ON THE STABILITY OF THE MOTION OF A VISCOUS LIQUID*

BY

FRANCIS ROBERT SHARPE

§1. *Introduction.* REYNOLDS † has established an energy criterion for the stability of the motion of a viscous liquid, and has applied it to the case of a liquid moving between two parallel planes. For a liquid of density ρ and viscosity μ, moving between two planes $2b$ apart with mean velocity U he found that the motion is unstable when

$$\frac{2b\rho U}{\mu} > 517.$$

Using the principle that the critical velocity is inversely proportional to the hydraulic mean depth he inferred that for a cylindrical pipe of radius a the critical velocity is given by

$$\frac{2a\rho U}{\mu} > 1034,$$

and he compared this with his experimental value 1900. In the following paper I discuss directly the case of the cylindrical pipe and find that the motion is unstable when

$$\frac{2a\rho U}{\mu} > 470.$$

I also find, by using a different solution of the equation of continuity and the boundary conditions, that the motion between two parallel planes is unstable when

$$\frac{2b\rho U}{\mu} > 167,$$

instead of 517 as found by REYNOLDS.

§ 2. *Rectilinear flow through a pipe.* When the motion through a pipe is symmetrical the equations of motion, expressed in cylindrical coördinates r, θ, z, are ‡

$$(1) \qquad \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z}\right) = \frac{\partial p_{rr}}{\partial r} + \frac{\partial p_{zr}}{\partial z} + \frac{p_{rr} - p_{\theta\theta}}{r},$$

$$(2) \qquad \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}\right) = \frac{\partial p_{rz}}{\partial r} + \frac{\partial p_{zz}}{\partial z} + \frac{p_{rz}}{r}.$$

The extensions are* $\partial u/\partial r$, u/r, $\partial w/\partial z$. and the shear $\partial u/\partial z + \partial w/\partial r$. Hence †

$$p_{rr} = -p + 2\mu\frac{\partial u}{\partial r}, \qquad p_{\theta\theta} = -p + 2\mu\frac{u}{r}, \qquad p_{zz} = -p + 2\mu\frac{\partial w}{\partial z},$$

$$p_{zr} = p_{rz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right).$$

The equation of continuity is ‡

$$(3) \qquad \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0,$$

and the boundary conditions are

$$(4) \qquad u = w = 0 \qquad \text{when} \qquad r = a.$$

The simplest solution of these equations is §

$$u = \frac{\partial w}{\partial t} = \frac{\partial w}{\partial z} = \frac{\partial p}{\partial r} = 0, \qquad \frac{\partial p}{\partial z} = \text{constant},$$

$$w = -\frac{1}{4\mu}\frac{\partial p}{\partial z}(a^2 - r^2),$$

and the mean velocity of flow is

$$U = -\frac{a^2}{8\mu}\frac{\partial p}{\partial z}.$$

§ 3. *Steady non-rectilinear motion.* For some minimum value of U, to be determined, permanent eddying motion is possible, that is, the rectilinear motion is unstable. The mean motion is still rectilinear but u is no longer zero at any particular instant. We assume then that

$$(5) \qquad u = u', \qquad w = \bar{w} + w',$$

where $\bar{w}$ is a function of r, and u' and w' are periodic functions of r and z whose mean values are zero.

The method used by REYNOLDS is to calculate the average rate of increase of the motion $u = u'$, $w = w'$, which is the motion relative to the mean motion $u = 0$, $w = \bar{w}$, for a volume of liquid extending to the boundaries of the liquid.

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* LOVE, loc. cit., p. 216.
† LAMB, *Hydrodynamics*, p. 512, 521.
‡ LAMB, loc. cit., p. 134.
§ LAMB, loc. cit., p. 521.

