element z such that $z \oplus z = z$, then z is called the *zero-element*, or zero.

DEFINITION 2. If there is a unique zero-element z (see definition 1), and if there is a uniquely determined element u, different from zero, and such that $u \odot u = u$, then u is called the *unit-element*, or unity.

DEFINITION 3. If there is a unique zero-element z (see definition 1), and if a given element a determines uniquely an element a' such that $a \oplus a' = z$, then a' is called the *negative of* a, and is denoted by $-a$.

DEFINITION 4. If there is a unique zero-element z and a unique unit-element u (see definitions 1 and 2), and if a given element a, different from z, determines uniquely an element a'' such that $a \odot a'' = u$, then a'' is called the *reciprocal of* a, and is denoted by $1/a$.

The first seven postulates, giving the general laws of operation in the system, are to be understood to hold only in so far as the elements, sums, and products involved are elements of K.

POSTULATE I,1. $a \oplus b = b \oplus a$.

POSTULATE I,2. $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.

POSTULATE I,3. If $a \oplus b = a \oplus b'$, then $b = b'$.

POSTULATE I,4. $a \odot b = b \odot a$.

POSTULATE I,5. $(a \odot b) \odot c = a \odot (b \odot c)$.

POSTULATE I,6. If $a \odot b = a \odot b'$, and $a \oplus a \neq a$, then $b = b'$.

POSTULATE I,7. $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$.

POSTULATE I,8. If a and b are elements of K, then $a \oplus b$ is an element of K.

POSTULATE I, 9. There is an element x in K such that $x \oplus x = x$.

POSTULATE I,10. If there is a unique zero-element z in K (see definition 1), then for every element a in K there is an element a' in K, such that $a \oplus a' = z$.

POSTULATE I,11. If a and b are elements of K, then $a \odot b$ is an element of K.

POSTULATE I,12. If there is a unique zero-element, z, in K (see definition 1), then there is an element y in K, different from z, and such that $y \odot y = y$.

POSTULATE I,13. If there is a unique zero-element, z, and a unique unit-element, u, different from z, in K (see definitions 1 and 2), then for every element a in K, provided $a \neq z$, there is an element a'' in K such that $a \odot a'' = u$.

The postulates I:1–13 *make the class K a field with respect to* $\oplus$ *and* $\odot$.

POSTULATE II,1. If a is an element of C, then a is an element of K.

POSTULATE II,2. The class C contains at least one element.

POSTULATE II,3. If a is an element of C, then there is an element b in C, such that $a \neq b$.

POSTULATE II,4. If a and b are elements of C, then $a \oplus b$, if it exists in K at all, is an element of C.

POSTULATE II,5. If a is an element of C, then its negative, $-a$ (see definition 3), if it exists in K at all, is an element of C.

POSTULATE II,6. If a and b are elements of C, then $a \odot b$, if it exists in K at all, is an element of C.

POSTULATE II,7. If a is an element of C, then its reciprocal $1/a$ (see definition 4), if it exists in K at all, is an element of C.

The postulates II:1–7, *taken with the postulates* I:1–13, *make the sub-class C, like the class K, a field with respect to* $\oplus$ *and* $\odot$.

POSTULATE III,1. If a and b are elements of C, and $a \neq b$, then either $a \lessdot b$ or else $a \gtrdot b$.

POSTULATE III,2. If $a \lessdot b$, then $a \neq b$.

POSTULATE III,3. If a, b, and c are elements of C, and if $a \lessdot b$ and $b \lessdot c$, then $a \lessdot c$.

POSTULATE III,4. If Γ is a non-empty subclass in C, and if there is an element b in C such that $\alpha \lessdot b$ for every element α of Γ, then there is an element X in C having the following two properties with regard to the sub-class Γ:

1°) if α is an element of Γ, then $\alpha \lessdot X$ or $\alpha = X$; while

2°) if x' is any element of C such that $x' \lessdot X$, there is an element ξ in Γ such that $\xi \gtrdot x'$.

The postulates III:1–4 *and* II:2–3, *taken with the redundant postulate* III,5 (*which is here omitted*), *make the sub-class C a one-dimensional continuum with respect to* $\lessdot$, *in the sense defined by* DEDEKIND.

