ON A CERTAIN SYSTEM OF CONJUGATE LINES ON A SURFACE CONNECTED WITH EULER'S TRANSFORMATION*

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In a paper published in vol. 26 of the American Journal of Mathematics I have proved a number of theorems concerning curves and two-dimensional surfaces in five-dimensional space which belong to a so-called asymptotic complex whose lines satisfy the differential equations

$$(1)\qquad \begin{aligned} dx_5 + x_2\,dx_1 - x_1\,dx_2 + x_4\,dx_3 - x_3\,dx_4 &= 0\,, \\ dx_1\,dx_2 + dx_3\,dx_4 &= 0\,. \end{aligned}$$

If (u) and (v) are the coördinate lines on a surface belonging to such a complex,† and if we make use of the transformation ‡

$$(2)\qquad x_1 = \frac{P_1}{2},\ x_2 = X_1,\ x_3 = \frac{P_2}{2},\ x_4 = X_2,\ x_5 + x_1x_2 + x_3x_4 = X_3,$$

where $X_1, X_2, X_3, P_1, P_2, -1$ are the coördinates of a surface-element in ordinary space, we obtain, as I have shown, a surface in three dimensional space on which the lines (u) and (v) are asymptotic lines.

The geometry of asymptotic complexes is thus seen to be closely connected with the general theory of surfaces; in fact, in five dimensions, to any geometric property of a two-dimensional point-manifoldness belonging to an asymptotic complex corresponds a property of surfaces in ordinary space.

In the first part of this paper it is shown that a certain single projective transformation of the complex (1) will lead to EULER's classical transformation

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† A surface is said to belong to the complex (1) whenever the linear tangents along the u and v lines are lines of the complex.

‡ This transformation, which was used by LIE, when generalized for $2r+1$ variables establishes a correspondence between two spaces M_{2r+1} and M_r of such a nature that the projective group of the linear complex $dx_{2r+1} + \Sigma(x_\nu\,dx_\mu - x_\mu\,dx_\nu) = 0$ becomes a group of irreducible contact-transformations in the space M_r. See LIE, *Theorie der Transformationsgruppen*, Abschnitt II, p. 522.

in three-dimensional space. This transformation, moreover, transforms the asymptotic lines on a surface into a definite system of conjugate lines having a well-defined geometric property which characterizes the system. I have called these lines *Euler's lines*, since they are inseparably bound up with the transformation that bears his name.

The problem to find all surfaces such that EULER's lines are lines of curvature is then considered, and it is found that the determination of such surfaces leads to the integration of a partial differential equation with equal invariants and quadratures. It follows as an immediate corollary that on these surfaces the lines of curvature correspond to asymptotic lines on the transformed surface by EULER's transformation.

While in LIE's sphere-geometry asymptotic lines correspond to lines of curvature by virtue of the well-known contact-transformation that transforms all the ∞^4 lines in space into ∞^4 spheres, in the case of EULER's transformation such correspondence is established only in the case of certain surfaces. It is also worthy of notice that while in LIE's sphere-geometry to a real surface corresponds in general an imaginary surface (the sphere being imaginary), in EULER's transformation corresponding elements are either both real or both imaginary.*

The second part has been devoted to the geometrical definition of EULER's lines and the derivation of their differential equation from this definition.

In the third part it is shown that EULER's transformation is only one among ∞^{10} which change asymptotic lines on a surface into EULER's lines on the corresponding surface. A group of contact-transformations leaving EULER's lines invariant is also considered, and it is shown that it contains ∞^{10} such transformations.

PART 1.

Let there be given in the space M_5 a two-dimensional surface belonging to the complex

$$dx_5 + x_2 dx_1 - x_1 dx_2 + x_4 dx_3 - x_3 dx_4 = 0, \tag{3}$$

and let it be written

$$x_i = \phi_i(u, v) \qquad (i = 1, 2, \cdots, 5). \tag{4}$$

The following conditions must evidently be fulfilled:

$$\begin{aligned} &\frac{\partial \phi_5}{\partial u} + \phi_2 \frac{\partial \phi_1}{\partial u} - \phi_1 \frac{\partial \phi_2}{\partial u} + \phi_4 \frac{\partial \phi_3}{\partial u} - \phi_3 \frac{\partial \phi_4}{\partial u} = 0, \\ &\frac{\partial \phi_5}{\partial v} + \phi_2 \frac{\partial \phi_1}{\partial v} - \phi_1 \frac{\partial \phi_2}{\partial v} + \phi_4 \frac{\partial \phi_3}{\partial v} - \phi_3 \frac{\partial \phi_4}{\partial v} = 0. \end{aligned} \tag{5}$$

* See LIE, *Geometrie der Berührungstransformationen*, vol. 1, pp. 411–480.

Using the transformation (2) we obtain in the space M_3 a surface

$$X_1 = \phi_2, \qquad X_2 = \phi_4, \qquad X_3 = \phi_1\phi_2 + \phi_3\phi_4 + \phi_5,$$

which is the image of the surface (4) considered as a point-locus in M_5. If the complex (3) be asymptotic, that is, if the additional Monge equation

$$dx_1 dx_2 + dx_3 dx_4 = 0$$

be satisfied, and if the surface (4) belongs to this complex, we obtain, as I have proved in the paper mentioned above, a surface in M_3 on which (u) and (v) are asymptotic lines. The analytical conditions which must be satisfied by the functions ϕ_i are, besides (5), the following

$$(6) \qquad \begin{gathered} \frac{\partial\phi_1}{\partial u}\frac{\partial\phi_2}{\partial u} + \frac{\partial\phi_3}{\partial u}\frac{\partial\phi_4}{\partial u} = 0, \qquad \frac{\partial\phi_1}{\partial v}\frac{\partial\phi_2}{\partial v} + \frac{\partial\phi_3}{\partial v}\frac{\partial\phi_4}{\partial v} = 0, \\ \frac{\partial\phi_2}{\partial u}\frac{\partial\phi_3}{\partial v} + \frac{\partial\phi_2}{\partial v}\frac{\partial\phi_3}{\partial u} = 0. \end{gathered}$$

From this it may be proved that the coordinates ϕ_2 and ϕ_4 must satisfy the differential equation with equal invariants *

$$(7) \qquad \frac{\partial^2\theta}{\partial u\,\partial v} + \tfrac{1}{2}\frac{\partial}{\partial v}\log R\cdot\frac{\partial\theta}{\partial u} + \tfrac{1}{2}\frac{\partial}{\partial u}\log R\cdot\frac{\partial\theta}{\partial v} = 0, \dagger$$

where

$$R = \frac{\dfrac{\partial\phi_3}{\partial u}}{\dfrac{\partial\phi_2}{\partial u}} = -\frac{\dfrac{\partial\phi_3}{\partial v}}{\dfrac{\partial\phi_2}{\partial v}}.$$

Conversely, whenever particular solutions ϕ_2 and ϕ_4 of this equation can be found, the other functions ϕ_1, ϕ_3 and ϕ_5 can be obtained by quadratures and the corresponding surfaces thus be determined ‡. It may also be proved that to

* A partial differential equation of the second order

$$\frac{\partial^2\phi}{\partial u\,\partial v} + a\frac{\partial\phi}{\partial u} + b\frac{\partial\phi}{\partial v} + c\phi = 0$$

is said to be one of equal invariants whenever

$$\frac{\partial a}{\partial u} + ab - c = \frac{\partial b}{\partial v} + ab - c.$$

See DARBOUX, *Théorie des Surfaces*, vol. II, chapter 2.

† It should also be noticed that that the function $X_3 = \phi_1\phi_2 + \phi_3\phi_4 + \phi_5$ satisfies (7).

‡ American Journal of Mathematics, vol. 26, pp. 130–134.

any surface in M_3 referred to its asymptotic lines corresponds in M_5 a two-dimensional surface belonging to an asymptotic complex.