§ 4. *The equations of mean motion.* Using a bar to denote a mean value we have, on substituting (5) in (1), (2) and (3),

$$\rho \left\{ \frac{\partial u'}{\partial t} + u' \frac{\partial u'}{\partial r} + (\bar{w} + w') \frac{\partial u'}{\partial z} \right\} = \frac{\partial(\bar{p}_{rr} + p'_{rr})}{\partial r} + \frac{\partial(\bar{p}_{zr} + p'_{zr})}{\partial z}$$

$$+ \frac{p_{rr} + p'_{rr} - \bar{p}_{\theta\theta} - p'_{\theta\theta}}{r},$$

$$\rho \left\{ \frac{\partial w'}{\partial t} + u' \frac{\partial w'}{\partial r} + (\bar{w} + w') \frac{\partial w'}{\partial z} + u' \frac{\partial \bar{w}}{\partial r} \right\} = \frac{\partial(\bar{p}_{rz} + p'_{rz})}{\partial r}$$

$$+ \frac{\partial(\bar{p}_{zz} + p'_{zz})}{\partial z} + \frac{\bar{p}_{rz} + p'_{rz}}{r},$$

(6) $$\frac{\partial u'}{\partial r} + \frac{u'}{r} + \frac{\partial w'}{\partial z} = 0.$$

Multiplying (6) by $\rho u'$ and $\rho w'$ respectively and adding to the preceding equations,

$$\rho \left\{ \frac{\partial u'}{\partial t} + \frac{\partial(u'u')}{\partial r} + \frac{\partial(u'w')}{\partial z} + \bar{w} \frac{\partial u'}{\partial z} + \frac{u'u'}{r} \right\} = \frac{\partial(\bar{p}_{rr} + p'_{rr})}{\partial r}$$

$$+ \frac{\partial(\bar{p}_{zr} + p'_{zr})}{\partial z} + \frac{\bar{p}_{rr} + p'_{rr} - \bar{p}_{\theta\theta} - p'_{\theta\theta}}{r},$$

$$\rho \left\{ \frac{\partial w'}{\partial t} + \frac{\partial(u'w')}{\partial r} + \frac{\partial(w'w')}{\partial z} + \bar{w} \frac{\partial w'}{\partial z} + u' \frac{\partial \bar{w}}{\partial r} + \frac{u'w'}{r} \right\} = \frac{\partial(\bar{p}_{rz} + p'_{rz})}{\partial r}$$

$$+ \frac{\partial(p_{zz} + p'_{zz})}{\partial z} + \frac{\bar{p}_{rz} + p'_{rz}}{r}.$$

Taking mean values, we have the equations of mean motion

(7) $$\rho \left\{ \frac{\partial(\overline{u'u'})}{\partial r} + \frac{\overline{u'u'}}{r} \right\} = \frac{\partial \bar{p}_{rr}}{\partial r} + \frac{\partial \bar{p}_{zr}}{\partial z} + \frac{\bar{p}_{rr}}{r},$$

(8) $$\rho \left\{ \frac{\partial(\overline{u'w'})}{\partial r} + \frac{\overline{u'w'}}{r} \right\} = \frac{\partial \bar{p}_{rz}}{\partial r} + \frac{\partial \bar{p}_{zz}}{\partial z} + \frac{\bar{p}_{rz}}{r} = -\frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 \bar{w}}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{w}}{\partial r} \right).$$

Multiplying (8) by r and integrating

$$r\rho\overline{u'w'} = -\frac{r^2}{2} \frac{\partial p}{\partial z} + \mu r \frac{\partial \bar{w}}{\partial r} + C.$$

Since this holds for $r = 0$, we have $C = 0$. Hence

(9) $$\mu \frac{\partial w}{\partial r} = \frac{r}{2} \frac{\partial p}{\partial z} + \rho \overline{u'w'}.$$