POSTULATE IV,1. If $a, x, y, a \oplus x$, and $a \oplus y$ are elements of C, and $x \lessdot y$, then $a \oplus x \lessdot a \oplus y$, whenever * $a \oplus x \neq a \oplus y$.

POSTULATE IV,2. If a, b, and $a \odot b$ are elements of C, and $a \gtrdot z$ and $b \gtrdot z$, then $a \odot b \gtrdot z$ (where z is the zero-element of definition 1).

The twenty-six postulates of groups I–IV *make the sub-class C equivalent to the class of all real numbers with respect to* $\oplus$, $\odot$, *and* $\lessdot$.

POSTULATE V,1. If K is a field with respect to $\oplus$ and $\odot$, then there is an element j in K such that $j \odot j = -u$, where $-u$ is the negative of the unit-element of the field (see definitions 2 and 3).

* Cf. footnote under postulate IV,1, above.

POSTULATE V,2. If K and also C are fields with respect to $\oplus$ and $\odot$, and if there is an element i such that $i \odot i = -u$ (see postulate V,1), then for every element a in K there are elements x and y in C such that $x \oplus (i \odot y) = a$.

These twenty-eight postulates make the class K equivalent to the class of all (ordinary) complex numbers with respect to $\oplus$, $\odot$, and $\circledcirc$. (See the theorem in § 8.)

§ 10. *The consistency of the postulates.*

To prove that these postulates are consistent—that is, to prove that contradictory propositions can never be deduced from them by the processes of formal logic—it is sufficient to show the existence of any system $(K, C, \oplus, \odot, \circledcirc)$ in which all the postulates are satisfied; for then the postulates themselves and all their logical consequences express properties of this system, and must therefore be free from contradiction (since no really existent system can have contradictory properties).*

The following systems, all of which are *equivalent*, or abstractly identical, with respect to K, C, $\oplus$, $\odot$, and $\circledcirc$, are examples of systems which satisfy all the postulates:

1) $K =$ the class of ordinary complex numbers (that is, the class of all couples of the form (x, y), where x and y are real numbers), with $\oplus$ and $\odot$ defined as the ordinary $+$ and $\times$; $C =$ the class of real numbers (that is, the class of DEDEKIND's "cuts," or the class of CANTOR's "fundamental sequences"), with $\circledcirc$ defined as the ordinary $<$. Here $z = 0$, $u = 1$, and $j = \pm\sqrt{-1}$.

2) $K =$ the class of all the points in a plane (the ordinary "complex plane"), with $\oplus$ and $\odot$ defined in the usual geometric manner;† $C =$ the points of a fixed straight line in the plane (the "axis of reals"), with $\circledcirc$ defined as "on the left of." Here z is the point $(0, 0)$, u the point $(1, 0)$, and j the point $(0, 1)$ or $(0, -1)$.

* This statement, of course, makes large assumptions in regard to the objective validity of our processes of logical deduction. In this connection see the second of the problems proposed by D. HILBERT at the Paris congress of 1900 (loc. cit.) and an article by A. PADOA in L'Enseignement Mathématique, vol. 5 (1903), pp. 85–91.

† Thus, the point $a + b$ is the point reached by starting from a and travelling a path equal in length and direction to the path from O to b; while the point $a \times b$ is the point whose "angle" (from OX) equals the *sum* of the angles of a and b, and whose "distance" (from O) equals the *product* of the distances of a and b. Here the product, x, of the two distances, α and β, may be defined geometrically by the proportion $x : \alpha = \beta : 1$; or it may be defined in terms of measurement, as follows: let

$$\alpha = \lim_{n=\infty}\left(\frac{p_n}{q_n}\right) \quad \text{and} \quad \beta = \lim_{n=\infty}\left(\frac{r_n}{s_n}\right)$$

where the p_n/q_n and r_n/s_n are successive rational approximations to the lengths of α and β; then the sequence of products $p_n r_n / q_n s_n$ will have a limit, and this limit will be the required product of α and β. [Cf. E. V. HUNTINGDON, Strassburg dissertation, *Die Grund-Operationen an absoluten und complexen Grössen in geometrischer Behandlung*, Braunschweig, 1901.]

System (1) is an "arithmetical" system, of which (2) is the "geometrical representation."

