DETERMINATION OF THE ABSTRACT GROUPS OF ORDER $p^2qr$; $p$, $q$, $r$ BEING DISTINCT PRIMES

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Since the publication† in 1899 of Professor Miller’s “Report on recent progress in the theory of groups of finite order,” Western‡ has published his determination of the groups of order $p^3q$, and Le Vasseur§ has discussed the order $p^2q^2$. This paper is devoted to the determination of all groups of the order $p^2qr$. It thus completes the discussion of the problem of groups whose orders are products of four primes. ||

With the exception of the group of order $2^2 \cdot 3 \cdot 5$, simply isomorphic with the icosahedron-group, all groups of order $p^2qr$ are solvable. The maximal self-conjugate subgroups will therefore serve as the basis of classification. The twelve possible arrangements of the factors of composition are

$$(1) \ ppqr, \ (2) \ pprq, \ (3) \ pqpr, \ (4) \ pqrp, \ (5) \ prpq, \ (6) \ prqp,$$

$$(7) \ qppr, \ (8) \ qprp, \ (9) \ qrpp, \ (10) \ rqpp, \ (11) \ rppq, \ (12) \ rpqp.$$  

If for a given type of group precisely the arrangements $(i), (j), (k), \ldots$, of the factors of composition are possible, then we symbolize the group $(i, j, k, \ldots)$. Two groups having distinct symbols cannot be simply isomorphic.

The group $G$ always contains a maximal invariant subgroup** of order $p^2q$, and may contain maximal subgroups†† of order $p^2r$ and $pqr$. We shall discuss

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in detail in this paper only two classes of groups: those possessing invariant subgroups of both the types \( H_{p^2 q} \) and \( H_{p^2 r} \), and those possessing maximal invariant subgroups of the type \( H_{p^2 q} \) only. A detailed summary of the results obtained in the other classes is given at the end. We shall thus be concerned principally with the subgroups \( H_{p^2 q} (\sigma = q, r) \) all types of which are given in the following table, in which \( \tau \) denotes the number of distinct types, while \( (p) \) signifies \((\text{modulo } p)\):

<table>
<thead>
<tr>
<th>( H_{p^2 q, i} )</th>
<th>( S_3^{-1} S_3 S_3^{-1} S_1 S_3 S_3^{-1} S_2 S_3 S_3^{-1} S_1 S_2 )</th>
<th>Parameters</th>
<th>( \tau )</th>
</tr>
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<tbody>
<tr>
<td>( i = I )</td>
<td>( S )</td>
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<td>1</td>
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<tr>
<td>( \Omega )</td>
<td>( S_1 )</td>
<td>( S_2 )</td>
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</tr>
<tr>
<td>( III )</td>
<td>( S_1^* )</td>
<td>( S_2 )</td>
<td>( S_1 )</td>
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<td>( IV )</td>
<td>( S_1^* )</td>
<td>( S_2^* )</td>
<td>( S_1 )</td>
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<tr>
<td>( V )</td>
<td>( S_1^* )</td>
<td>( S_2^* )</td>
<td>( S_1 )</td>
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<td>( VI )</td>
<td>( S_2 )</td>
<td>( S_1^{-1} S_2 S_1 )</td>
<td>( S_1 )</td>
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</table>

\( \sigma = q, r; S_1 = 1, S_2 = 1, S_3 = 1. \)

§ 1. Determination of \( \rho_{\alpha, \gamma} \).

By Sylow's theorem, \( N_\sigma = qr/\sigma, p, p^2, p^2 q \sigma, p^2 q r/\sigma \) or 1. If \( N_{\sigma_1} = 1 \) then \( \rho_{\alpha, \gamma} = 1 \), \( \Omega \) being any operator of prime order in \( G \). When \( N_\sigma > 1 \), the result of transforming the single conjugate set of \( N_\sigma \) subgroups

\[ g_1, g_2, g_3, \ldots, g_{N_\sigma} \]

by \( \Omega \) is to permute them among themselves. Hence

\[ \Omega^{-1}(g_1, g_2, \ldots, g_{N_\sigma}) \Omega = (g_1^1, g_2^1, \ldots, g_{N_\sigma}^1) = J_{\Omega, \sigma}. \]

It follows that \( J_{\Omega, \sigma} = 1 \) and

\[ N_\sigma - \rho_{\alpha, \sigma} \equiv 0 \pmod{\omega}; \quad \rho_{\alpha, \sigma} \equiv 1. \]

Next let \( \omega = \sigma \). Then \( N_p = (p^2 - 1)/(p - 1) = p + 1 \), and

\[ p + 1 - \rho_{\alpha, p} \equiv 0 \pmod{\sigma}. \]

Hence either \( \rho_{\alpha, p} \equiv 0 \) or else \( \rho_{\alpha, p} \equiv 2 \) (\( \omega = q, r \)). Now if the subgroup \( I_{\alpha_3} \) of \( H_{p^2 q, i} \) is cyclical the order of its group of isomorphisms is

\[ I = \phi(p^2) = p(p - 1). \]

*Throughout the paper \( \iota \) denotes a non-integral mark of the GF \( p^2 \). Thus \( \iota^\sigma = 1(p) \) is an abbreviation for \( \iota^\sigma \equiv 1 \pmod{p, p}, P \) being any quadratic function irreducible modulo \( p \).