§ 5. *The energy of the relative motion.* Subtracting (7) and (8) from the preceding equations we have for the relative motion,

$$\rho\left\{\frac{\partial u'}{\partial t} + \frac{\partial}{\partial r}(u'u' - \overline{u'u'}) + \bar{w}\frac{\partial u'}{\partial z} + \frac{\partial(u'w')}{\partial z} + \frac{u'u' - \overline{u'u'}}{r}\right\}$$

$$= \frac{\partial p'_{rr}}{\partial r} + \frac{\partial p'_{zr}}{\partial z} + \frac{p'_{rr} - p'_{\theta\theta}}{r},$$

$$\rho\left\{\frac{\partial w'}{\partial t} + u'\frac{\partial w'}{\partial r} + \frac{\partial}{\partial r}(u'w' - \overline{u'w'}) + \bar{w}\frac{\partial w'}{\partial z} + u'\frac{\partial \bar{w}}{\partial r} + \frac{u'w' - \overline{u'w'}}{r}\right\}$$

$$= \frac{\partial p'_{rz}}{\partial r} + \frac{\partial p'_{zz}}{\partial z} + \frac{p'_{rz}}{r}.$$

Multiplying these equations by u', and w' respectively, adding, and taking mean values,

$$\left(\frac{\partial}{\partial t} + \bar{w}\frac{\partial}{\partial z}\right)\tfrac{1}{2}\rho(\overline{u'^2} + \overline{w'^2}) + \rho\overline{u'w'}\frac{\partial \bar{w}}{\partial r} =$$

$$\text{mean value of } u'\left(\frac{\partial p'_{rr}}{\partial r} + \frac{\partial p'_{zr}}{\partial z} + \frac{p'_{rr} - p'_{\theta\theta}}{r}\right) + w'\left(\frac{\partial p'_{rz}}{\partial r} + \frac{\partial p'_{zz}}{\partial z} + \frac{p'_{rz}}{r}\right).$$

Integrating over the section of the pipe and by parts on the right hand side, dividing by 2π, and remembering that u', w' vanish on the boundary, we obtain the equation

$$\int_0^a \left(\frac{\partial}{\partial t} + \bar{w}\frac{\partial}{\partial z}\right)\tfrac{1}{2}\rho(\overline{u'^2} + \overline{w'^2})\cdot r\,dr = \int_0^a -\rho\overline{u'w'}\frac{\partial \bar{w}}{\partial r}\cdot r\,dr$$

$$-\mu\int_0^a\left[2\left(\frac{\partial u'}{\partial r}\right)^2 + 2\left(\frac{\partial w'}{\partial z}\right)^2 + 2\frac{u'^2}{r^2} + \left(\frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial r}\right)^2\right]r\,dr,$$

where, in the second integral on the right, mean values are to be taken.

The left hand side of this equation is the average rate of increase of the energy of the relative motion of a length $1/2\pi$ of liquid in the pipe.

Hence we have REYNOLD's criterion for the possibility of permanent eddying motion expressed in cylindrical coördinates, namely,

$$(10)\quad -\int_0^a \rho\overline{u'w'}\frac{\partial \bar{w}}{\partial r}\cdot r\,dr \gtreqless \mu\int_0^a\left[2\left(\frac{\partial u'}{\partial r}\right)^2 + 2\left(\frac{\partial w'}{\partial z}\right)^2\right.$$

$$\left. + 2\frac{u'^2}{r^2} + \left(\frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial r}\right)^2\right]r\,dr,$$

where, in the integrand on the right, mean values are to be taken.

The interpretation of equations (9) and (10) is that in order to maintain the relative as well as the mean motion an increased pressure gradient is necessary and the additional energy is dissipated in the relative motion, the right-hand

integrand of (10) being STOKES's dissipation function expressed in cylindrical coördinates.

§ 6. *Determination of a possible relative motion.* The solution of the continuity equation (6) with the boundary conditions $u' = w' = 0$, for $r = a$, which leads to a minimum value of U is

$$u' = LA\frac{\pi r}{2a}(\sin p + \sin 3p)\sin\frac{\pi Lz}{2a} - LB\frac{\pi r}{2a}(\sin 2p + \tfrac{1}{2}\sin 4p)\cos\frac{\pi Lz}{2a},$$

$$w' = A\left\{2(\sin p + \sin 3p) + \frac{\pi r}{2a}(\cos p + 3\cos 3p)\right\}\cos\frac{\pi Lz}{2a}$$

$$+ B\left\{2(\sin 2p + \tfrac{1}{2}\sin 4p) + \frac{\pi r}{2a}(2\cos 2p + 2\cos 4p)\right\}\sin\frac{\pi Lz}{2a},$$

as is easily seen on substitution, p being $\pi r/2a$.