Closely associated with an asymptotic complex is a complex defined by the equations

$$(8)\qquad \begin{gathered} dx_5 + x_2 dx_1 - x_1 dx_2 + x_4 dx_3 - x_3 dx_4 = 0, \\ dx_1 dx_2 - dx_3 dx_4 = 0; \end{gathered}$$

in fact, if we employ the transformation

$$(9)\qquad x_1 = \frac{\bar{x}_2}{2},\ x_2 = -2\bar{x}_1,\ x_3 = \bar{x}_3,\ x_4 = \bar{x}_4,\ x_5 = \bar{x}_5,$$

we obtain an asymptotic complex. In the space M_3 this transformation is equivalent to the well-known EULER's transformation *

$$(9')\qquad P_1 = \bar{\bar{X}}_1,\ X_1 = -\bar{P}_1,\ P_2 = \bar{P}_2,\ X_2 = \bar{\bar{X}}_2,\ X_3 = \bar{\bar{X}}_3 - \bar{P}_1 \bar{\bar{X}}_1$$

which transforms the surface-elements of M_3 into the surface-elements of the corresponding space $\bar{\bar{M}}_3$; moreover, it is a contact-transformation, since

$$(10)\qquad dX_3 - P_1 dX_1 - P_2 dX_2 = d\bar{\bar{X}}_3 - \bar{P}_1 d\bar{\bar{X}}_1 - \bar{P}_2 d\bar{\bar{X}}_2.$$

Let there now be given a surface in M_5 belonging to the complex (8). We have

$$(11)\qquad \begin{gathered} \frac{\partial \phi_5}{\partial u} + \phi_2 \frac{\partial \phi_1}{\partial u} - \phi_1 \frac{\partial \phi_2}{\partial u} + \phi_4 \frac{\partial \phi_3}{\partial u} - \phi_3 \frac{\partial \phi_4}{\partial u} = 0, \\ \frac{\partial \phi_5}{\partial v} + \phi_2 \frac{\partial \phi_1}{\partial v} - \phi_1 \frac{\partial \phi_2}{\partial v} + \phi_4 \frac{\partial \phi_3}{\partial v} - \phi_3 \frac{\partial \phi_4}{\partial v} = 0. \end{gathered}$$

$$(12)\qquad \frac{\partial \phi_1}{\partial u}\frac{\partial \phi_2}{\partial u} - \frac{\partial \phi_3}{\partial u}\frac{\partial \phi_4}{\partial u} = 0, \qquad \frac{\partial \phi_1}{\partial v}\frac{\partial \phi_2}{\partial v} - \frac{\partial \phi_3}{\partial v}\frac{\partial \phi_4}{\partial v} = 0.$$

In order that $d\phi_5$ shall be an exact differential, we must also have

$$\frac{\partial \phi_1}{\partial u}\frac{\partial \phi_2}{\partial v} - \frac{\partial \phi_1}{\partial v}\frac{\partial \phi_2}{\partial u} + \frac{\partial \phi_3}{\partial u}\frac{\partial \phi_4}{\partial v} - \frac{\partial \phi_3}{\partial v}\frac{\partial \phi_4}{\partial u} = 0,$$

which, by the aid of the two equations (12), reduces to the form

$$\left(\frac{\partial \phi_3}{\partial u}\frac{\partial \phi_1}{\partial v} + \frac{\partial \phi_3}{\partial v}\frac{\partial \phi_1}{\partial u}\right)\left(\frac{\partial \phi_4}{\partial u}\frac{\partial \phi_1}{\partial v} - \frac{\partial \phi_4}{\partial v}\frac{\partial \phi_1}{\partial u}\right) = 0.$$

* See, for example, LIE, *Berührungstransformationen*, p. 645.

We shall assume that the second factor is different from zero,* so that we may put

$$\frac{\partial \phi_3}{\partial u}\frac{\partial \phi_1}{\partial v}+\frac{\partial \phi_3}{\partial v}\frac{\partial \phi_1}{\partial u}=0, \tag{13}$$

which may be replaced by the equivalent one

$$\frac{\partial \phi_1}{\partial u}\frac{\partial \phi_2}{\partial v}+\frac{\partial \phi_3}{\partial u}\frac{\partial \phi_4}{\partial v}=0. \tag{13'}$$

obtained by eliminating $\partial\phi_1/\partial v$ and $\partial\phi_3/\partial v$ from (13) and the second of (12).

If now we transform the surface $x_i=\phi_i\,(u, v)$ by the transformation (2) *we obtain in M_3 a surface on which (u) and (v) are conjugate lines.* In fact, introducing the coördinates of M_3 in (13′) we get

$$\frac{\partial X_1}{\partial v}\frac{\partial P_1}{\partial u}+\frac{\partial X_2}{\partial v}\frac{\partial P_2}{\partial u}=0,$$

which is the condition that (u) and (v) shall be conjugate lines. Hence the

THEOREM. *To a surface in M_5 belonging to the complex*

$$dx_5+x_2dx_1-x_1dx_2+x_4dx_3-x_3dx_4=0,$$

$$dx_1dx_2-dx_3dx_4=0$$

corresponds by virtue of the transformation

$$x_1=\frac{P_1}{2},\ x_2=X_1,\ x_3=\frac{P_2}{2},\ x_4=X_2,\ x_1x_2+x_3x_4+x_5=X_3$$

a surface in M_3 on which (u) and (v) are conjugate lines.

Suppose now that $\overline{M}_5$ be a space with coördinates $\bar{x}_i$ and let a one-to-one correspondence be established between it and the space M_5 by means of the transformation (9). Since the complex (8) is transformed into an asymptotic complex in $\overline{M}_5$, any surface belonging to the former is transformed into a surface belonging to the latter, and conversely; hence, if we obtain the images in M_3 and $\overline{M}_3$ of the respective surfaces S_3 and $\overline{S}_3$, using the transformation (2), these will be of such a nature that EULER's transformation transforms S_3 into $\overline{S}_3$ and, moreover, to the conjugate lines on S_3 correspond asymptotic lines on $\overline{S}_3$ and conversely, so that we *may* say:

By means of Euler's transformation a one-to-one correspondence is established between two spaces M_3 and $\overline{M}_3$ such that all the surfaces S_3 in M_3 whose images in M_5 are surfaces belonging to the complex

*It may easily be proved that if this factor vanishes the surface $x_i=\phi_i$ $(i=1,\cdots,5)$ will degenerate into a curve; hence the assumption.

$$dx_5 + x_2 dx_1 - x_1 dx_2 + x_4 dx_3 - x_3 dx_4 = 0, \qquad dx_1 dx_2 - dx_3 dx_4 = 0,$$

are transformed into surfaces in $\bar{M}_3$ *whose images in* $\bar{M}_5$ *are surfaces belonging to an asymptotic complex. To the conjugate curves* (u) *and* (v) *on* S_3 *correspond the asymptotic lines* (u) *and* (v) *on* $\bar{S}_3$, *and conversely, to the asymptotic lines* (u) *and* (v) *on* $\bar{S}_3$ *correspond a set of conjugate lines* (u) *and* (v) *on* S_3.

If now we put

$$R = \frac{\dfrac{\partial \phi_2}{\partial u}}{\dfrac{\partial \phi_4}{\partial u}} = -\frac{\dfrac{\partial \phi_2}{\phi v}}{\dfrac{\partial \phi_4}{\partial v}}, \tag{14}$$

the conditions (12) and (13′) may be written

$$\frac{\partial \phi_3}{\partial u} = R\frac{\partial \phi_1}{\partial u}, \qquad \frac{\partial \phi_3}{\partial v} = -R\frac{\partial \phi_1}{\partial v}, \tag{15}$$

$$\frac{\partial \phi_2}{\partial u} = R\frac{\partial \phi_4}{\partial u}, \qquad \frac{\partial \phi_2}{\partial v} = -R\frac{\partial \phi_4}{\partial v}. \tag{16}$$

Eliminating $\partial\phi_3/\partial u$, $\partial\phi_3/\partial v$, $\partial\phi_2/\partial u$, $\partial\phi_2/\partial v$ we find that ϕ_4 and ϕ_1 must satisfy the differential equation

$$\frac{\partial^2 \phi}{\partial u \partial v} + \tfrac{1}{2}\frac{\partial}{\partial v}\log R \cdot \frac{\partial \phi}{\partial u} + \tfrac{1}{2}\frac{\partial}{\partial u}\log R \cdot \frac{\partial \phi}{\partial v} = 0. \tag{17}$$

Conversely, if we know two particular solutions, ϕ_4 and ϕ_1, of this equation, ϕ_3 and ϕ_2 may be obtained by quadratures from (15) and (16) and ϕ_5 likewise from (11).* The surface

$$X_1 = \phi_2, \qquad X_2 = \phi_4, \qquad X_3 = \phi_5 + \phi_1\phi_2 + \phi_3\phi_4, \tag{18}$$

has (u) and (v) for conjugate lines and is transformed by EULER's transformation into a surface on which (u) and (v) are asymptotic lines.

Since the surface (18) is referred to a set of conjugate lines the differential equation of its asymptotic lines must take the form †

$$A du^2 + B dv^2 = 0;$$

in fact, we find by an easy calculation that this equation is

$$\frac{\partial \phi_1}{\partial u}\frac{\partial \phi_2}{\partial u} du^2 + \frac{\partial \phi_1}{\partial v}\frac{\partial \phi_2}{\partial v} dv^2 = 0. \tag{19}$$

* It may also be shown that the function $\phi_5 + \phi_3\phi_4 - \phi_1\phi_2$ satisfies the above equation.

† See DARBOUX, *Leçons*, vol. 1, p. 139, § 110.

Example I. Let $R=$ const., then $\phi_1=\rho_1(u)+\sigma_1(v)=c_1(u-v)$, and $\phi_4=\rho_4(u)+\sigma_4(v)$. From (15) we find that ϕ_3 must have the form $\phi_3=cc_1(u+v)$; likewise from (16) we get $\phi_2=c[\rho_4(u)-\sigma_4(v)]$ and from (11)

$$\phi_5+\phi_1\phi_2+\phi_3\phi_4=4cc_1\int u\rho_4'du+4cc_1\int v\sigma_4'dv.$$

If now we put

$$\int\rho_4du=-F(u),\qquad \int\sigma_4dv=-F_1(v),\qquad c=k,\qquad c_1=\frac{1}{k},$$

we obtain in M_3 the surface

$$(20)\quad X_1=-\frac{k}{4}(F'-F_1'),\ X_2=-F'-F_1',\ X_3=-uF'-vF_1'+F+F_1.$$

By means of EULER's transformation we obtain the surface

$$(21)\quad \begin{aligned} \bar{X}_1&=\frac{u-v}{k},\qquad \bar{X}_2=-F'-F_1',\\ \bar{X}_3&=-\frac{u+v}{2}F'-\frac{u+v}{2}F_1'+F+F_1,\end{aligned}$$

which may also be obtained directly by considering a translation surface in M_3 belonging to an asymptotic complex.* DARBOUX † has derived the same surface by a different method.

The equation of asymptotic lines reduces in the case of the surfaces (20) to the form

$$F''(u)du^2-F_1''(v)dv^2=0$$

which may be integrated by quadratures; it follows, therefore, that the corresponding set of conjugate lines on (21) may be so obtained. ‡

If we take note of the fact that the surface (20) is the most general form of a translation surface whose generating curves are in planes forming a constant angle $\theta=\tan^{-1}8k/(k^2-16)$ with each other, we have the

THEOREM. *The asymptotic lines of all translation surfaces whose generating curves lie in two intersecting planes may be found by quadratures.* If $k=4$, the planes are perpendicular to each other for which special case this theorem has been proved by BIANCHI. §

*American Journal of Mathematics, vol. 26, p. 131.

† *Leçons*, vol. 1, pp. 141–142.

‡ As is shown by DARBOUX's method the lines $(u-v)/k=\alpha$, $(u+v)/2=\beta$ are also a set of conjugate lines which may be constructed by KOENIG's method. On all surfaces (21) we know therefore two different sets of conjugate lines.

§ *Lezioni di geometria differenziale*, p. 111; German edition, p. 113.

Example II. Let $R = 1$; $\phi_2 = u - v$, $\phi_4 = u + v$, $\phi_1 = -\phi_1(v)$, $\phi_3 = \phi_1(v)$ and

$$\phi_5 + \phi_1\phi_2 + \phi_3\phi_4 = -4\int \phi_1(v)\,dv.$$

In M_3 we get the cylinder

$$X_1 = u - v, \qquad X_2 = u + v, \qquad X_3 = \xi(v) \tag{22}$$

whose elements are $v = \text{const.}$ In $\overline{M}_3$ we obtain the ruled surface

$$\overline{X}_1 = 2\phi_1(v),\ \overline{X}_2 = u + v,\ \overline{X}_3 = \xi(v) - 2(u - v)\phi_1(v) \tag{22'}$$

on which (u) and (v) are asymptotic lines. In this case then we observe that to the lines of curvature (u) and (v) on the cylinder correspond asymptotic lines on the ruled surface (22′).

This particular example raises the question whether it is possible to determine all the surfaces on which there is a one-to-one correspondence between lines of curvature and asymptotic lines by virtue of EULER's transformation. To do this we must introduce the condition that the conjugate lines shall be at right angles. We find

$$F = (1 + 4\phi_1^2)\frac{\partial\phi_2}{\partial u}\frac{\partial\phi_2}{\partial v} + (1 + 4\phi_3^2)\frac{\partial\phi_4}{\partial u}\frac{\partial\phi_4}{\partial v} = 0, \tag{23}$$

which by virtue of (14) may be written

$$(1 + 4\phi_1^2)R^2 - (1 + 4\phi_3^2) = 0. \tag{24}$$

This equation may be satisfied if we assume ϕ_2 and ϕ_4 functions of only one variable, but we shall exclude this case, since the surface then degenerates into a curve. Introducing the value of R obtained from (24) in (15), we obtain the two equations

$$\begin{aligned} \tfrac{1}{2}\frac{\partial}{\partial u}\log\left(2\phi_1 + \sqrt{1 + 4\phi_1^2}\right) &= \frac{\dfrac{\partial\phi_3}{\partial u}}{\sqrt{1 + 4\phi_3^2}}, \\ \tfrac{1}{2}\frac{\partial}{\partial v}\log\left(2\phi_1 + \sqrt{1 + 4\phi_1^2}\right) &= \frac{\dfrac{\partial\phi_3}{\partial v}}{\sqrt{1 + 4\phi_3^2}}, \end{aligned} \tag{25}$$

from which we obtain the following differential equation for ϕ_3,

$$\frac{\partial^2\phi_3}{\partial u\partial v} + \tfrac{1}{2}\frac{\partial}{\partial v}\log\frac{1}{\sqrt{1 + 4\phi_3^2}}\cdot\frac{\partial\phi_3}{\partial u} + \tfrac{1}{2}\frac{\partial}{\partial u}\log\frac{1}{\sqrt{1 + 4\phi_3^2}}\cdot\frac{\partial\phi_3}{\partial v} = 0.$$

When simplified, this becomes

$$\frac{\partial^2 \phi_3}{\partial u \partial v} - \frac{4\phi_3}{1+4\phi_3^2}\frac{\partial \phi_3}{\partial v}\frac{\partial \phi_3}{\partial u} = 0,$$

which integrated gives

$$\phi_3 = \frac{\rho^2(u)\sigma^2(v) - 1}{4\rho(u)\sigma(v)}.$$

Substituting this value in (25) and integrating, we also find

$$\phi_1 = \frac{\rho^2 - \sigma^2}{4\rho\sigma},$$

ρ and σ being arbitrary functions of u and v respectively. The value of R is now found to be

$$R = \frac{1+\rho^2\sigma^2}{\rho^2+\sigma^2}.$$

In order to find ϕ_2 and ϕ_4 we substitute this value of R in (16) and eliminate $\partial\phi_4/\partial u$ and $\partial\phi_4/\partial v$. We obtain a differential equation of the form

$$\frac{\partial^2 \phi_2}{\partial u \partial v} + \frac{\sigma\sigma'(1-\rho^4)}{(\rho^2+\sigma^2)(1+\sigma^2\rho^2)}\frac{\partial \phi_2}{\partial u} + \frac{\rho\rho'(1-\sigma^4)}{(\rho^2+\sigma^2)(1+\sigma^2\rho^2)}\frac{\partial \phi_2}{\partial v} = 0,$$

which has equal invariants and of which we know one particular solution, ϕ_3, as may be easily verified. Having obtained a particular solution ϕ_2 (different from ϕ_3) we may obtain ϕ_4 from (16) by quadratures and ϕ_5 may then be found from (11) as before. *We have thus found a surface on which the lines of curvature by* EULER'S *transformation correspond to asymptotic curves on the transformed surface.*

If we put $\rho(u) = u$ and $\sigma(v) = v$, remembering that $\phi_2 = X_2 = \overline{X}_2$ and $2\phi_3 = P_1 = \overline{X}_1$, we may state the preceding result thus:

If a surface, referred to its lines of curvature, (u), (v), is transformed by EULER'S *transformation into a new surface on which (u) and (v) are asymptotic lines, the cartesian coördinates $\overline{X}_1$ and $\overline{X}_2$ of the surface must satisfy the differential equation*

$$(26)\qquad \frac{\partial^2 \phi}{\partial u \partial v} + \frac{v(1-u^4)}{(u^2+v^2)(1+u^2v^2)}\frac{\partial \phi}{\partial u} + \frac{u(1-v^4)}{(u^2+v^2)(1+u^2v^2)}\frac{\partial \phi}{\partial v} = 0.$$

Conversely, whenever two particular solutions, $\overline{X}_1$ and $\overline{X}_2$, of (26) can be obtained, a surface can be found such that Euler's transformation transforms the surface into a new surface on which the asymptotic lines (u) and (v) correspond to the lines of curvature on the original surface.

There is one particular case not included in the above theorem, viz., when either ρ or σ is a constant. We shall consider this case later. The case where $R = \text{const.}$ was considered on p. 457, where for $k = 4$ lines of curvature on a cylinder were obtained.

Surfaces of this class are, as a rule, transcendental. It is not at all difficult to obtain a particular solution of (26) differing from ϕ_3 and depending on an arbitrary constant. In fact, putting

$$\phi_2 = \phi_3 + k\left[\xi(u) + \eta(v)\right] \qquad (k = \text{const.}),$$

and substituting in (26) we obtain

$$v(1-u^4)\frac{d\xi}{du} + u(1-v^4)\frac{d\eta}{dv} = 0,$$

which may be satisfied by putting

$$\frac{d\xi}{du} = \frac{-u}{1-u^4}, \qquad \frac{d\eta}{dv} = \frac{v}{1-v^4},$$

so that ϕ_2 will have the form

$$\phi_2 = \phi_3 - k\int\frac{udu}{1-u^4} + k\int\frac{vdv}{1-v^4}.$$

We may now determine ϕ_4 from (16) by quadratures. We find

$$\phi_4 = \frac{u^2-v^2}{4uv} - k\int\left[\frac{u^2+v^2}{1+u^2v^2}\cdot\frac{u}{1-u^4}du + \frac{u^2+v^2}{1+u^2v^2}\,\frac{v}{1-v^4}dv\right];$$

so that we have the following functions

$$\phi_1 = \frac{u^2-v^2}{4uv}, \qquad \phi_2 = \frac{u^2v^2-1}{4uv} + \frac{k}{2}\log\frac{1-u^2}{1+u^2}\cdot\frac{1+v^2}{1-v^2},$$

$$\phi_3 = \frac{u^2v^2-1}{4uv}, \qquad \phi_4 = \frac{u^2-v^2}{4uv} - \frac{k}{4}\log\frac{(1+u^2v^2)^2}{(1-u^4)(1-v^4)}.$$

ϕ_5 may then be calculated without difficulty from (11). In M_3 we obtain the surface

$$X_1 = \frac{u^2v^2-1}{4uv} + \frac{k}{2}\log\frac{1-u^2}{1+u^2}\cdot\frac{1+v^2}{1-v^2},$$

$$(27) \qquad X_2 = \frac{u^2-v^2}{4uv} - \frac{k}{4}\log\frac{(1+u^2v^2)^2}{(1-u^4)(1-v^4)},$$

$$X_3 = \frac{(u^2v^2-1)(u^2-v^2)}{8u^2v^2} + k\tan^{-1}uv,$$

on which (u) and (v) are lines of curvature. For $k=0$ we obtain the quadric surface $2X_1X_2=X_3$ whose rectilinear generators are $(u^2v^2-1)/4uv=u'$, $(u^2-v^2)/4uv=v'$. Transforming (27) into $\overline{M}_3$ we get the surface

$$(28)\qquad \begin{aligned} \bar{X}_1 &= \frac{u^2-v^2}{2uv}, \qquad \bar{X}_2 = \frac{u^2-v^2}{4uv} - \frac{k}{4}\log\frac{1-u^2}{1+u^2}\cdot\frac{1+v^2}{1-v^2}, \\ \bar{X}_3 &= -\frac{k}{4}\,\frac{u^2-v^2}{uv}\log\frac{(1-u^2)(1+v^2)}{(1+u^2)(1-v^2)} + k\tan^{-1}uv, \end{aligned}$$

on which (u) and (v) are asymptotic lines. For $k=0$ we obtain the straight line $\bar{X}_3=0$, $\bar{X}_1=\bar{X}_2$ which is the transform of the quadric $2\bar{X}_1\bar{X}_2-\bar{X}_3=0$.

We shall now consider the case where R is a function of one variable only, say v. For this purpose it will be sufficient to put $\rho=1/\sqrt{c}$ and $\sigma^2=v^2/c$, so that R will take the form

$$R=\frac{c^2+v^2}{c(1+v^2)}.$$

Introducing this value in (16) and eliminating $\partial\phi_4/\partial u$ and $\partial\phi_4/\partial v$, we obtain the differential equation

$$\frac{\partial^2\phi}{\partial u\partial v}+\tfrac{1}{2}\frac{\partial}{\partial v}\log\frac{c(1+v^2)}{c^2+v^2}\cdot\frac{\partial\phi}{\partial u}=0.$$

By integrating this we find, putting the arbitrary function of u which occurs in the integral equal to u,

$$\phi_2=u\sqrt{R}+\int\sigma_2'R\,dv,$$

and from (25) and (16),

$$\phi_3=\tfrac{1}{4}\left(\frac{c}{v}-\frac{v}{c}\right),\qquad \phi_1=\tfrac{1}{4}\left(v-\frac{1}{v}\right),\qquad \phi_4=\frac{u}{\sqrt{R}}-\sigma_2,$$

where σ_2 is an arbitrary function of v. The surface in M_3 is

$$\begin{aligned} X_1 &= u\sqrt{R}+\int\sigma_2'R\,dv, \qquad X_2=\frac{u}{\sqrt{R}}-\sigma_2, \\ X_3 &= \frac{c^2-1}{\sqrt{c}}\,\frac{uv}{\sqrt{(1+v^2)(c^2+v^2)}}+\frac{1}{c}\int\frac{(v^4-c^2)\sigma_2'}{v(1+v^2)}\,dv; \end{aligned}$$

a ruled surface on which (u) and (v) are lines of curvature. It is, moreover, developable as may easily be proved by forming the differential equation of the asymptotic curve.* In $\overline{M}_3$ we get the ruled surface,

* Since $\partial\phi_1/\partial u=0$, the differential equation (19) becomes

$$\frac{\partial\phi_1}{\partial v}\frac{\partial\phi_2}{\partial v}\,dv^2=0,$$

which means that the surface is developable.

$$\overline{X}_1 = \tfrac{1}{2}\left(v - \frac{1}{v}\right), \quad \overline{X}_2 = \frac{u}{\sqrt{R}} - \sigma_2,$$

$$\overline{X}_3 = \frac{c^2-1}{\sqrt{c}} \frac{uv}{\sqrt{(1+v^2)(c^2+v^2)}} - \frac{(v^2-1)u\sqrt{R}}{2v}$$
$$+ \frac{1}{c}\int \frac{(v^4-c^2)\sigma_2' dv}{v(1+v^2)} - \tfrac{1}{2}\frac{v^2-1}{v}\int \sigma_2' R dv,$$

on which $v =$ const. are the rectilinear generators and $u =$ const. the family of asymptotic lines corresponding to the generators. It will be noticed that all the rectilinear generators lie in a plane parallel to the X_2X_3-plane. For the special value $c = 1$, R becomes equal to unity and we obtain in M_3 a cylinder

$$X_1 = u + \sigma_2, \qquad X_2 = u - \sigma_2, \qquad X_3 = \xi(v)$$

whose elements are parallel to the plane $X_3 = 0$. If we put $\sigma_2(v) = -v'$ this surface takes the form

$$X_1 = u - v, \qquad X_2 = u + v, \qquad X_3 = \xi(v)$$

which is the cylinder obtained on p. 457. The transform of this surface is the surface (22′).

Part 2.

Since through any point on a surface there pass two asymptotic lines, and since Euler's transformation establishes a one-to-one correspondence between the asymptotic lines on a surface in $\overline{M}_3$ and a set of conjugate lines on the transform in M_3, it is clear that this system must be a definite one of all the ∞^1 pairs of conjugate lines that can be made to pass through a given point. The question is, therefore, what geometric property distinguishes this system from all the others. To answer this question we proceed as follows:

Suppose given in M_3 any surface S_3 and let it be referred to a family of conjugate lines (u) and (v). This surface, considered as an ensemble of ∞^2 surface-elements, will when subjected to the transformation (2) become a two-dimensional point-locus in M_5 whose coördinates (u), (v) belong to the null-system

$$dx_5 + x_2 dx_1 - x_1 dx_2 + x_4 dx_3 - x_3 dx_4 = 0.$$

If the coördinates of S_3 be written

$$X_1 = \phi_2, \qquad X_2 = \phi_4, \qquad X_3 = \phi, \tag{30}$$

the image in M_5 will be

$$x_1 = \frac{P_1}{2} = \phi_1, \qquad x_2 = X_1 = \phi_2, \qquad x_3 = \frac{P_2}{2} = \phi_3,$$

$$x_4 = X_2 = \phi_4, \qquad x_5 = X_3 - \frac{P_1}{2} X_1 - \frac{P_2}{2} X_2 = \phi_5,$$

where

$$P_1 = \frac{\partial X_3}{\partial X_1}, \qquad \text{and} \qquad P_2 = \frac{\partial X_3}{\partial X_2}.$$

Since (u) and (v) are conjugate lines, we have

$$(31) \qquad \begin{aligned} \frac{\partial X_3}{\partial u} &= P_1 \frac{\partial X_1}{\partial u} + P_2 \frac{\partial X_2}{\partial u}, \\ \frac{\partial X_3}{\partial v} &= P_1 \frac{\partial X_1}{\partial v} + P_2 \frac{\partial X_2}{\partial v}, \end{aligned}$$

$$(32) \qquad \frac{\partial P_1}{\partial u} \frac{\partial X_1}{\partial v} + \frac{\partial P_2}{\partial u} \frac{\partial X_2}{\partial v} = 0,$$

from which it also follows, since dX_3 must be an exact differential,

$$(33) \qquad \frac{\partial P_1}{\partial v} \frac{\partial X_1}{\partial u} + \frac{\partial P_2}{\partial v} \frac{\partial X_2}{\partial u} = 0.$$

We now put

$$(34) \qquad \frac{\dfrac{\partial X_1}{\partial u}}{\dfrac{\partial X_2}{\partial u}} = R_1(u, v), \qquad \frac{\dfrac{\partial X_1}{\partial v}}{\dfrac{\partial X_2}{\partial v}} = R_2(u, v),$$

so that the equations (32) and (33) take the form

$$(35) \qquad R_2 \frac{\partial P_1}{\partial u} + \frac{\partial P_2}{\partial u} = 0, \qquad R_1 \frac{\partial P_1}{\partial v} + \frac{\partial P_2}{\partial v} = 0.$$

Eliminating $\partial X_1/\partial u$, $\partial X_1/\partial v$ from (34) and $\partial P_2/\partial u$, $\partial P_2/\partial v$ from (35), we find that X_2 and P_1 must satisfy the differential equation

$$(36) \qquad \frac{\partial^2 \theta}{\partial u \partial v} + \frac{\dfrac{\partial R_1}{\partial v}}{R_1 - R_2} \cdot \frac{\partial \theta}{\partial u} - \frac{\dfrac{\partial R_2}{\partial u}}{R_1 - R_2} \cdot \frac{\partial \theta}{\partial v} = 0.$$

Conversely, if two particular solutions of this equation can be found, we obtain from (34), (35) and (31) by means of quadratures a surface on which (u) and (v) are conjugate lines. Now suppose, in particular, that $-R_2 = R_1 = R$. Since R_1 and R_2 are the respective slopes of the tangents drawn to a point of intersection of the projections of any pair of conjugate lines on the $X_1 X_2$-plane,

this relation expresses the geometrical property that *the triangle formed by any pair of such tangents and either axis is isosceles.**

The equation (36) may now be written

$$\frac{\partial^2 \theta}{\partial u \partial v} + \tfrac{1}{2}\frac{\partial}{\partial v}\log R \cdot \frac{\partial \theta}{\partial u} + \tfrac{1}{2}\frac{\partial}{\partial u}\log R \cdot \frac{\partial \theta}{\partial v} = 0.$$

But, since $X_2 = \phi_4(u, v)$, and $P_1 = 2\phi_1$, this is the equation (17) on page 455. We have therefore the

THEOREM. *If a surface possesses a set of conjugate lines* (u) *and* (v) *such that the pair of tangents drawn to the point of intersection of the projection of the curves on the* $X_1 X_2$*-plane form with either axis an isosceles triangle, then will* EULER'S *transformation transform it into a surface on which the asymptotic lines correspond to this family of conjugate lines.* We shall call such a set of conjugate lines *Euler's lines.*

The relation $R_2 = -R_1$ may be written

$$\frac{\partial X_2}{\partial u}\frac{\partial X_1}{\partial v} + \frac{\partial X_2}{\partial v}\frac{\partial X_1}{\partial u} = 0$$

from which, by the use of (32) and (33) we deduce the equations

$$\frac{\partial X_1}{\partial u}\frac{\partial P_1}{\partial u} - \frac{\partial X_2}{\partial u}\frac{\partial P_2}{\partial u} = 0, \qquad \frac{\partial X_1}{\partial v}\frac{\partial P_1}{\partial v} - \frac{\partial X_2}{\partial v}\frac{\partial P_2}{\partial v} = 0.$$

That is to say, the image of the surface in M_5 belongs to the complex (7) which we shall call a *conjugate complex.* Conversely, the relation $R_2 = -R_1$ is satisfied for all surfaces in M_3 whose images in M_5 by the transformation (2) belong to a conjugate complex, so that we may say:

The necessary and sufficient condition that a surface in M_3 *referred to a set of conjugate lines shall have as image in* M_5 *a surface belonging to a conjugate complex is that the conjugate lines shall be a family of* EULER'S *lines.*

Let there be given on any surface in M_3 a system of lines satisfying the differential equation

$$dX_1 dP_1 - dX_2 dP_2 = 0, \tag{37}$$

which expanded may be written

* This may also be expressed thus: The conjugate lines, when projected on the $X_1 X_2$-plane form infinitesimal parallelograms whose areas equal

$$2\frac{\partial X_1}{\partial u}\frac{\partial X_2}{\partial v}\,du dv;$$

this may easily be proved geometrically.

$$(38)\quad \left(\frac{\partial X_1}{\partial u}\frac{\partial P_1}{\partial u}-\frac{\partial X_2}{\partial u}\frac{\partial P_2}{\partial u}\right)du^2$$
$$+\left(\frac{\partial X_1}{\partial v}\frac{\partial P_1}{\partial u}+\frac{\partial X_1}{\partial u}\frac{\partial P_1}{\partial v}-\frac{\partial X_2}{\partial v}\frac{\partial P_2}{\partial u}-\frac{\partial X_2}{\partial u}\frac{\partial P_2}{\partial v}\right)dudv$$
$$+\left(\frac{\partial X_1}{\partial v}\frac{\partial P_1}{\partial v}-\frac{\partial X_2}{\partial v}\frac{\partial P_2}{\partial v}\right)dv^2=0.$$

We shall investigate some of the properties of these lines. Since the differential equation is of the second degree it is satisfied by an infinite system of curves $\phi_1=c_1$, $\phi_2=c_2$. Let us suppose that the surface be referred to this system; since now $u=$ const., $v=$ const. must be integrals of the equation we have

$$(39)\quad \frac{\partial X_1}{\partial u}\frac{\partial P_1}{\partial u}-\frac{\partial X_2}{\partial u}\frac{\partial P_2}{\partial u}=0,\qquad \frac{\partial X_1}{\partial v}\frac{\partial P_1}{\partial v}-\frac{\partial X_2}{\partial v}\frac{\partial P_2}{\partial v}=0.$$

But we also have

$$\frac{\partial X_3}{\partial u}=P_1\frac{\partial X_1}{\partial u}+P_2\frac{\partial X_2}{\partial u},\qquad \frac{\partial X_3}{\partial v}=P_1\frac{\partial X_1}{\partial v}+P_2\frac{\partial X_1}{\partial v}.$$

Eliminating $\partial X_3/\partial u$ and $\partial X_3/\partial v$ we get

$$(40)\quad \frac{\partial X_1}{\partial u}\frac{\partial P_1}{\partial v}+\frac{\partial X_2}{\partial u}\frac{\partial P_2}{\partial v}-\frac{\partial P_1}{\partial u}\frac{\partial X_1}{\partial v}-\frac{\partial P_2}{\partial u}\frac{\partial X_2}{\partial v}=0,$$

and eliminating $\partial P_1/\partial u$, $\partial P_1/\partial v$ from equations (39) and (40) we obtain the equation

$$\left(\frac{\partial X_2}{\partial u}\frac{\partial X_1}{\partial v}+\frac{\partial X_2}{\partial v}\frac{\partial X_1}{\partial u}\right)\left(\frac{\partial X_1}{\partial u}\frac{\partial P_2}{\partial v}-\frac{\partial X_1}{\partial v}\frac{\partial P_2}{\partial u}\right)=0.$$

If the second factor vanishes we find that combined with (39) it will cause the coefficient of $dudv$ in (38) to vanish, in which case every line on the surface will satisfy the equation; this can happen only for special surfaces. Excluding this case we have

$$(41)\quad \frac{\partial X_2}{\partial u}\frac{\partial X_1}{\partial v}+\frac{\partial X_2}{\partial v}\frac{\partial X_1}{\partial u}=0;$$

combining this again with the second of equations (39) we obtain

$$\frac{\partial P_1}{\partial u}\frac{\partial X_1}{\partial v}+\frac{\partial P_2}{\partial u}\frac{\partial X_2}{\partial v}=0.$$

This is the condition that (u) and (v) shall be conjugate lines. Moreover,

these lines constitute a family of EULER's lines, since the condition (41) is just the relation $R_2 = -R_1$. Hence the

THEOREM. *If an infinite family of curves* $\phi_1 = c_1$, $\phi_2 = c_2$ *on a surface satisfy the differential equation*

$$\begin{aligned}(38)\qquad &\left(\frac{\partial P_1}{\partial u}\frac{\partial X_1}{\partial u} - \frac{\partial P_2}{\partial u}\frac{\partial X_2}{\partial u}\right)du^2 \\ &+\left(\frac{\partial X_1}{\partial v}\frac{\partial P_1}{\partial u} + \frac{\partial X_1}{\partial u}\frac{\partial P_1}{\partial v} - \frac{\partial X_2}{\partial v}\frac{\partial P_2}{\partial u} - \frac{\partial X_2}{\partial u}\frac{\partial P_2}{\partial v}\right)du\,dv \\ &+\left(\frac{\partial P_1}{\partial v}\frac{\partial X_1}{\partial v} - \frac{\partial P_2}{\partial v}\frac{\partial X_2}{\partial v}\right)dv^2 = 0,\end{aligned}$$

then will these lines be a family of EULER's *lines.**

Conversely, *a family of* EULER's *lines on a surface satisfies the differential equation* (38).

PROOF. Let the curves be $\xi_1(u, v) = c_1$, $\xi_2(u, v) = c_2$, which by hypothesis are EULER's lines. Introducing the curvilinear coördinates $\xi_1 = u'$, $\xi_2 = v'$ in the rectangular coördinates of the surface, we have by hypothesis

$$(42)\qquad \begin{aligned}&\frac{\partial P_1}{\partial u'}\frac{\partial X_1}{\partial v'} + \frac{\partial P_2}{\partial u'}\frac{\partial X_2}{\partial v'} = 0,\\ &\frac{\partial X_2}{\partial u'}\frac{\partial X_1}{\partial v'} + \frac{\partial X_2}{\partial v'}\frac{\partial X_1}{\partial u'} = 0,\end{aligned}$$

to which we may also add the condition

$$(43)\qquad \frac{\partial P_1}{\partial v'}\frac{\partial X_1}{\partial u'} + \frac{\partial P_2}{\partial v'}\frac{\partial X_2}{\partial u'} = 0,$$

since dX_3 must be an exact differential also after the introduction of the new coördinates. The conditions (42) and (43) reduce to the following

$$(43')\qquad \frac{\partial P_1}{\partial u'}\frac{\partial X_1}{\partial u'} - \frac{\partial P_2}{\partial u'}\frac{\partial X_2}{\partial u'} = 0, \qquad \frac{\partial P_1}{\partial v'}\frac{\partial X_1}{\partial v'} - \frac{\partial P_2}{\partial v'}\frac{\partial X_2}{\partial v'} = 0.$$

Expressing $\partial P_i/\partial u'$, $\partial P_i/\partial v'$, $\partial X_i/\partial u$, $\partial X_i/\partial v$ $(i = 1, 2)$ in terms of $\partial P_i/\partial u$, $\partial P_i/\partial v$, $\partial X_i/\partial u$, $\partial X_i/\partial v$, ξ_1 and ξ_2 and substituting in (43′) we get

* If u and v are a set of asymptotic lines on the surface this equation is simplified; in fact, an easy calculation will show that it reduces to

$$\frac{\partial P_2}{\partial u}\frac{\partial X_2}{\partial u}du^2 + \frac{\partial P_2}{\partial v}\frac{\partial X_2}{\partial v}dv^2 = 0.$$

$$\left(\frac{\partial P_1}{\partial v}\frac{\partial X_1}{\partial v}-\frac{\partial X_2}{\partial v}\frac{\partial P_2}{\partial v}\right)\left(\frac{\partial \xi_1}{\partial v}\right)^2$$
$$-\left(\frac{\partial X_2}{\partial u}\frac{\partial P_2}{\partial v}+\frac{\partial X_2}{\partial v}\frac{\partial P_2}{\partial u}-\frac{\partial X_1}{\partial u}\frac{\partial P_1}{\partial v}-\frac{\partial X_1}{\partial v}\frac{\partial P_1}{\partial u}\right)\frac{\partial \xi_1}{\partial u}\frac{\partial \xi_1}{\partial v}$$
$$+\left(\frac{\partial P_1}{\partial u}\frac{\partial X_1}{\partial u}-\frac{\partial P_2}{\partial u}\frac{\partial X_2}{\partial u}\right)\left(\frac{\partial \xi_1}{\partial u}\right)^2=0,$$

and an equation of identically the same form in $\partial\xi_2/\partial u$, $\partial\xi_2/\partial v$. But, since $\xi_1=c_1$ and $\xi_2=c_2$, we have

$$\frac{du}{dv}=-\frac{\dfrac{\partial \xi_1}{\partial u}}{\dfrac{\partial \xi_1}{\partial v}}=-\frac{\dfrac{\partial \xi_2}{\partial u}}{\dfrac{\partial \xi_2}{\partial v}},$$

and substituting these ratios in the above, we obtain the differential equation (38). Q. E. D.

The theorem on p. 458 may now be stated thus:

The problem of finding all surfaces on which Euler's lines are lines of curvature depends on the integration of the equation (26) *and quadratures. These surfaces have the following geometric property. The projection of the lines of curvature on the X_1X_2-plane form a system such that the two tangents drawn at a point of intersection of any pair of curves form with either axis an isosceles triangle.*

If the system S_1 be referred to a set of asymptotic lines, the equation (38) of EULER's lines takes the simple form

$$\frac{\partial P_2}{\partial u}\frac{\partial X_2}{\partial u}du^2+\frac{\partial P_2}{\partial v}\frac{\partial X_2}{\partial v}dv^2=0. \tag{38'}$$

If S_1 be transformed into S_2 the asymptotic curves $u=c$, $v=c$ become EULER's lines on S_2, and the equation of asymptotic lines on S_2 will be identically of the same form as (38′), since by EULER's transformation $X_2=X_2$ and $P_2=P_2$. It follows therefore that if on any surface S_1 we know the asymptotic lines, the EULER lines are known on the corresponding surface S_2.

PART 3.

The following question now presents itself: Are there contact-transformations other than EULER's that will change asymptotic lines into EULER's lines? We shall find that there exist ∞^{10} such transformations. To establish this it will be convenient to go back to the space M_5 from which we started.

A null-system in M_5, defined by the equations

$$(44)\qquad \begin{aligned} x_i &= \rho_i x_5 + \sigma_i \qquad (i=1,2,3,4),\\ \rho_2\sigma_1 - \rho_4\sigma_2 + \rho_4\sigma_3 - \rho_3\sigma_4 + 1 &= 0, \end{aligned}$$

will when subjected to the transformation (2) become an ensemble of surface-elements consisting of element-bands formed by ∞^5 parabolæ

$$(45)\qquad \begin{aligned} \rho_4 X_1 - \rho_2 X_2 &= \sigma_2\rho_4 - \sigma_4\rho_2,\\ X_3 &= \frac{\rho_1\rho_2 + \rho_3\rho_4}{\rho_2^2}(X_1 - \sigma_2)^2 + \frac{2(\rho_2\sigma_1 + \rho_4\sigma_3)}{\rho_2}(X_1 - \sigma_2) + \sigma_1\sigma_2 + \sigma_3\sigma_4, \end{aligned}$$

lying in planes parallel to the X_3-axis.* Moreover, the coördinates of the plane of each surface-element are subject to the conditions

$$(46)\qquad \begin{aligned} \rho_2 \frac{P_1}{2} &= \rho_1 X_1 + \sigma_1\rho_2 - \sigma_2\rho_1,\\ \rho_2 \frac{P_2}{2} &= \rho_3 X_1 + \sigma_3\rho_2 - \sigma_2\rho_3. \end{aligned}$$

To each line of the nullsystem corresponds one of the parabolæ considered as an element-band, or, in other words, to the ∞^1 points of the line correspond the ∞^1 surface-elements of the parabolæ. We shall now consider the effect of EULER's transformation on these element-bands when the nullsystem becomes a conjugate complex, that is, when the additional MONGE equation

$$dx_1\,dx_2 - dx_3\,dx_4 = 0$$

is satisfied. Now in order that the lines (44) shall be lines of such a complex we must have

$$\rho_1\rho_2 - \rho_3\rho_4 = 0.$$

Introducing this condition in (45) we obtain the parabolæ

$$(47)\qquad \begin{aligned} \rho_4 X_1 - \rho_2 X_2 &= \sigma_2\rho_4 - \sigma_4\rho_2,\\ X_3 &= \frac{2\rho_1}{\rho_2}(X_1 - \sigma_2)^2 + \frac{2(\rho_2\sigma_1 + \rho_4\sigma_3)}{\rho_2}(X_1 - \sigma_2) + \sigma_1\sigma_2 + \sigma_3\sigma_4, \end{aligned}$$

while the conditions (46) become

$$(46')\qquad \begin{aligned} \rho_2 \frac{P_1}{2} &= \rho_1 X_1 + \sigma_1\rho_2 - \sigma_2\rho_1,\\ \rho_4 \frac{P_2}{2} &= \rho_1 X_1 + \sigma_3\rho_4 - \sigma_2\rho_1. \end{aligned}$$

* American Journal of Mathematics, vol. 26.

Next I shall show that *Euler's transformation transforms these* ∞^5 *parabolæ into* ∞^4 *straight lines.* In fact, transforming (46′) and (47) and reducing, we have

$$(48)\qquad \begin{aligned} &\rho_4\bar{X}_1 - 2\rho_1\bar{X}_2 = 2(\sigma_1\rho_4 - \rho_1\sigma_4),\\ &\bar{X}_3 = \frac{\rho_4\sigma_3 - \rho_1\sigma_2}{\rho_1}\bar{X}_1 - \frac{2\sigma_1\rho_4\sigma_3}{\rho_1} + \sigma_1\sigma_2 + \sigma_3\sigma_4, \end{aligned}$$

which represent ∞^4 straight lines in M_3.

In the same way we may prove without difficulty that if the null-system (44) becomes asymptotic so that $\rho_1\rho_2 + \rho_3\rho_4 = 0$, the ∞^4 straight lines

$$(49)\qquad \begin{aligned} &\rho_4 X_1 - \rho_2 X_2 = \sigma_2\rho_4 - \sigma_4\rho_2,\\ &X_3 = \frac{2(\rho_2\sigma_1 + \rho_4\sigma_3)}{\rho_2}(X_1 - \sigma_2) + \sigma_1\sigma_2 + \sigma_3\sigma_4. \end{aligned}$$

are by EULER's transformation transformed into ∞^5 parabolæ of a form similar to (47).

Since all the contact-transformations that transform (47) into (48) also transform EULER's lines into asymptotic lines and vice versa our problem is now simplified. Let us consider an asymptotic complex in M_5. There exist ∞^{10} projective transformations which leave it invariant.* In M_3 we get a group of ∞^{10} contact transformations which leave invariant the differential equations

$$(50)\qquad \begin{aligned} dX_3 - P_1 dX_1 - P_2 dX_2 &= 0,\\ dX_1 dP_1 + dX_2 dP_2 &= 0. \end{aligned}$$

These transformations have the form

$$(51)\qquad \begin{aligned} \bar{X}_1 &= a_1 X_1 + b_1 X_2 + d_1,\\ \bar{X}_2 &= a_2 X_1 + b_2 X_2 + d_2,\\ \bar{X}_3 &= a_3 X_1 + b_3 X_2 + c_3 X_3 + d_3,\\ \bar{P}_1 &= \frac{b_2 c_3}{\Delta} P_1 - \frac{a_2 c_3}{\Delta} P_2 + \frac{a_3 b_2 - a_2 b_3}{\Delta},\\ \bar{P}_2 &= \frac{-b_1 c_3}{\Delta} P_1 + \frac{a_1 c_3}{\Delta} P_2 + \frac{a_1 b_3 - a_3 b_1}{\Delta}, \end{aligned}$$

where $\Delta = a_1 b_2 - a_2 b_1$. We notice that they are projective point-transformations since the coördinates $\bar{X}_1$, $\bar{X}_2$, $\bar{X}_3$ do not involve P_1 and P_2; all these

*American Journal of Mathematics, vol. 26, p. 146.

transformations form a subgroup of the general linear group. We shall denote any one of them by the symbol T_a.

If now we superpose EULER's transformation on (51) we obtain ∞^{10} contact-transformations

$$
(52)\qquad
\begin{aligned}
\bar{\bar{X}}_1 &= a_1 P_1 + b_1 X_2 + d_1,\\
\bar{\bar{X}}_2 &= a_2 P_1 + b_2 X_2 + d_2,\\
\bar{\bar{X}}_3 &= a_3 P_1 + b_3 X_2 + b_3(X_3 - P_1 X_1) + d_3,\\
\bar{\bar{P}}_1 &= -\frac{b_2 c_3}{\Delta} X_1 - \frac{a_2 c_3}{\Delta} P_2 + \frac{a_3 b_2 - a_2 b_3}{\Delta},\\
\bar{\bar{P}}_2 &= \frac{b_1 c_3}{\Delta} X_1 + \frac{a_1 c_3}{\Delta} P_2 + \frac{a_1 b_3 - a_3 b_1}{\Delta},
\end{aligned}
$$

which change the differential equation

$$(53)\qquad d\bar{X}_1 d\bar{P}_1 + d\bar{X}_2 d\bar{P}_2 = 0,$$

into

$$(54)\qquad dX_1 dP_1 - dX_2 dP_2 = 0,$$

and, therefore, also asymptotic lines into EULER's *lines.* They also change ∞^5 parabolæ (47) into the ∞^4 straight lines (48); we shall denote them by $T_a E$. The transformations (52) do not form a group, for, if so we should have $T_a E T_b E = T_k E$, where T_a, T_b, T_k are transformations of the group (51), that is, $T_a E T_b = T_k$. But the succession of T_a, E and T_b cannot be equivalent to a projective transformation.

All the transformations leaving the equation (53) invariant are included in the group (51) and LEGENDRE's transformation.* The transformation EL will, therefore, have the same property as any one of (52). But the succession of two EULER's transformations is a LEGENDRE's transformation so that the combination of a Legendre transformation with any one of the transformations (52), or, what is the same thing, *the succession of T_a and an odd number of* EULER's *transformations will transform* EULER's *lines $u = c$, $v = c$ on S_1 into the asymptotic lines $u = c$, $v = c$ on S_2.*

If we transform the coördinates on the left hand side in (52) by EULER's transformation we obtain ∞^{10} contact-transformations which leave invariant the equation (54). *These transformations form a group;* in fact calling any one of them T_e we have $T_e = E^{-1} T_a E$ and $E^{-1} T_a E\, E^{-1} T_b E = E^{-1} T_c E$. This

*See p. 148 of my article, American Journal of Mathematics, vol. 26. By a curious mistake the name EULER's transformation is there given to the well-known one of LEGENDRE:

$$\bar{X}_1 = P_2, \qquad \bar{X}_2 = -P_1, \qquad \bar{X}_3 = X_3 - P_1 X_1 - P_2 X_2, \qquad \bar{P}_1 = -X_2, \qquad \bar{P}_2 = X_1.$$

group is a subgroup of a group of contact-transformations connected with the projective group of the non-special nullsystem in M_5

$$dx_5 + x_2 dx_1 - x_1 dx_2 + x_4 dx_3 - x_3 dx_4 = 0,$$

which has been discussed by LIE in the second volume of his *Theorie der Transformationsgruppen.* The transformations are

$$\begin{aligned}
\bar{P}_1 &= a_1 P_1 + b_1 X_2 + d_1, \\
P_2 &= \frac{c_3 b_1 X_1}{\Delta} + \frac{a_1 c_3}{\Delta} P_2 + \frac{a_1 b_3 - a_3 b_1}{\Delta}, \\
\bar{X}_1 &= \frac{b_2 c_3 X_1}{\Delta} + \frac{a_2 c_3}{\Delta} P_2 - \frac{a_3 b_2 - a_2 b_3}{\Delta}, \\
\bar{X}_2 &= a_2 P_1 + b_2 X_2 + d_2, \\
\bar{X}_3 - \bar{P}_1 \bar{X}_1 &= a_3 P_1 + b_3 X_2 + c_3 (X_3 - P_1 X_1) + d_3.
\end{aligned} \tag{55}$$

Now since LEGENDRE's transformation also leaves (54) invariant, all the transformations that leave this equation invariant are obtained by superposing LEGENDRE's on any one of the ∞^{10} transformations (55). We have thus found all the transformations which change the ∞^5 parabolæ (47) into themselves. It should be noticed that the group (55) is similar to the projective group (51).*

In conclusion we shall give the following resumé putting in evidence the relations of the spaces M_5 and M_3:

M_5.	M_3.
I. $\begin{cases}(a)\ dx_5 + x_2 dx_1 - \cdots = 0, \\ (b)\ dx_1 dx_2 + dx_3 dx_4 = 0.\end{cases}$	I. $\begin{cases}(a)\ dX_3 - P_1 dX_1 - P_2 dX_2 = 0, \\ (b)\ dX_1 dP_1 + dX_2 dP_2 = 0.\end{cases}$
II. Projective group of nullsystem (a) (21 parameters).	II. Irreducible group, C_3, of contact transformations.
III. Sub-group G_a of ∞^{10} transformations leaving the asymptotic complex I invariant.	III. Subgroup C_a of C_3 leaving I invariant, i. e., transforming asymptotic lines (u), (v) on S_1 into asymptotic lines (u), (v) on S_2.
IV. Special projective transformation not included in II leaving I invariant ($\bar{x}_2 = 2x_3$, $\bar{x}_4 = -2x_1$, $2\bar{x}_1 = -x_4$, $2\bar{x}_3 = x_2$, $\bar{x}_5 = x_5$).	IV. LEGENDRE's transformation leaving I invariant.

* See LIE-SCHEFFER's *Continuierliche Gruppen*, p. 427.

M_5.	M_3.
V. $\begin{cases}(a) & dx_5 + x_2 dx_1 - \cdots = 0, \\ (b) & dx_1 dx_2 - dx_3 dx_4 = 0.\end{cases}$	V. $dX_3 - P_1 dX_1 - P_2 dX_2 = 0,$ $dX_1 dP_1 - dX_2 dP_2 = 0.$
VI. Linear projective group of ∞^{10} transformations leaving V invariant (denoted by G_c).	VI. Group of contact transformations similar to C_3, leaving V invariant, i. e., changing EULER's lines (u), (v) on S_1 into EULER's lines (u), (v) on S_2.
VII. Special projective transformation not included in VI, changing I into V.	VII. EULER's transformation changing EULER's lines into asymptotic lines.
VIII. ∞^{10} projective transformations transforming an asymptotic complex I into a conjugate complex V.	VIII. ∞^{10} contact transformations changing I into V, or changing EULER's lines (u), (v) on S_1 into asymptotic lines (u), (v) on S_2.

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