In the following systems, the "complex numbers" employed may be interpreted in either way, the geometric phraseology being perhaps the most convenient. All these systems are obtained from (1) or (2) by a projective transformation of the plane, as explained below.

3) $K =$ the class of ordinary complex numbers; $a \oplus b = a + b$; $a \odot b = kab$, where k is any real number not zero; $C =$ the class of real numbers, with $\circledcirc$ defined as $<$ or as $>$, according as k is positive or negative.

Here $z = 0$, $u = 1/k$, and $j = \pm(1/k)\sqrt{-1}$.

4) $K =$ the class of ordinary complex numbers; $a \oplus b = ab/(a + b)$, except that $a \oplus b = a + b$ whenever a or b or $a + b$ is zero; $a \odot b = ab$; $C =$ the class of real numbers, with $\circledcirc$ defined as the ordinary $<$, except that $(a \circledcirc b) = (a > b)$ whenever a and b are both positive or both negative.

Here $z = 0$, $u = 1$, and $j = \pm\sqrt{-1}$.

It will be noticed that the ordinary meaning of addition is preserved in system (3), and that of multiplication in system (4).

5) $K =$ the class of ordinary complex numbers; $a \oplus b = a + b - h$, and $a \odot b = kab - hk(a + b) + h(1 + hk)$, where h and k are any real or complex numbers, provided $k \neq 0$;* $C =$ the class of complex numbers whose corresponding points, in the complex plane, lie on the straight line through the two points h and $(1/k) + h$; with $\circledcirc$ defined as the relation of order along this line, the forward direction being so chosen that $h \circledcirc (1/k) + h$.

Here $z = h$, $u = (1/k) + h$, $-u = -(1/k) + h$, and $j = \pm(\sqrt{-1}/k) + h$.

6) Let α, β, γ, δ be any real or complex numbers such that $\alpha\delta - \beta\gamma \neq 0$; and let F, for the moment, stand for the class comprising all the ordinary complex numbers together with an extra element to be denoted by ω. (Geometrically speaking, F is the class of all the points on the complex sphere, including the "North Pole," ω.)† We then define K, $\oplus$, $\odot$, C and $\circledcirc$ as follows:

$K =$ all the elements of F, *excluding one point*, P, where if $\gamma \neq 0$, $P = \alpha/\gamma$, and if $\gamma = 0$, $P = \omega$.

If $a \neq \omega$ and $b \neq \omega$, then

$$a \oplus b = \frac{(2\alpha\gamma\delta - \beta\gamma^2)ab - \alpha^2\delta(\alpha + b) + \alpha^2\beta}{\gamma^2\delta ab - \beta\gamma^2(a + b) + 2\alpha\beta\gamma - \alpha^2\delta},$$

except when the denominator is zero, in which case $a \oplus b = \omega$.

If $a \neq \omega$ and $b \neq \omega$, then

* For an especially simple case, take $h = -1$ and $k = 1$.

† The complex sphere, omitting the North Pole, is the stereographic projection of the complex plane.

$$a \odot b = \frac{(\beta\gamma^2 + \alpha\delta^2)ab - (\alpha\beta\gamma + \alpha\beta\delta)(a+b) + (\alpha^2\beta + \alpha\beta^2)}{(\gamma\delta^2 + \gamma^2\delta)ab - (\beta\gamma\delta + \alpha\gamma\delta)(a+b) + (\beta^2\gamma + \alpha^2\delta)},$$

except when the denominator is zero, in which case $a \odot b = \omega$.

If $a = \omega$, or $b = \omega$, or both, then $a \oplus b$ and $a \odot b$ are defined as the limits of the expressions above, when ω is replaced by $1/x$ and x approaches 0; it being understood that (on the sphere)

$$\lim_{x=0}\left[\frac{1}{x}\right] = \omega.$$

With these definitions of $\oplus$ and $\odot$, the class K proves to be a field* in which z and u have the following values:

$$\text{if } \delta \neq 0,\ z = \frac{\beta}{\delta}; \qquad \text{if } \delta = 0,\ z = \omega;$$

$$\text{if } \gamma + \delta \neq 0,\ u = \frac{\alpha + \beta}{\gamma + \delta}; \qquad \text{if } \gamma + \delta = 0,\ u = \omega.$$

We then take $C =$ the class of all the points on the circumference of a circle drawn on the sphere through the three points P, z, and u, excluding the point P;† the relation $\circledcirc$ being defined as the relation of order along this curve in the direction $P - z - u - P$.