†Sylow, Mathematische Annalen, vol. 5 (1872).
If $I_{pq}$ is of type $[1, 1]$ its group of isomorphisms is simply isomorphic with the congruence group $\{S_1, S_2 \cdots \}$ of order $I = p(p - 1)^2(p + 1)$, where $S_1$ is
\[
y_1 \equiv a_{11}x_1 + a_{12}x_2, \quad y_2 \equiv a_{21}x_1 + a_{22}x_2 \pmod{p},
\]
or say
\[
S_1 = (a_{11}x_1 + a_{12}x_2, a_{21}x_1 + a_{22}x_2).
\]
Since $\Omega$ corresponds to an isomorphism of $G$, $\{\Omega\}$ corresponds to a subgroup of the group of isomorphisms of $G$ and $\omega$ divides $I$. Hence when $I_{pq}$ is cyclical, or when $I_{pq} = [1, 1]$ and $p \equiv 1(\sigma)$, $\rho_{\omega, p} \equiv 2$. But when $p \equiv -1(\sigma)$ and $p$ is odd, $\rho_{\omega, p} = 0$. Also since $\rho_{\omega, \sigma} \equiv 1$, $J_{\omega, \sigma}$ and $J_{\omega, \sigma}$ may be permutable. If
\[
S_1 = (b_{11}x_1 + b_{12}x_2, b_{21}x_1 + b_{22}x_2)
\]
the necessary and sufficient conditions that $S_1S_2 = S_2S_1$ are
\[
(3) \quad \delta_{12} = \begin{vmatrix}
   a_{12} & b_{12} \\
   a_{11} - a_{22} & b_{11} - b_{22}
\end{vmatrix} = 0,* \quad \delta'_{12} = \begin{vmatrix}
   a_{21} & b_{21} \\
   a_{11} - a_{22} & b_{11} - b_{22}
\end{vmatrix} = 0,
\]
\[
d_{12} = \begin{vmatrix}
   a_{12} & a_{21} \\
   b_{12} & b_{21}
\end{vmatrix} = 0.
\]

§ 2. Class (9, 10), $p > q > r$.

We now consider the groups whose symbol is (9, 10), having the maximal subgroups $H_{pq}$, and $H_{pqr}, (i, j = IV, V, VI)$. Since $I_{pq}$ is invariant in $G$ the existence of a subgroup of type IV excludes the possibility of a subgroup of type V or VI, and vice versa. There are thus five cases to consider.

[1] $i = j = IV$. Here $I_{pq} = \{P\}$ is cyclical and $P$ may be regarded as the generator of order $p^2$ in both $H$-subgroups. Since $\rho_{q, r} \equiv 1$, we may choose $\{R\}$ permutable with $Q$ and, since $q > r$, $QR = RQ$, so that $G$ is defined by $P^aq = Q^r = R^r = 1$, $Q^{-1}PQ = P^a$, $R^{-1}PR = P^b$, $QR = RQ$; or for brevity $G = (\alpha : \beta : 1)$, where
\[
\alpha^q = 1, \quad \beta^r = 1(p^2), \quad p \equiv 1(qr), \quad \tau = 1.
\]

[2] $i = j = V$. Let $H_{pq}, (i = IV, V, VI)$. $H_{pqr}, (i = IV, V, VI)$, wherein $QR = RQ$. We may write
\[
R^{-1}P_1R = P_1, \quad R^{-1}P_2R = P_2^b, \quad \alpha^r = 1(p), \quad \beta = \alpha^q.
\]
\[
Q^{-1}P_1Q = P_1^{a_1}P_2^{a_2}, \quad Q^{-1}P_2Q = P_1^{a_1}P_2^{a_2},
\]
and from the permutable isomorphisms of $I_{pq}$
\[
J_Q = \left( \begin{array}{c}
P_1^{a_1}P_2^{a_2} \\
P_1^{a_2}P_2^{a_1} + a_{22}P_2^{a_2}
\end{array} \right), \quad J_R = \left( \begin{array}{c}
P_1^{a_1}P_2^{a_2} \\
P_1^{a_2}P_2^{a_1} + a_{22}P_2^{a_2}
\end{array} \right),
\]

* All congruences are taken modulo $n$ unless otherwise indicated.
\[ \delta_{12} = a_{12}(\alpha - \beta) \equiv 0, \quad \delta'_{12} = a_{21}(\alpha - \beta) \equiv 0. \]

Reserving for later treatment the ambiguous case \( h = 1 \), we deduce \( a_{12} = a_{21} = 0 \). Suppose next that

\[
R^{-1} P'_i R = P_{1i}^{b_{1i}} P_{2i}^{b_{2i}} \quad (i = 1, 2).
\]

Then

\[
(RQ)^{-1} P'_1 (RQ) = P_{1i}^{b_{1i}} P_{2i}^{b_{2i}} = (QR)^{-1} P'_1 (QR) = P_{1i}^{b_{1i}} P_{2i}^{b_{2i}},
\]

\[
b_{11}(a_{11} - \gamma) \equiv 0, \quad b_{21}(a_{22} - \gamma) \equiv 0, \quad \gamma' \equiv 1,
\]

\[
b_{12}(a_{11} - \delta) \equiv 0, \quad b_{22}(a_{22} - \delta) \equiv 0, \quad \delta' \equiv \gamma'k.
\]

Thus when \( h \equiv 1, k \equiv 1 \) we have one of the two equivalent results

\[
a_{11} \equiv \gamma, \quad a_{22} \equiv \delta \quad \text{or} \quad a_{11} \equiv \delta, \quad a_{22} \equiv \gamma.
\]

In case \( h \equiv 1, k \equiv 1 \), the set (5) becomes

\[
b_{11}(a_{11} - \gamma) \equiv 0, \quad b_{21}(a_{22} - \gamma) \equiv 0,
\]

\[
b_{12}(a_{11} - \gamma) \equiv 0, \quad b_{22}(a_{22} - \gamma) \equiv 0,
\]

and there are three possibilities to consider, viz.,

(i) \( a_{11} \equiv \gamma, \quad b_{11} \equiv 0, \quad b_{12} \equiv 0, \quad b_{21} \equiv 0, \quad b_{22} \equiv 0, \quad a_{22} \equiv \gamma; \)

(ii) \( a_{11} \equiv \gamma, \quad a_{22} \equiv \gamma, \quad b_{21} \equiv b_{22} \equiv 0, \quad b_{11} \equiv 0, \quad b_{12} \equiv 0; \)

(iii) \( a_{11} \equiv \gamma, \quad a_{22} \equiv \gamma. \)

Case (i) implies

\[
R^{-1} P'_1 R = P_{21}^{b_{21}}, \quad R^{-1} P'_2 R = P_{22}^{b_{22}},
\]

\[
R^{-1} P_{1i}^{b_{1i}} R = R^{-1} P_{2i}^{b_{2i}} R \quad \text{or} \quad P_{1i}^{b_{1i}} = P_{2i}^{b_{2i}},
\]

contrary to the independence of \( P'_1 \) and \( P'_2 \). Likewise, case (ii) is excluded.

Hence \( a_{11} \equiv a_{22} \equiv \gamma \).

In a similar manner, when \( h \equiv 1, k \equiv 1 \), we get \( a_{11} \equiv a_{22} \equiv \alpha \).

Next let \( h = 1, k = 1 \), so that

\[
R^{-1} P'_i R = P_{1i}^{a_{1i}}, \quad Q^{-1} P'_i Q = P_{1i}^{a_{2i}} \quad (i = 1, 2).
\]

One of the operations \( P'_1 \), \( P'_2 \) must be independent of \( P_1 \). As \( \gamma' \equiv 1 \mod p \), we may assume that \( P_1 \) and \( P'_2 \) are independent. These will generate \( I_{ps} \), so that

\[
Q^{-1} P_1 Q = P_{1i}^{a_{1i}} P_{2i}^{a_{2i}}, \quad R^{-1} P'_2 R = P_{1i}^{b_{1i}} P_{2i}^{b_{2i}}.
\]

The abelian conditions from \( J_q \) and \( J_R \) are [Eq. (3)]

\[
\delta_{12} = b_{12}(a_{11} - \delta) \equiv 0, \quad \delta'_{12} = a_{21}(b_{22} - \alpha) \equiv 0, \quad d_{12} = a_{21}b_{12} \equiv 0.
\]

Thus three possibilities arise, viz.,
For (i), let $P'_1 = P'_1 P'_2$, $P'_2 = P'_1 P'_2$, whence

\[ Q^{-1} P'_1 Q = P'^{\gamma_1} P'^{\gamma_2} = P^{h_1} P^{h_2}, \]

\[ R^{-1} P'_2 R = P^{h_2} P^{h_3} = P_{a_1+1} P_{a_2+1}, \]

\[ (\gamma - \delta) x \equiv 0, \quad (\gamma - \delta) y \equiv 0, \]

\[ w(b_{22} - \beta) \equiv 0, \quad z(\alpha - \beta) + b_{12} w \equiv 0. \]

Hence $\gamma \equiv \delta$ and $k = 1$; but as $P'_1, P'_2$ are independent, $w \equiv 0$, $b_{22} \equiv \beta$, $\alpha \equiv \beta$ and $h \equiv 1$, contrary to hypothesis. Since (ii) is likewise excluded, we have $a_{21} \equiv b_{12} \equiv 0$,

\[ Q^{-1} P'_1 Q = P'^{\gamma_1}, \quad R^{-1} P'_2 R = P^{h_2}, \]

\[ x(a_{11} - \gamma) \equiv 0, \quad y(\delta - \gamma) \equiv 0, \]

\[ z(\beta - \alpha) \equiv 0, \quad w(b_{22} - \beta) \equiv 0, \]

where $x \equiv 0$, $w \equiv 0$. Hence when $\alpha \equiv \beta$, $\delta \equiv \gamma$ there results $a_{11} \equiv \gamma$, $b_{22} \equiv \alpha$. We are thus led to a single set of defining relations:

\[ P'_1 = P'_2 = Q = R' = 1, \quad P'_1 P'_2 = P'_2 P'_1, \quad Q^{-1} P'_1 Q = P'_1, \]

\[ Q^{-1} P'_2 Q = P'_2, \quad R^{-1} P'_1 R = P'_1, \quad R^{-1} P'_2 R = P'_2, \quad R Q = Q R, \]

\[ \alpha' = 1(p), \quad \gamma' = 1(p) \quad (h = 1, 2, \ldots, r - 1; k = 1, 2, \ldots, q - 1), \]

or, briefly, say $G = (1: \gamma 0: 0\gamma^k: \alpha 0: 0\alpha^k: 1)$. Proceeding to the determination of $\tau$ we observe that there are, by hypothesis, two subgroups, $\{P'_1\}, \{P'_2\}$, both permutable with $Q$ and $R$. In any isomorphism of $G$ with itself either $\{P'_1\} \sim \{P'_2\}, \{P'_2\} \sim \{P'_1\}$ or else $\{P'_1\} \sim \{P'_1\}, \{P'_2\} \sim \{P'_2\}$. Hence there are two choices of generators of order $p$. Every element of $G$ is of the form $\Omega = R^{x} Q^{\alpha} P'^{\gamma_1} P'^{\gamma_2}$. Hence $\Omega' = R^{x'} Q^{\alpha'} P'^{\gamma_1'} P'^{\gamma_2'}$, so that $\Omega$ is of order $r$ only when $y \equiv 0 \pmod{q}$ and of order $q$ when $x \equiv 0 \pmod{r}$. Thus the most general operator of order $q$ is $Q'_0 = Q^{r} P'^{\gamma_1} P'^{\gamma_2}$, which transforms $G$ in the same manner as $Q = Q^{r}$. Similarly $R'_0 = R^{r}$. Employing the new generators $R'_1, Q'_0, P'_{10} = P'_1, P'_{20} = P'_2$, we get

\[ (1: \gamma 0: 0\gamma^k: \alpha 0: 0\alpha^k: 1) \sim (1: \gamma 0: 0\gamma^k: \alpha 0: 0\alpha^k: 1). \]

Hence any set of relations involving arbitrary primitive roots $(\alpha', \gamma')$ can be transformed into the original set. Next let $P'_{10} = P'_2, P'_{20} = P'_1$. Then

\[ (1: \gamma 0: 0\gamma^k: \alpha 0: 0\alpha^k: 1) \sim (1: \gamma 0: 0\gamma^k: \alpha 0: 0\alpha^k: 1). \]
if

\[ ky \equiv 1 \pmod{q}, \quad hx \equiv 1 \pmod{r}. \]

The group characterized by \([h, k]\) is thus isomorphic with \([x, y]\) when (6) is satisfied. Further \(\tau\) equals the number of distinct solutions of (6), e.g., when \(r = 2\), \(\tau = \frac{1}{2}(q + 1)\), and when \(r\) is odd, \(\tau = \frac{1}{2}(qr + q + r + 1)\).

\[ [3] \quad i = VI, j = V. \quad \text{When} \quad h = 1 \quad \text{we have} \quad Q^{-1}P_jQ = P_{2j}^\nu (j = 1, 2). \]

Assuming that

\[ R^{-1}P_1 R = P_1 P_1^\nu, \quad R^{-1}P_2 R = P_1 P_2^\nu, \]

we derive

\[ a_{11} x - z \equiv 0, \quad x - (\nu + \iota - a_{11})z \equiv 0, \]

\[ a_{22} y - w \equiv 0, \quad y - (\nu + \iota - a_{22})w \equiv 0. \]

The elimination of \(x, y, z, w\) gives

\[ a_{jj}^2 - (\nu + \iota) a_{jj} + 1 \equiv 0 \quad (j = 1, 2), \]

whence \(a_{jj} = \nu\) or \(\iota\). Hence \(a_{11}, a_{22}\) are galoisian imaginaries* and \(G\), for \(i = VI, j = V\), does not exist.

Before considering the ambiguous case \(h = 1\) a few general results must be established.

Let \(S\) and \(T\) be any set of generators of \(I_{x^*}\), so that \(G = \{S, T, Q, R\}\).

We may write

\[ P_1' = S^*T^\nu, \quad P_2' = S^*T^\iota, \]

\[ Q^{-1}SQ = S^{a_{11}}T^{a_{21}}, \quad Q^{-1}TQ = S^{a_{12}}T^{a_{22}}. \]

Hence

\[ Q^{-1}P_1'Q = P_2' = S^*T^\iota = S^{a_{11}x + a_{12}y}T^{a_{21}x + a_{22}y}, \]

\[ Q^{-1}P_2'Q = P_1'^{-1}P_2'^{\nu + \iota} = S^{-x + (\nu + \iota)x}T^{-y + (\nu + \iota)y} = S^{a_{11}x + a_{12}y}T^{a_{21}x + a_{22}y}, \]

whence results the eliminant

\[
\begin{array}{cccc}
 x & y & z & w \\
 a_{11} & a_{12} & -1 & 0 \\
 a_{21} & a_{22} & 0 & -1 \\
 1 & 0 & a_{11} - t & a_{12} \\
 0 & 1 & a_{21} & a_{22} - t \\
\end{array}
\equiv 0 \pmod{p},
\]

where \(t = \nu + \iota\). Its expansion gives

\[
D_{12}^2 - t(a_{11} + a_{22} - t)D_{12} + a_{22}^2 - a_{11}^2 + t(a_{11} - a_{22}) + 2a_{12}a_{21} + 1 \equiv 0.
\]

Now assume \(S = P_1\). Then, since \(p \equiv -1 \pmod{q}\), \(\rho_{Q, P} = 0\) and we may take \(Q^{-1}P_1Q \equiv U\) as \(T\). Then

*Serret, Cours d'Algèbre Supérieur, cinq. ed. (1885), tome 2, sec. 3, chap. 3. See also Dickson, Linear Groups, pp. 14–19.
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$J_q = \left( \frac{P_{r^2} U_{r^2}}{P_{r^2} U_{r^2} + a^2 r^2} \right)$, $J_q' = 1$,

$D_{12} = \begin{vmatrix} 0 & a_{12} \\ 1 & a_{22} \end{vmatrix} = (-a_{12})^q \equiv 1 \pmod{p}$.

Now $-a_{12}$ cannot be a primitive root of this congruence; for, if so $p \equiv 1 \pmod{q}$, whereas $p \equiv -1 \pmod{q}$ and $q > r$. It follows that $a_{12} \equiv 1 \pmod{p}$ and

$D \equiv (a_{22} - t)^2 \equiv 0$, $a_{22} \equiv t \equiv v^p + t$.

This gives $I_p = \{ P_1 U \}$ and

$Q^{-1} P_1 Q = U$, $Q^{-1} U P = P^{-1}_1 U v^p + t$, $R^{-1} P_1 R = P'_1$, $R^{-1} U R = P'_1 U^n$,

(7)

$\delta_{12} = \begin{vmatrix} -1 & \xi \\ -v^p - t & \alpha - \eta \end{vmatrix} \equiv 0$, $\delta_{12}' = \begin{vmatrix} 1 & 0 \\ -v^p - t & \alpha - \eta \end{vmatrix} \equiv 0$,

and thus, when $h = 1$, $\gamma \equiv \alpha$, $\xi \equiv 0 \pmod{p}$.

Inversely let $P_2 = P'_1 U^v$. Then

$R^{-1} P_2 R = P'_1 U^{v^p + t}$, $P = P'_1 U^{v^p + t}$

and hence $h = 1$. Thus when $h = 1$ there exists a group

$G = \{ P_1, U, Q, R \} = (1:0:1:-1v^p + t:a0:0\alpha:1)$,

where $\alpha' \equiv 1 \pmod{p}$, $P \equiv 1 \pmod{r}$, $\tau = 1$. Also $p \equiv -1 \pmod{q}$ and, in the

$GF[p^2]$, $\xi \equiv 1 \pmod{p}$.

[4] $i = V$, $j = VI$. Since $r$ is necessarily an odd prime, the argument of

[3] again gives for $G$ a single type, $G = (1:1:0:0:01:-1v^p + t:1)$, with

$\gamma' \equiv 1 \pmod{p}$, $P \equiv 1 \pmod{q}$, $\tau = 1$. Likewise $p \equiv -1 \pmod{r}$; and $v \equiv 1 \pmod{p}$

in the $GF[p^2]$.


$D$ we are led to the same equations (7), viz.,

$Q^{-1} P_1 Q = U$, $Q^{-1} U P = P^{-1}_1 U v^p + t$, $v' \equiv 1 \pmod{p}$.

Let us assume that

$R^{-1} P_1 R = P'_2 = P'_1 U^\gamma$, $R^{-1} U R = P'_1 U^\gamma$.

Then

$\delta_{12} = \begin{vmatrix} -1 & x \\ -v^p - t & x - \omega \end{vmatrix} \equiv 0$, $\delta_{12}' = \begin{vmatrix} 1 & \gamma \\ -v^p - t & x - \omega \end{vmatrix} \equiv 0$.

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Thus
\[ d_{12} = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} \equiv 0, \quad D_{12} = \begin{vmatrix} x & z \\ y & w \end{vmatrix} \equiv 0. \]

Thus
\[ z \equiv -y, \quad w \equiv x + (\iota_1^p + \iota_1)y, \quad D_{12} \equiv x^2 + (\iota_1^p + \iota_1)xy + y^2. \]

Since
\[ R^{-1}P_2R = P_1^{-1}P_2^{\iota_1^p + \iota_2}, \quad \iota_2' \equiv 1 (p), \]
so that
\[ R^{-1}U^\nu R = P_1^{-1}U^-^{\iota_1^p + \iota_2}U^\nu + (\iota_1^p + \iota_2)U^\nu + (\iota_1^p + \iota_2)U^\nu. \]

Since \( P_1 \) and \( P_2 \) are independent, \( y \equiv 0 \); hence
\begin{align*}
(8) & \quad 2x + (\iota_1^p + \iota_1)y - (\iota_2^p + \iota_2) \equiv 0, \\
(9) & \quad y^2 - x^2 + (\iota_2^p + \iota_2)x - 1 \equiv 0.
\end{align*}

From the latter we at once derive
\[ D_{12} = x^2 + (\iota_1^p + \iota_1)xy + y^2 \equiv 1, \]
\begin{align*}
(10) & \quad (\iota_2 - \iota_1^p \iota_2^p)^2x^2 - (1 - \iota_1^p)(\iota_2 - \iota_2^p)x + (1 - \iota_1^p \iota_2^p)(\iota_2 - \iota_1^p) \equiv 0, \\
(11) & \quad (\iota_1 - \iota_1^p)^2y^2 - (\iota_2 - \iota_2^p) \equiv 0.
\end{align*}

There always exist integral solutions of (10) and (11), \( x = \epsilon_j, y = \sigma_j (j = 1, 2) \). Thus
\[ R^{-1}P_1R = P_1^{(\iota_1^p + \iota_2^p)}U^{-\sigma_1}, \quad R^{-1}UR = P_1^{\sigma_1}U^{\epsilon_1}. \]

**Theorem.** The two general types of \( G \) characterized by the two distinct sets of solutions of (10) and (11), viz. \([\epsilon_1, \sigma_1]\) and \([\epsilon_2, \sigma_2]\) are simply isomorphic.

In proof, \( \sigma_2 \equiv -\sigma_1 \), and congruence (8) gives
\[ 2\epsilon_2 - (\iota_1^p + \iota_1)\epsilon_1 - (\iota_2^p + \iota_2) \equiv 0, \quad \epsilon_2 \equiv \epsilon_1 + (\iota_1^p + \iota_1)\epsilon_1. \]

Hence the two types of \( G \) are characterized by
\[ R^{-1}P_1R = P_1^{(\iota_1^p + \iota_2^p)\sigma_1}U^{-\sigma_1}, \quad R^{-1}UR = P_1^{\sigma_1}U^{\epsilon_1}, \]
and
\[ R^{-1}P_1R = P_1^{\epsilon_1}U^{\sigma_1}, \quad R^{-1}UR = P_1^{-\sigma_1}U^{\epsilon_1}(\iota_1^p + \iota_2^p)\sigma_1. \]

Let us select a new operation of order \( q \) from \( \{ Q \} \), e.g. \( Q' = Q^{-1} \). Then
\[ Q'R = RQ', \quad Q'^{-1}UQ' = P_1, \]
\[ Q'^{-1}P_1Q' = U^n P_1^{\iota_2^p + \iota_1^p}, \quad r_j = \frac{\iota_1^p - \iota_2^p}{\iota_1^p - \iota_2^p}. \]

The result of selecting \( Q' \) and \( (\epsilon_2, \sigma_2) \) is thus to interchange \( P_1 \) and \( U \) and to reproduce the relations given by \( Q \) and \( (\epsilon_1, \sigma_1) \). Hence \([\epsilon_2, \sigma_2] \sim [\epsilon_1, \sigma_1]\).

The quantities \( \iota_1 \) and \( \iota_2 \) are marks of the \( GF[p^2] \) and in that field appertain...
OF ORDER $p^2qr$; $p, q, r$ BEING DISTINCT PRIMES

respectively to the exponents $q$ and $r$. Let $\rho$ be any primitive root in the $GF[p^2]$. It is easy to show that $\tau = 1$ and hence we may select

$$\tau_1 = \rho^{(p^2-1)/q}, \quad \tau_2 = \rho^{(p^2-1)/r},$$

thus

$$G = (1: 01; -1, \tau_1^q + \tau_1; \epsilon + (\tau_1^q + \tau_1)\sigma, -\sigma: \sigma\epsilon: 1),$$

where

$$\tau_1 = \rho^{(p^2-1)/q}, \quad \tau_2 = \rho^{(p^2-1)/r}, \quad \rho^{p^2-1} = 1; \quad p \equiv -1 \pmod{qr}, \quad \tau = 1,$$

$$(\tau_1 - \tau_1^q)^2\sigma^2 - (\tau_2 - \tau_2^q)^2 = 0, \quad 2\epsilon + (\tau_1^q + \tau_1)\sigma - (\tau_2^q + \tau_2) = 0.$$

§ 3. The generating function $[k]$. Consider the relation $R^{-1}P_1R^* = P_2^{\mu_1}U^{\nu}$. From it

$$u_{z+1} - (2x + t_1y)u_z + (x^2 + t_1xy + y^2)u_{z-1} = 0,$$

$$u_{z+1} - t_2u_z + u_{z-1} = 0 \quad (t_j = t^j + y; j = 1, 2).$$

These recurring formulae give

$$u_k = [k]_2x - [k - 1]_2, \quad v_k = [k]_2y,$$

where

$$[k]_j = \frac{t^j_2 - t^j_1}{t^j_2 - t^j_1}.$$

Following are some of the properties of the generating function $[k]_j$.

$$(12) \quad \frac{[k + 1]_j}{[k]_j} = 1 + \frac{1}{t_j + \tau_j + \cdots k \text{ terms}},$$

$$(13) \quad [k]_j^2 - [k + 1]_j[k - 1]_j - 1 \equiv 0,$$

$$(14) \quad [0]_j \equiv 0, \quad [1]_j \equiv 1, \quad [-k]_j \equiv -[k]_j,$$

$$(15) \quad [k + 1]_j \equiv [2]_j[k]_j - [k - 1]_j,$$

$$(16) \quad ([k + 1]_j - [k - 1]_j - [2]_j)t^k_j \equiv (t^k_j - 1)(t^{k-1}_j - 1).$$

§ 4. Class (10), $p > q > r$.

We shall consider next groups possessing a single maximal self-conjugate subgroup $H_{pq}^r$ of non-abelian type ($i = \text{III}, \text{IV}, \text{V}, \text{VI}$). It is readily shown that class (10, 12), with $i = \text{III}$, must contain an invariant subgroup $H_{pq}^r$. Class (10) remains to be considered.

$$(1) \quad i = \text{IV}. \text{ Here } H_{pq}^{r, \text{IV}} = \{ P, Q \} \text{ and since } \{ P \} \text{ is self-conjugate in } G, R^{-1}PR = P^\beta. \text{ Since } \rho_{R, s} \equiv 1 \pmod{R, Q, \text{VR}. Hence}

(QR)^{-1}P(QR) = P^\alpha = (QR)^{-1}P(RQR) = P^\beta = 1(p^2), \quad \alpha^2 \equiv 1(p^2), \quad \alpha\beta(\alpha^{-1} - 1) \equiv 0 \pmod{p^2}, \quad \gamma \equiv 1 \pmod{q}.$$

* Dickson, Linear Groups, p. 13.
Hence \( \{P_1, P_2, R\} \) is self-conjugate in \( \{P_1, P_2, Q, R\} = G \), contrary to hypothesis.

\[ (2) \quad i = V. \quad \text{Let } H_{p^r, v} = \{P_1', P_2', Q\}. \quad \text{Assuming that} \]
\[ R^{-1}P_1' R = P_1^* P_2', \quad R^{-1}P_2 R = P_1'^* P_2^*, \]
we deduce
\[ a_{11} \alpha (\alpha^{-1} - 1) \equiv 0, \quad a_{21} (\beta r - \alpha) \equiv 0, \]
\[ a_{22} \beta (\beta^{-1} - 1) \equiv 0, \quad a_{12} (\alpha r - \beta) \equiv 0, \]
where \( \alpha^r \equiv 1(p) \), \( \beta \equiv \alpha^h \). Hence \( \gamma \equiv 1 \mod q \). Hence
\[ a_{11} \equiv 0, \quad a_{22} \equiv 0, \quad \alpha^h \equiv \alpha, \quad \alpha r \equiv \alpha^h \mod p, \]
\[ \gamma \equiv h \mod q, \quad \alpha^r \equiv \alpha \mod p, \quad \gamma^2 \equiv 1 \mod q. \]

But \( \gamma \) appertains to the exponent \( r \) modulo \( q \), and therefore \( r = 2 \) and \( \gamma = 1 \mod q \). Thus
\[ R^{-1}P_1' R = P_2^*, \quad R^{-1}P_2 R = P_1'^*, \quad a_{12} a_{21} \equiv 1 \mod p. \]

Then \( P_1 = P_1'^*, \ P_2, \ Q, \ R \), generate a group of order \( 2p^t q \), viz.,
\[ G = \langle 1 : 0 : 0 : 0 : a^r - 1 : 0 : 1 : 0 : - 1 \rangle. \] Also \( p = 1(q), \ r = 1. \)

\[ (3) \ i = VI. \quad \text{It has been shown [§ 1], that } p \equiv \pm 1 \mod r. \]

(a) First let \( p = 1(r) \). Then \( P_1, P_2 \) may be selected which are permutable with \( R \). If
\[ Q^{-1}P_1 Q = P_2, \quad Q^{-1}P_2 Q = P_1'^{-1} P_2^{p+1}, \]
then
\[ R^{-1}P_1 R = P_1^* , \quad R^{-1}Q R = Q'_r, \quad \gamma \equiv 1 \mod q. \]

Since \( I_r \), is invariant in \( G \) we may assume that
\[ P_3 = P_1 P_2^* , \quad R^{-1}P_2 R = P_1^* P_2^*, \]
Hence
\[ (QR)^{-1}P_1 (QR) = P_1 P_2^* = (RQ'^{-1} R_1 (RQ') = P_1^{-1} P_2^* [\gamma], \]
\[ (QR)^{-1}P_2 (QR) = P_1^{-1} P_2^* [\gamma] = (RQ'^{-1} P_2 (RQ') = P_1^{-1} P_2^* [\gamma], \]
\[ x = - [\gamma] \beta , \quad y = [\gamma] \beta , \]
\[ [\gamma]^2 = [\gamma - 1]^2 + [2] [\gamma - 1] + 1, \]
\[ [\gamma] \{ [\gamma + 1] - [\gamma - 1] - [2] \} \equiv 0. \]

Now \( [\gamma] \equiv 0 \mod q \). Since \( [- k] = - [k] \) and
\[ [\gamma + 1] - [\gamma - 1] - [2] \equiv (\gamma + 1 - 1)(\gamma - 1 - 1) \equiv 0 \mod q, \]
there results \( \gamma \equiv 1 \mod q, \gamma' \equiv (1)^r \equiv 1 \mod q \), whence \( r = 2 \). If
\[ R^{-1}P_3 R = P_3^* , \] then \( \alpha \equiv \pm 2 \mod p \).
First let the upper sign hold. If \( \beta = 1 \), then \( w = 0 \) which is impossible, since \( P_1, P_2 \) are independent. Hence \( \beta = -1, x = -[2], y = +[1] \equiv +1 \).
Likewise if we use the lower sign, \( \beta = +1, x = +[2], y = -[1] \equiv -1 \).
We thus obtain the two sets of defining relations:

\[
(1:01: -1^\tau P + \iota: \equiv 10: 1^\tau + \iota^2, \pm 1: -1).
\]

To determine \( \tau \), let \( Q_0 = Q^\tau, R_0 = R, P_{10} = P_1, P_{20} = P_1^{-[s-1]}P_2^s \); there results

\[
\{ P_{10}, P_{20}, Q_0, R_0 \} = (1:01: -1^\tau + \iota^2: \equiv 10: \pm [x-1] \equiv [2][x], \pm [x]: -1).
\]

But

\[
\pm [x-1] \equiv [2][x] \equiv [x+1] \equiv (\tau^\tau + \iota^2) \equiv [x-1],
\]

[Eq. (15)]. Hence

\[
\{ P_{10}, P_{20}, Q_0, R_0 \} = (1:01: -1^\tau + \iota^2: \equiv 10: (\tau^\tau + \iota^2), \pm 1: -1) \sim G.
\]

Thus the same defining relations are reproduced with \( \iota \) replaced by \( \iota^s \), and so \( \tau = 1 \).

It will now be proved that these two types are simply isomorphic. Select new operators as follows:

\[
q_1 = Q, \ r_1 = R, \ p_1 = P_1 \times P_1, \ p_2 = P_1^{-[s-1]}P_2^s = q_1^{-1} p_1 q_1.
\]

Then using the first set of defining relations we will have

\[
q_1^{-1} p_2 q_1 = p_1^{-1} P_2^{s+\iota}, \ r_1^{-1} p_1 r_1 = P_1, \ r_1^{-1} p_2 r_1 = P_1^{s+\iota} P_1^{-1}, \ r_1^{-1} q_1 r_1 = q_1^{-1}
\]

if

\[
2a + [2]b \equiv 0 \ (\text{mod} \ p).
\]

Hence when a new operator \( p_1 = P_1 P_2^b \) is selected, where \( a \) and \( b \) are solutions of \( 2a + (s+\iota)b \equiv 0 \ (\text{mod} \ p) \), the first type is transformed into the second.
They are therefore isomorphic.

(b) When \( p = -1(r), r \ odd, \rho_{R,p} = 0 \). As before, we deduce

\[
Q^{-1} P_1 Q = P_2, \quad Q^{-1} P_2 Q = P_1^{-1} P_2^{s+\iota}, \quad \iota_1 \equiv 1(p),
\]

\[
R^{-1} P_1 R = P_3, \quad R^{-1} P_2 R = P_1^{-1} P_2^{s+\iota}, \quad \iota_2 \equiv 1(p),
\]

Let \( P_3 = P_1^{-1} P_2^s \) and \( R^{-1} P_2 R = P_4 = P_1^s P_2^s \). Then

\[
(17) \quad R^{-1} P_2 R = P_1^{-[\gamma+1]}(s+\iota)^s P_2^{[s]} = P_1^{-[\gamma-1]} [s]^{*} \cdot [s]^{[\gamma]} P_1^{[\gamma]} [s]^{*} \cdot [s]^{[\gamma+1]} [s]^{*}.
\]

In addition to the latter, but not independent of them, we have the congruences derived from
(18) \[(QR)^{-1}P_\gamma(QR) = (RQ_r)^{-1}P_\gamma(RQ_r)\].

The equations (17) and (18) give us the dialytic eliminant
\[
\Delta_{12} = \{i_2 + i_2\} \{[\gamma]_1^2 - (i_2 + i_2)[\gamma]_1 + 1\} \{(i_1^r + 1)(i_1^{-1} - 1)\}^2 = 0.
\]
Now \([\gamma]_1\) is an integer, and since \(r \neq 2\), and \(\gamma \neq -1\), it follows that \(\gamma \equiv 1\pmod{q}\), contrary to hypothesis. Hence when \(p \equiv -1\pmod{r}\) and \(r\) is odd, no corresponding group \(G\) exists.

The results of this section may be summarized in the following

**Theorem.** A group \(G_{pqr}\) \((p > q > r)\) always contains a maximal self-conjugate subgroup \(H\) of order \(p^2q\). If \(H\) is the only maximal invariant subgroup of \(G\) and if \(r\) is odd, then \(N_q = 1\) and \(H\) is necessarily abelian. If \(r\) is even \((r = 2)\) and \(p \equiv 1\pmod{q}\) there exists one type whose subgroup \(H_{p^2q}\) is non-abelian, and if \(r\) is even and \(p \equiv -1\pmod{q}\) there exists a second type possessing a non-abelian \(H_{p^2q}\). These two types of \(G\) contain respectively \(q\) and \(pq\) operators (and subgroups) of order 2, and in each type \(N_q = p^2\). Moreover, with exception of the two types just described, every group of order \(p^2qr\) \((p > q > r)\), in which \(N_r \equiv 0\pmod{q}\), possesses an abelian maximal self-conjugate subgroup \(H_{p^2q}\).

A general summary of all the existent types of \(G\) follows. Except for \(i\) and \(p\), every parameter occurring in the tables is an integer; while \(i\) and \(p\) are marks of the \(GF[p^2]\). See footnote on the second page of the paper.
Table 1. \( p > q > r \).

<table>
<thead>
<tr>
<th>Class</th>
<th>(Q^{-1}P_1Q)</th>
<th>(Q^{-1}P_2Q)</th>
<th>(R^{-1}P_1R)</th>
<th>(R^{-1}P_2R)</th>
<th>Parameters.</th>
<th>Arith. Rel.</th>
<th>(\tau)</th>
</tr>
</thead>
<tbody>
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<td>(P_1)</td>
<td>(P_1)</td>
<td>(P_1)</td>
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<td>(\beta^s \equiv 1(p))</td>
<td>(\gamma^s \equiv 1(p))</td>
</tr>
<tr>
<td>[12 \cdot 12]</td>
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<td>(P_2)</td>
<td>(P_2)</td>
<td>(P_2)</td>
<td>(p \equiv 1(q))</td>
<td>(p \equiv 1(q))</td>
<td>(p \equiv 1(q))</td>
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<td>(P_1)</td>
<td>(P_1)</td>
<td>(P_1)</td>
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<td>(p \equiv 1(q))</td>
<td>(1)</td>
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<td>(P_2)</td>
<td>(P_2)</td>
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<td>(P_1)</td>
<td>(P_1)</td>
<td>(P_1)</td>
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<td>(p \equiv 1(q))</td>
<td>(\frac{1}{2}(q + 1))</td>
</tr>
<tr>
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<td>(P_2)</td>
<td>(P_2)</td>
<td>(P_2)</td>
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<td>(p \equiv 1(q))</td>
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<tr>
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<td>(P_1)</td>
<td>(P_1)</td>
<td>(P_1)</td>
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<td>(p \equiv 1(q))</td>
<td>(1)</td>
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<td>[9101112]</td>
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<td>(P_2)</td>
<td>(P_2)</td>
<td>(P_2)</td>
<td>(\alpha^s \equiv 1(p))</td>
<td>(p \equiv 1(q))</td>
<td>(\frac{1}{2}(r + 1))</td>
</tr>
<tr>
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<td>(P_1)</td>
<td>(P_1)</td>
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<td>(p \equiv 1(q))</td>
<td>(q - 1)</td>
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<td>[91012]</td>
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<td>(P_2)</td>
<td>(P_2)</td>
<td>(P_2)</td>
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<td>(p \equiv 1(q))</td>
<td>(r - 1)</td>
</tr>
<tr>
<td>[910]</td>
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<td>(P_1)</td>
<td>(P_1)</td>
<td>(P_1)</td>
<td>(\gamma^s \equiv \alpha^s \equiv 1(p))</td>
<td>(p \equiv 1(q))</td>
<td>(\frac{1}{2}(q + 1)) or (\frac{1}{2}(r + 1)(q + 1))</td>
</tr>
<tr>
<td>(\rho = \text{prim. root in})</td>
<td>(GF[p^s])</td>
<td>(\ell_1, \ell_2 = \rho^{s-1}q, r)</td>
<td>(2s + [2], \sigma - [2] = \ell_1^s \equiv 0) ((\ell_1 - \ell_2)^2 \sigma^2 - (\ell_1 - \ell_2) \ell_2 \equiv 0)</td>
<td>(p \equiv -1(qr))</td>
<td>(1)</td>
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Case (b). \( R^{-1}QR = Q^\gamma; \gamma' = 1(q) \).

<table>
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<th>Class.</th>
<th>( Q^{-1}P_2Q )</th>
<th>( Q^{-1}P_3Q )</th>
<th>( R^{-1}P_2R )</th>
<th>Parameters</th>
<th>Arith. rel.</th>
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<td>( q = 1(r) )</td>
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<td>( P_3 )</td>
<td>( P_1 )</td>
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<td>( h = 1 )</td>
<td>( q = 1(r) )</td>
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<td>( P_1 )</td>
<td>( P_3 )</td>
<td>( h = 1, 2 \ldots r - 1 )</td>
<td>( p = q = 1(r) )</td>
</tr>
<tr>
<td>[101112]</td>
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<td>( P_1 )</td>
<td>( P_2 )</td>
<td>( h = 1, 2 \ldots r - 1 )</td>
<td>( p = q = 1(r) )</td>
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<tr>
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<td>( P_2 )</td>
<td>( P_1 )</td>
<td>( P_2 )</td>
<td>( h = 1, 2 \ldots r - 1 )</td>
<td>( p = q = 1(r) )</td>
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<td>( h = 1, r = 1 )</td>
<td>( q = 1(p) )</td>
</tr>
<tr>
<td>&quot;</td>
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<td>( P_2 )</td>
<td>( P_2 )</td>
<td>( P_1 )</td>
<td>( h = 1, r = 1 )</td>
<td>( q = 1(p) )</td>
</tr>
</tbody>
</table>

Table 2. \( q > p > r \).

\( I_p \) non-cyclical; \( P_i^p = Q^p = R^p = 1 \) (\( i = 1, 2 \)), \( P_i P_2 = P_2 P_1, RP_2 = P_2 R \),

\( I_p \) cyclical; \( P_i^p = Q^p = R^p = 1, RP_1 = P_1 R \).

<table>
<thead>
<tr>
<th>Class.</th>
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<th>( P^{-1}QP_2 )</th>
<th>( R^{-1}QR )</th>
<th>Parameters</th>
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<th>( r )</th>
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<td>( Q^b )</td>
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<td>( q = 1(p^\gamma) )</td>
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<td>( Q )</td>
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<td>( \alpha^a = 1(q) )</td>
<td>( q = 1(p) )</td>
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<td>( \alpha^a = 1(q) )</td>
<td>( q = 1(p^\gamma) )</td>
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<td>( q = 1(p^\gamma) )</td>
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<td>( Q^a )</td>
<td>( Q^a )</td>
<td>( P_1 )</td>
<td>( \alpha^a = \gamma^a = 1(q) )</td>
<td>( q = 1(p^\gamma) )</td>
</tr>
</tbody>
</table>

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Table 3. $q > r > p$.

**Case (a).**

$I_p$, non-cyclical; $P_1^a = Q = R^e = 1(i = 1, 2)$, $P_1P_2 = P_2P_1$, $RQ = QR$,

$I_p$, cyclical; $P_1^a = Q^a = R^e = 1$, $QR = RQ$.

<table>
<thead>
<tr>
<th>Class</th>
<th>$P_1^{-1}QP_1$</th>
<th>$P_2^{-1}QP_2$</th>
<th>$P_1^{-1}RP_1$</th>
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<th>Parameters</th>
<th>Arith. Rel.</th>
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<td>$R^a$</td>
<td>.</td>
<td>$\alpha^a = 1(q)$</td>
<td>$q = r = 1(p)$</td>
<td>$p - 1$</td>
</tr>
<tr>
<td>[125]</td>
<td>$Q^a$</td>
<td>.</td>
<td>$R^a$</td>
<td>.</td>
<td>$\beta^a = 1(r)$</td>
<td>$q = 1(p^3)$</td>
<td>$p - 1$</td>
</tr>
<tr>
<td>[234]</td>
<td>$Q^a$</td>
<td>.</td>
<td>$R^a$</td>
<td>.</td>
<td>$\beta^a = 1(q)$</td>
<td>$r = 1(p^3)$</td>
<td>$p - 1$</td>
</tr>
<tr>
<td>[12]</td>
<td>$Q^a$</td>
<td>.</td>
<td>$R^a$</td>
<td>.</td>
<td>$\beta^a = 1(q)$</td>
<td>$q = r = 1(p^3)$</td>
<td>$p^3 - 1$</td>
</tr>
<tr>
<td>[12345678]</td>
<td>$Q$ $Q$</td>
<td>$R$ $R^a$</td>
<td>$\alpha^a = 1(r)$</td>
<td>$r = 1(p)$</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>[123]</td>
<td>$Q$ $Q^a$</td>
<td>$R$ $R^a$</td>
<td>$\beta^a = 1(r)$</td>
<td>$q = r = 1(p)$</td>
<td>$p - 1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>[1235]</td>
<td>$Q$ $Q^a$</td>
<td>$R$ $R^a$</td>
<td>$\beta^a = 1(q)$</td>
<td>$q = r = 1(p)$</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Case (b).** The simple group $G_{15}$, $p = 2$, $q = 5$, $r = 3$.

$Q^a = 1$, $P^a = 1$, $(QP)^3 = 1$, $[R = QP]$. 

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