These give

$$\overline{u'w'} = \frac{LAB\pi^2 r^2}{32a^2}(3\sin p + 3\sin 3p - \sin 5p - \sin 7p).$$

Hence from (9) we find, neglecting $\overline{u'w'}^2$, the left-hand side of equation (10) equal to

$$-\frac{LAB\pi^2\rho U}{64a^2\mu}\int_0^a r^4(3\sin p + 3\sin 3p - \sin 5p - \sin 7p)\,dr = .215\,LABa\rho U.$$

Substituting the values of u' and w' in the right hand side of (10), taking mean values, and integrating, we obtain the condition

$$\frac{.215\,LABa\rho U}{\mu} \geqq (.304A^2 + .127B^2)L^4$$
$$+ (4.26A^2 + 6.67B^2)L^2 + 35.7A^2 + 16.34B^2.$$

Hence for a minimum value of U we have to minimize

$$\left(\frac{.304}{\lambda} + .127\lambda\right)L^3 + \left(\frac{4.26}{\lambda} + 6.67\lambda\right)L + \left(\frac{35.7}{\lambda} + 16.34\lambda\right)\frac{1}{L}$$

where $\lambda = B/A$.

Differentiating with respect to L and λ we find

$$3(.304 + .127\lambda^2)L^4 + (4.26 + 6.67\lambda^2)L^2 - 35.7 - 16.34\lambda^2 = 0,$$

$$(-.304 + .127\lambda^2)L^4 + (-4.26 + 6.67\lambda^2)L^2 - 35.7 + 16.34\lambda^2 = 0.$$

Eliminating λ^2,

$$.4633L^8 + 20.55L^6 + 151.67L^4 - 2333.35 = 0.$$

Hence $L = 1.798$, $\lambda = 1.158$, approximately.   The minimum value of U is therefore given by

$$\frac{2a\rho U}{\mu} = 470.$$

REYNOLDS' experiments gave

$$\frac{2a\rho U}{\mu} = 1900.$$

REYNOLDS' result for the flow between two parallel planes a distance $2b$ apart is

$$\frac{2b\rho U}{\mu} = 517,$$

and because the hydraulic mean depth for a cylinder radius a is $\frac{1}{2}a$ and for two parallel planes is b he inferred that for a cylinder

$$\frac{2a\rho U}{\mu} = 1034.$$

§ 7. *Flow between parallel planes.*   When the motion is the same in planes parallel to the xy plane the equations of motion are *

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y},$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = \frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{yy}}{\partial y},$$

where

$$p_{xx} = -p + 2\mu\frac{\partial u}{\partial x}, \; p_{yy} = -p + 2\mu\frac{\partial u}{\partial y},$$

$$p_{yx} = p_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right).$$

The equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

In the case of liquid flowing between two parallel planes $y = \pm b$ the boundary conditions are $u = v = 0$ when $y = \pm b$.

The simplest solution of these equations is

$$v = \frac{\partial u}{\partial x} = \frac{\partial p}{\partial y} = 0, \qquad \frac{\partial p}{\partial x} = \text{constant}, \qquad u = \frac{1}{2\mu}\frac{\partial p}{\partial x}(y^2 - b^2),$$

and the mean velocity of flow is

$$U = -\frac{b^2}{3\mu}\frac{\partial p}{\partial x}.$$

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* LAMB, *Hydrodynamics*, pp. 512–520.

Using the method of the preceding problem we find for the equation of mean motion

$$(11) \qquad \rho \frac{\partial (u'v')}{\partial y} = - \frac{\partial p}{\partial x} + \mu \frac{\partial^2 \bar{u}}{\partial y^2},$$

and the criterion of stability,

$$(12) \quad - \int_{-b}^{b} \rho \overline{u'v'} \frac{\partial \bar{u}}{\partial y} dy \geqq \mu \int_{-b}^{+b} \left\{ 2\left(\frac{\partial u'}{\partial x}\right)^2 + 2\left(\frac{\partial v'}{\partial y}\right)^2 + \left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right)^2 \right\} dy.$$

§ 8. *Determination of a possible relative motion.* The solution of the continuity equation, with the boundary conditions, which leads to a minimum value of U is

$$u' = A(\sin p + \sin 3p)\cos\frac{\pi L x}{2b} + B(2\sin 2p + 4\sin 4p)\sin\frac{\pi L x}{2b},$$

$$v' = -LA(\cos p + \tfrac{1}{3}\cos 3p)\sin\frac{\pi L x}{2b} + LB(\cos 2p + \cos 4p)\cos\frac{\pi L x}{2b},$$

where $p = \pi y/2b$.

These give

$$4\overline{u'v'} = -LAB\left\{ \tfrac{11}{3}\sin p + 6\sin 3p + \tfrac{8}{3}\sin 5p + \tfrac{1}{3}\sin 7p \right\}.$$

To determine $\partial\bar{u}/\partial y$ we have, on integrating (11),

$$\rho\overline{u'v'} = -y\frac{\partial p}{\partial x} + \mu\frac{\partial \bar{u}}{\partial y} + C.$$

Now $\overline{u'v'}$ is an odd function of y and so is $\partial \bar{u}/\partial y$ since the mean motion is symmetrical with respect to the planes $y = \pm b$. Hence C is zero and approximately

$$\frac{\partial \bar{u}}{\partial y} = \frac{y}{\mu}\frac{\partial p}{\partial x} = -\frac{3Uy}{b^2}.$$

Hence the left hand side of (12) reduces to

$$\frac{-18.6 LAB\rho U}{\pi^2}.$$

Substituting the values of u' and v' in the right hand side of (12), taking mean values, and integrating, we have the condition

$$\frac{-18.6 LAB\rho U}{\pi^2}$$

$$\geqq \frac{\mu b}{2}\left\{ \frac{\pi^2 L^4}{4b^2}(\tfrac{10}{9}A^2 + 2B^2) + \frac{2\pi^2 L^2}{4b^2}(2A^2 + 20B^2) + \frac{\pi^2}{4b^2}(10A^2 + 272B^2) \right\},$$

or

$$\frac{3b\rho U}{\mu} \geqq \left(\frac{\pi}{2}\right)^4 \frac{(\tfrac{10}{9}A^2 + 2B^2)L^4 + (4A^2 + 40B^2)L^2 + 10A^2 + 272B^2}{-3.1 LAB}.$$

REYNOLDS takes the solution

$$u' = A\left(\cos p + 3\cos 3p\right)\cos\frac{\pi Lx}{2b} + B\left(2\cos 2p + 2\cos 4p\right)\sin\frac{\pi Lx}{2b},$$

$$v' = LA\left(\sin p + \sin 3p\right)\sin\frac{\pi Lx}{2b} - LB\left(\sin 2p + \tfrac{1}{2}\sin 4p\right)\cos\frac{\pi Lx}{2b},$$

and obtains the condition

$$\frac{3b\,U\rho}{\mu} \geqq \left(\frac{\pi}{2}\right)^4\frac{(2A^2 + \tfrac{5}{4}B^2)\,L^4 + (20A^2 + 16B^2)\,L^2 + 82A^2 + 80B^2}{-1.325\,LAB}.$$

The coefficients of A^2 and B^2 being nearly equal, he determines the minimum value of U by assuming $A = B$.

It is interesting to notice that his value for v' vanishes midway between the planes and u' is there a maximum whereas the values here adopted give a maximum value of v' midway between the planes, which seem to be more in accordance with experiments.

Using the method of the preceding problem we find a minimum value of U given by

$$\frac{3b\,U\rho}{\mu} = 41\left(\frac{\pi}{2}\right)^4,$$

when $L = 1.398$ and $A/B = 4.03$.

Hence our final result is that the motion is unstable when

$$\frac{2b\,U\rho}{\mu} \geqq 167.$$

REYNOLDS' result is 517.

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