[This system (6), which includes the preceding systems as special cases, was suggested to me by Professor C. L. Bouton, who had noticed that the general formulæ

$$a \oplus b = f[f^{-1}(a) + f^{-1}(b)],$$

and

$$a \odot b = f[f^{-1}(a) \times f^{-1}(b)],$$

in which f and its inverse f^{-1} are single-valued functions, provide a pair of operations which satisfy, in general, the postulates for a field with respect to $\oplus$ and $\odot$. In the present example,

$$f(x) = \frac{\alpha x + \beta}{\gamma x + \delta} \qquad (\alpha\delta - \beta\gamma \neq 0).$$

This transformation (which is the general projective transformation) carries the system (1) into the system (6), the points 0, 1, and the excluded point ω in system (1) going over into the points z, u, and the excluded point P in system (6).

To obtain the system (3), use $f(x) = x/k$. To obtain (4), take $f(x) = 1/x$, and afterwards denote the point ω by the symbol 0 (the excluded point being here $P = 0$). To obtain (5), use the transformation $f(x) = (x/k) + h$.]

* It is especially important to notice that the values of $a \oplus b$ and $a \odot b$, as given by the definitions, are determinate, and belong to the class K, whenever a and b are elements of K.

† It should be noticed that under the given conditions no two of these three points can coincide.

§ 11. *The independence of the postulates.**

The independence of the postulates is proved by the following twenty-eight systems, in each of which K, $\oplus$, $\odot$, C, and $\circledcirc$ are so defined as to satisfy the twenty-seven other postulates, but not the one in question. No one of the postulates, then, can be deducible from the others; for if it were, every system which had the other properties would have this property also, which is not the case.†

For I,1. $K =$ all positive real numbers; $a \oplus b = b$; $a \odot b = ab$; $C = K$; $\circledcirc = <$.

For I,2. $K =$ all real numbers; $a \oplus b = (a + b)/3$; $a \odot b = ab$; $C = K$; $\circledcirc = <$.

For I,3. $K =$ all real numbers; $a \oplus b = 0$; $a \odot b = ab$; $C = K$; $\circledcirc = <$.

For I,4. $K =$ all real numbers; $a \oplus b = a + b$; $a \odot b = b$; $C = K$; $\circledcirc = <$.

For I,5. $K =$ the class of all couples (α, β) in which α and β are real numbers; with

$$(\alpha_1, \beta_1) \oplus (\alpha_2, \beta_2) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2)$$

and

$$(\alpha_1, \beta_1) \odot (\alpha_2, \beta_2) = (\alpha_1\alpha_2 - \beta_1\beta_2, -\alpha_1\beta_2 - \alpha_2\beta_1);$$

$C =$ the class of real numbers, with $\circledcirc = <$. (If we represent these couples as points $\alpha + i\beta$ in the complex plane, $\oplus$ will be the ordinary addition of complex numbers, and $\odot$ will be the ordinary multiplication followed by reflection in the axis of α.)

For I,6. The same system as for I,5, except that here $(\alpha_1, \beta_1) \odot (\alpha_2, \beta_2) = (\alpha_1\alpha_2, \beta_1\alpha_2 + \alpha_1\beta_2 + \beta_1\beta_2)$.

For I,7. $K =$ all real numbers; $a \oplus b = a + b$; $a \odot b = a + b + 1$; $C = K$; $\circledcirc = <$.

For I:8–13. In constructing the proof-systems for these postulates, $\oplus$, $\odot$, and $\circledcirc$ are defined as the ordinary $+$, $\times$, and $<$ (except as to $\odot$ in the case of I,11), and the subclass C is taken identical with K; class K itself being defined appropriately for each case, as follows:

8) $K =$ the class of the three real numbers -1, 0, and 1.

9) $K =$ all positive real numbers.

10) $K =$ all positive real numbers with 0.

11) $K =$ all real numbers; $a \odot b = ab$ when $ab = 0$ or 1; otherwise $a \odot b$ not in the class.

---

* I am greatly indebted to Mr. G. D. BIRKHOFF, of the Harvard Graduate School, who has read this section in manuscript, and checked the correctness of the proofs of independence.

† This is the method of proving independence which has become familiar during the last decade, especially through the work of PEANO, PADOA, PIERI, and HILBERT; but the remark made in the first footnote in § 10, concerning the proof of consistency, applies also here.

12) $K =$ all even integers.

13) $K =$ all integers.

For II,1. $K =$ all complex numbers of the form (x, y), where x and y are rational; $C =$ all real uumbers; with $\oplus$, $\odot$, and $\circledcirc$ defined as the ordinary $+$, $\times$, and $<$.

For II:2–7. In constructing the proof-systems for these postulates we take $K =$ all complex numbers, and define $\oplus$ and $\odot$ as the ordinary $+$ and $\times$; the subclass C and the relation $\circledcirc$ are then chosen appropriately for each postulate, as follows:

2) $C =$ an empty class.

3) $C =$ the class containing the single element 0; with $0 \circledcirc 0$ false.

4) $C =$ the class of all real numbers x such that $-1 \leqq x \leqq 1$; with $\circledcirc = <$.

5) $C =$ all positive real numbers, with or without zero; $\circledcirc = <$.

6) $C =$ all pure imaginary numbers, with 0; $\circledcirc$ defined so that $a \circledcirc b$ is true whenever $a/i < b/i$ (where $i = \sqrt{-1}$).

7) $C =$ all integers, or all even integers; $\circledcirc = <$.

For III:1–3. As the proof-systems for these postulates, take $K =$ all complex numbers, and $C =$ all real numbers, with $\oplus$ and $\odot$ defined as the ordinary $+$ and $\times$, and with $\circledcirc$ defined appropriately for each case, as follows:

1) $a \circledcirc b$ always false.

2) $a \circledcirc b$ always true within C.

3) $a \circledcirc b$ always true within C when $a \neq b$; $a \circledcirc a$ false.

For III,4. $K =$ all complex numbers of the form (x, y), where x and y are rational; $C =$ all rational real numbers; and $\oplus$, $\odot$, and $\circledcirc$ defined as the ordinary $+$, $\times$, and $<$.

For IV,1. $K =$ all complex numbers; $\oplus$ and $\odot$ defined as the ordinary $+$ and $\times$; $C =$ all real numbers, with $\circledcirc$ defined as the ordinary $<$, except that $(a \circledcirc b) = (a > b)$ whenever a and b are both positive.

For IV,2. $K =$ all complex numbers; $a \oplus b = a + b$; $a \odot b = -ab$; $C =$ all real numbers, with $\circledcirc$ defined as the ordinary $<$.

For V,1. $K =$ all real numbers; $C = K$; $\oplus$, $\odot$, and $\circledcirc$ defined as the ordinary $+$, $\times$ and $<$.

For V,2.* $K =$ the class of all expressions T of the form

$$T = A_m t^m + A_{m+1} t^{m+1} + A_{m+2} t^{m+2} + \cdots,$$

where t is a parameter, and m any integer (positive, negative or zero), while the A's are ordinary complex numbers. The operations $\oplus$ and $\odot$ are defined as the ordinary $+$ and $\times$ for such (finite or infinite) expressions. The class C is

* The systems for V,2 were suggested by the non-archimedean number-system given by D. HILBERT in his *Grundlagen der Geometrie*, 1899, § 33. In this connection compare also H. HANKEL, *Theorie der complexen Zahlensysteme*, 1867, § 31.

the class of all those elements T in which all the coefficients are zero except A_0, and A_0 is real; that is, $C =$ the class of real numbers. Within the class C the relation $\odot$ is defined as the ordinary $<$.

This system contains the system of ordinary complex numbers as a part of itself, just as the system of the ordinary complex numbers contains the system of real numbers. A still more inclusive system, still satisfying all the postulates except V,2, may be constructed by replacing the A's in the expression above by expressions S_m, S_{m+1}, S_{m+2}, $\cdots$, of the form

$$S = B_n s^n + B_{n+1} s^{n+1} + B_{n+2} s^{n+2} + \cdots,$$

where s is another parameter, and n any integer, while the B's are ordinary complex numbers.

The list of proofs of independence is thus complete.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS.