DYNAMICAL TRAJECTORIES: THE MOTION OF A PARTICLE IN AN ARBITRARY FIELD OF FORCE*

BY

EDWARD KASNER

The object of the present paper is to work out for space of three dimensions a geometric theory analogous to that given for the two-dimensional case in an earlier paper. †

We consider a particle moving freely in space under the action of any positional force, and write its equations of motion in the form

$$(1)\qquad \ddot{x}=\phi(x, y, z), \qquad \ddot{y}=\psi(x, y, z), \qquad \ddot{z}=\chi(x, y, z),$$

where the dots indicate derivatives with respect to the time. The functions ϕ, ψ, χ are assumed to have partial derivatives of first and second order in the region of space considered. The case where the force vanishes everywhere is excluded.

The motion of the particle is determined when its initial position and initial velocity are given. By taking all possible initial conditions we obtain a definite quintuply infinite system of trajectories. Our object is to study the properties of such systems of curves with a view to obtaining a *complete geometric characterization.* The main result is stated in article 39 at the end of the paper.

The first properties derived are consequences of the elementary fact that the osculating plane at any point of a trajectory is determined by its initial direction and by the direction of the force (articles 4–11). The next set relate to osculating spheres.‡ With each lineal element there are associated ∞^1 osculating spheres whose centers lie on a straight line (articles 12–15). The straight lines corresponding to all the elements at a point form a congruence of order one and class three determined by a twisted cubic curve (articles 16–23). The properties obtained at this stage belong to a more general class of curve systems than the dynamical class. The final characterization is attained by introducing certain related systems of plane curves, here termed S-systems, which have all the

* Presented to the Society September 6, 1906. Received for publication December 12, 1906.

† *The Trajectories of Dynamics*, Transactions of the American Mathematical Society, vol. 7 (1906), pp. 401–424. We shall refer to this as *Trajectories.*

‡ Properties I_1, I_2, and II of the present paper were given in a note published in the Bulletin of the American Mathematical Society, vol. 12 (1905), pp. 71–74.

properties of plane systems of trajectories (articles 29–39). The S-system in a given plane may be obtained dynamically by constructing, at each point of the plane, the component of the force (1) acting in that plane; the two-dimensional field thus derived generates the required system.

It will be seen that the spatial properties are not direct extensions of those given in the plane theory; and that the present theory requires the previous discussion of the two-dimensional case. The earlier theory is thus not absorbed by the present paper.

The logical relations of the various properties are settled by converting the main theorems, thus bringing to light general systems of curves of interest apart from their connections with dynamics. Of the incidental results we mention only the interpretation of the curl of the force given in article 21, and the special characterization of conservative forces in articles 22 and 40.*

Differential equations of the trajectories

1. In order to derive the geometrical properties of the trajectories, it is necessary to eliminate the time. We distinguish the two types of derivatives involved by means of dots and primes. Thus

$$(2)\qquad \begin{aligned} &\dot{x}=\frac{dx}{dt}, \qquad \dot{y}=\frac{dy}{dt}, \qquad \dot{z}=\frac{dz}{dt}, \text{ etc.,}\\ &y'=\frac{dy}{dx}, \qquad z'=\frac{dz}{dx}, \text{ etc.} \end{aligned}$$

The geometric derivatives may be expressed in terms of the kinematic as follows:

$$(3)\qquad y'=\frac{\dot{y}}{\dot{x}}, \qquad z'=\frac{\dot{z}}{\dot{x}}, \qquad y''=\frac{\dot{x}\ddot{y}-\dot{y}\ddot{x}}{\dot{x}^3}, \qquad z''=\frac{\dot{x}\ddot{z}-\dot{z}\ddot{x}}{\dot{x}^3}.$$

Combining these equations with (1), we obtain two expressions for $\dot{x}^2$, namely

$$(4)\qquad \dot{x}^2=\frac{\psi-y'\phi}{y''}, \qquad \dot{x}^2=\frac{\chi-z'\phi}{z''}.$$

Thus one result free from the time is

$$(5)\qquad \frac{\psi-y'\phi}{y''}=\frac{\chi-z'\phi}{z''}.$$

Differentiating equations (4) with respect to x, we find two other equations

*The transformation theory given at the conclusion of *Trajectories* (articles 26–29) may be extended to space, but is not considered in the present paper. In the discussion of the Appell transformation (article 30), the formulas should be expressed, not in terms of the times t and t_1, but, as in Appell's treatment, in terms of the differentials dt and dt_1. The finite representation of page 423 yields merely the affine transformation. The appropriate converse question was solved by Painlevé, Journal de Mathématiques (1894).

free from the time.

$$(6)\qquad (\psi - y'\phi)y''' = \begin{vmatrix} 1 & \phi_x + y'\phi_y + z'\phi_z \\ y' & \psi_x + y'\psi_y + z'\psi_z \end{vmatrix} y'' - 3\phi y''^2,$$

$$(7)\qquad (\chi - z'\phi)z''' = \begin{vmatrix} 1 & \phi_x + y'\phi_y + z'\phi_z \\ z' & \chi_x + y'\chi_y + z'\chi_z \end{vmatrix} z'' - 3\phi z''^2,$$

where the subscripts denote partial derivatives.

The quintuply infinite system of trajectories is completely represented by the equations (5) *and* (6). We write these in the form

$$(8)\qquad \begin{aligned} z'' &= Ky'', \\ y''' &= Py'' + Qy''^2, \end{aligned}$$

where

$$(8')\qquad K = \frac{\chi - z'\phi}{\psi - y'\phi}, \qquad Q = \frac{-3\phi}{\psi - y'\phi},$$
$$P = \frac{\begin{vmatrix} 1 & \phi_x + y'\phi_y + z'\phi_z \\ y' & \psi_x + y'\psi_y + z'\psi_z \end{vmatrix}}{\psi - y'\phi}.$$

All other equations between the geometric derivatives, including (7), are indeed consequences of (8). This may be seen from the fact that a trajectory is completely determined by the assignment of the initial values of x, y, z, y', z', y''; for this is equivalent, according to (4), to the assignment of initial position and initial velocity. The corresponding values of z'', y''' are given by (8), and those of z''', y^{iv}, z^{iv}, etc., are obtained by successive differentiation of (8).

2. It will be convenient, in part of the subsequent discussion, to define the force whose components are ϕ, ψ, χ by the functions

$$(9)\qquad \omega_1 = \frac{\psi}{\phi}, \qquad \omega_2 = \frac{\chi}{\phi}, \qquad \Phi = \log\phi.$$

The first two of these define the direction of the force; the equations of the lines of force are, in fact,

$$(10)\qquad dx : dy : dz = 1 : \omega_1 : \omega_2.$$

The third depends upon the intensity of the force, and completely defines it when the other two are known.

The coefficients in the fundamental equations (8) may now be written

$$(8'')\qquad K = \frac{z' - \omega_2}{y' - \omega_1}, \qquad Q = \frac{3}{y' - \omega_1},$$
$$P = \Phi_x + y'\Phi_y + z'\Phi_z - \frac{\omega_{1x} + y'\omega_{1y} + z'\omega_{1z}}{y' - \omega_1}.$$

By differentiating the first equation (8) and combining with the second, we obtain

$$z''' = Ky''' + K'y'' = \left(P + \frac{K'}{K}\right)z'' + \frac{Q}{K}z''.$$

This may be written in the form

$$z''' = \bar{P}z'' + \bar{Q}z''^2, \tag{10}$$

where

$$\bar{P} = \Phi_x + y'\Phi_y + z'\Phi_z - \frac{\omega_{2x} + y'\omega_{2y} + z'\omega_{2z}}{z' - \omega_2}, \qquad \bar{Q} = \frac{3}{z' - \omega_2}, \tag{10'}$$

and is, of course, equivalent to (7).

3. It may be shown, as in *Trajectories*, that if two fields ϕ, ψ, χ and ϕ_1, ψ_1, χ_1, lead to the same trajectories, that is, the same equations (8), then they can differ only by a constant factor. The system of trajectories thus determines the field.*

Osculating planes.

4. *Property* I. The osculating plane of a trajectory at a given point is determined by its initial direction and by the direction of the force acting at the given point. This familiar fact is easily verified analytically by noting that the general equation of the osculating plane is

$$\begin{vmatrix} X & Y & Z \\ 1 & y' & z' \\ 0 & y'' & z'' \end{vmatrix} = 0, \tag{11}$$

where the current coördinates X, Y, Z refer to axes passing through the given point. This is satisfied, in virtue of (5), by $X : Y : Z = \phi : \psi : \chi$.

Property I. — *The osculating planes of all trajectories passing through a given point form a pencil.*

The axis of the pencil has the direction of the force acting at the given point.

5. *Converse of* I. It is easy to determine all quintuply infinite systems of curves possessing property I. Let the direction, fixed for each point, be defined by the ratios

$$1 : \omega_1 : \omega_2,$$

where ω_1 and ω_2 are arbitrary functions of x, y, z. By hypothesis, the osculating plane (11) passes through this direction for all possible values y', z', y'', z''. Hence

$$\begin{vmatrix} 1 & \omega_1 & \omega_2 \\ 1 & y' & z' \\ 0 & y'' & z'' \end{vmatrix} = 0,$$

* A geometric construction of the field will appear in a later paper, *The inverse problem of dynamics.*

which gives

$$\frac{z''}{y''} = \frac{z' - \omega_2}{y' - \omega_1}.$$

Conversely, when this relation is fulfilled, the curves have the required property. Hence:

All quintuply infinite systems of curves possessing property I *are represented by differential equations of the form*

$$(12) \qquad \begin{gathered} z'' = \frac{z' - \omega_2}{y' - \omega_1} y'' = Ky'', \\ y''' = f(x, y, z, y', z', y''). \end{gathered}$$

The result thus contains two arbitrary functions ω_1, ω_2 of x, y, z and an arbitrary function f of the six arguments indicated.

6. *Property* I_1. Before proceeding to essentially new properties, we derive a few corollaries of interest. If the initial position and direction, fixed by the values of x, y, z, y', z', be given, then y'' is arbitrary and we thus have ∞^1 trajectories. For these we may state the elementary result:

The ∞^1 *trajectories corresponding to a given lineal element have* (*at the given point*) *a common osculating plane.*

7. *Property* I_2. The torsion of any curve is given by the general formula

$$(13) \qquad \frac{1}{\rho} = \frac{y''z''' - z''y'''}{(y'y'' + z'z'')^2 - (1 + y'^2 + z'^2)(y''^2 + z''^2)}.$$

For trajectories, we have

$$z'' = Ky''.$$

Hence

$$z''' = Ky''' + K'y'',$$

where

$$K' = \frac{\omega_1'(z' - \omega_2) - \omega_2'(y' - \omega_1)}{(y' - \omega_1)^2}.$$

Introducing these values in (13), we find

$$(14) \qquad \frac{1}{\rho} = \frac{K'}{(y' + Kz')^2 - (1 + y'^2 + z'^2)(1 + K^2)}.$$

This does not involve y'' or z''. Hence the result:

The ∞^1 *trajectories determined by a given lineal element have* (*at the given point*) *the same torsion.*

8. *Converse of* I_1 *and* I_2. These properties hold not only for dynamical systems, but for the more general systems (12). They are thus consequences of I and might in fact have been derived synthetically from it. The converse,

however, does not hold. There are in fact systems possessing both I_1 and I_2 (for all lineal elements) but not possessing I. We omit the discussion and state the result:

A system of space curves possesses property I_1 *when the relation between* z'' *and* y'' *is of the form*

$$z'' = k(x, y, z, y', z')y'', \tag{15}$$

where k *is arbitrary. It also possesses property* I_2 *provided the function* k *satisfies the equation*

$$k_{y'} + kk_{z'} = 0. \tag{16}$$

The function K, appearing in (8) and (12), is seen to be a special solution of (16).

8′. *Related helices.* At each point of a space curve it is possible to construct a unique helix having in common with the given curve its direction, its osculating plane, its curvature, and its torsion.* We term this the *related helix.* If the current coördinates X, Y, Z refer to axes determined by the tangent, principal normal, and binormal at the given point, then the axis of the helix is given by the formulas

$$Y = \frac{r\rho^2}{r^2 + \rho^2}, \qquad \frac{X}{Z} = -\frac{r}{\rho}, \tag{17}$$

where r and ρ are the radii of curvature and torsion respectively.

Consider now the ∞^1 trajectories passing through a given point in a given direction. These have a common osculating plane and a common torsion. The curvature varies from curve to curve. The elimination of r from (17) gives

$$Y(X^2 + Z^2) + \rho XZ = 0. \tag{18}$$

This is recognized as PLÜCKER's normal equation for a cylindroid.† Hence:

The ∞^1 *trajectories determined by a given lineal element give rise to related helices whose axes generate a cylindroid.*

The discussion shows that this holds for the more general systems obtained in article 8. The result is thus a consequence of I_1 and I_2.

9. *Distribution of torsion.* In article 7 it was shown that for each lineal element there is a definite value of ρ. If we consider the ∞^2 elements at a given point, they will subdivide into sets of ∞^1 according to the values of the parameter ρ. If ρ has a given value, say c, the corresponding elements are

---

* SCHEFFERS, *Theorie der Curven*, p. 197. The contact is not of the third order unless dr/ds vanishes. Only in this case does an *osculating helix* exist.

† A cylindroid presents itself in a different connection in SCHEFFER's discussion (l. c., p. 196). There r is fixed, while ρ varies. The elimination of ρ from (17) leads to $Y(X^2 + Z^2) - rZ^2 = 0$.

found, from (14), to be

$$(20)\quad (\omega_1 z' - \omega_2 y')^2 + (y' - \omega_1)^2 + (z' - \omega_2)^2 - c\begin{vmatrix} y' - \omega_1 & \omega_{1x} + y'\omega_{1y} + z'\omega_{1z} \\ z' + \omega_2 & \omega_{2x} + y'\omega_{2y} + z'\omega_{2z} \end{vmatrix}.$$

The elements thus determine the cone

$$(20')\quad (\omega_1 Z - \omega_2 Y)^2 + (Y - \omega_1 X)^2 + (Z - \omega_2 X)^2 - c\begin{vmatrix} Y - \omega_1 X & \omega_{1x}X + \omega_{1y}Y + \omega_{1z}Z \\ Z - \omega_2 X & \omega_{2x}X + \omega_{2y}Y + \omega_{2z}Z \end{vmatrix}.$$

The trajectories which pass through a given point and have there the same torsion are arranged so that their initial directions form a quadric cone.

10. When the given torsion is zero, (20′) becomes

$$(21)\qquad \begin{vmatrix} Y - \omega_1 X & \omega_{1x}X + \omega_{1y}Y + \omega_{1z}Z \\ Z - \omega_2 X & \omega_{2x}X + \omega_{2y}Y + \omega_{2z}Z \end{vmatrix} = 0.$$

In this case the osculating plane has four, instead of three, consecutive points in common with the curve.

Of the ∞^3 trajectories through a given point there are ∞^2 with hyperosculating planes. The lineal elements of these trajectories generate the quadric cone (21).

11. By varying c in (20′) we obtain a pencil of cones. It may be shown that all pass through the element defined by the ratios

$$(22)\qquad 1 : \omega_1 : \omega_2,$$

and that along this element they have a common tangent plane

$$(23)\qquad \begin{vmatrix} 1 & \omega_1 & \omega_2 \\ 0 & \omega_{1x} + \omega_1\omega_{1y} + \omega_2\omega_{1z} & \omega_{2x} + \omega_1\omega_{2y} + \omega_2\omega_{2z} \\ X & Y & Z \end{vmatrix} = 0.$$

The element (22) has the direction of the line of force and the plane (23) is the osculating plane of the line of force.

Osculating spheres.

12. The properties thus far obtained are consequences of property I. To obtain independent properties we now consider osculating spheres. If we take the given point as origin and let X, Y, Z denote the coördinates of the center of the sphere, the equation of the sphere is

$$x^2 + y^2 + z^2 - 2Xx - 2Xy - 2Zz = 0.$$

The conditions for osculation with a given curve are

$$(24)\qquad \begin{aligned} X + y'Y + z'Z &= 0, \\ y''Y + z''Z &= 1 + y'^2 + z'^2, \\ y'''Y + z'''Z &= 3(y'y'' + z'z''). \end{aligned}$$

From these we obtain the general formulas

$$X = -\frac{(1 + y'^2 + z'^2)(y'z''' - z'y''') - 3(y'z'' - z'y'')(y'y'' + z'z'')}{y''z''' - z''y'''},$$

$$(25)\quad Y = \frac{(1 + y'^2 + z'^2)z''' - 3z''(y'y'' + z'z'')}{y''z''' - z''y'''},$$

$$Z = -\frac{(1 + y'^2 + z'^2)y''' - 3y''(y'y'' + z'z'')}{y''z''' - z''y'''},$$

13. *Property* II. Consider the ∞^1 trajectories determined by a given lineal element. For these y'' is arbitrary and z'', y''', z''' are determined as functions of y'' by the fundamental differential equations (8) and the derived equation (10). If we substitute these values in (25) or, as is more convenient, in (24), we may eliminate the parameter y'' and obtain two relations between X, Y, Z. One of these is already given by the first equation (24); the other is found to be

$$(26)\qquad PY + K\bar{P}Z + Q(1 + y'^2 + z'^2) - 3(y' + Kz') = 0.$$

The coefficients in both these equations are constants since they depend only upon the given lineal element. Since the equations are linear in X, Y, Z, we have

PROPERTY II. *The osculating spheres of the ∞^1 trajectories passing through a given point in a given direction have their centers on a straight line.*

The straight line is necessarily in the plane perpendicular to the given direction at the given point, i. e., in the common normal plane of the ∞^1 trajectories. Since the spheres all pass through the given point, it follows from the above that they pass through a common circle. Hence property II may be restated in this form:

The osculating spheres of the ∞^1 trajectories described above form a pencil.

14. *Converse of* II. It is clear that II is independent of I. In fact the proof of I depended upon only the first of the fundamental equations (8), while that of II involved also the second.

We now find all systems of curves possessing I and II. By article 5 the differential equations are necessarily of the form (12). It remains to determine the form of the function f. The hypothesis is that for each lineal element the locus of the centers (X, Y, Z) of the osculating spheres shall be a straight line. Since the centers are necessarily in the plane normal to the element, the equations of the straight line may be taken in of the form

$$X + y'Y + z'Z = 0,$$

$$A_1X + A_2Y + A_3Z + A_4 = 0,$$

where the coefficients A involve x, y, z, y', z' in any way. Substituting the general values (25) in the second of these equations, we find a relation of the form

$$(1+y'^2+z'^2)(B_1z'''-B_2y''')-3(B_1z''-B_2y'')(y'y''+z'z'')+B_3(y''z'''-z''y''')=0,$$

where the B's involve only x, y, z, y', z'. Introducing the values of z'', y''' given in (12) and the value of z''' found by differentiation, we obtain

$$f = Gy'' + Hy''^2,$$

where G and H are functions of x, y, z, y', z'.

It is easy to verify that this condition is sufficient, that is, that property II holds for all values of G and H. Hence:

The most general quintuply infinite system of space curves possessing properties I *and* II *is defined by equations of the form*

$$(27) \qquad z'' = Ky'' = \frac{z' - \omega_2}{y' - \omega_1} y'', \qquad y''' = Gy'' + Hy''^2;$$

the result thus involves two arbitrary functions ω_1, ω_2 *of* x, y, z, *and two arbitrary functions* G, H *of* x, y, z, y', z'.

15. *Correspondence of centers.* We now prove a property which holds not only for dynamical systems but for the general systems just obtained.

For each of the ∞^1 curves corresponding to a given lineal element we may construct a circle of curvature and an osculating sphere. Denote their centers by O' and O'' respectively. By property II the locus of O'' is a straight line. The locus of O' is evidently also a straight line, namely, the principal normal which, by property I_1, is common to all the curves. We now prove that *the ranges described by* O' *and* O'' *are similar.*

Take the given point as origin and the given initial direction as that of the axis of abscissas. Then the distance from O' to the given point, that is, the radius of curvature, is found to be

$$(28) \qquad r = \frac{1}{y''\sqrt{1+K^2}}.$$

The ordinate of O'' is

$$(29) \qquad Z = \frac{G + Hy''}{K'y''}.$$

Eliminating y'', we obtain a relation of the form

$$Z = ar + b,$$

which proves the result stated. In the dynamical case the property may be expressed as follows:

If a particle is projected from a given point in a given direction, and if for each of the ∞^1 trajectories obtained by varying the initial velocity we construct the center O' of the circle of curvature and the center O'' of the osculating sphere, then the points O' and O'' describe similar ranges.

There is no difficulty in proving this synthetically as a consequence of I and II.

The congruence Ω and the cubic Γ.

16. *Equations of* Ω. We have seen that to each lineal element there corresponds a definite straight line, the locus of the centers of the osculating spheres. If the point is kept fixed and the direction of the element varied, then ∞^2 of these straight lines are obtained. What is the character of the congruence Ω thus generated?

The equations of the straight line corresponding to a given element, found in article 13, may be written

$$X + y'Y + z'Z = 0,$$

$$(30) \quad \begin{vmatrix} 1 & \phi_x + y'\phi_y + z'\phi_z \\ y' & \psi_x + y'\psi_y + z'\psi_z \end{vmatrix} Y + \begin{vmatrix} 1 & \phi_x + y'\phi_y + z'\phi_z \\ z' & \chi_x + y'\chi_y + z'\chi_z \end{vmatrix} Z = 3(\phi + y'\psi + z'\chi).$$

Here x, y, z have fixed values and y', z' are arbitrary parameters leading to the ∞^2 straight lines of Ω. The quadratic terms in y', z' may be eliminated by combining the equations, giving the result:

The congruence Ω *is defined by the equations*

$$(31) \qquad X + y'Y + z'Z = 0, \qquad AX + BY + CZ = 3D,$$

where

$$(31') \qquad \begin{aligned} A &= y'\phi_y + z'\phi_z, & B &= \psi_x + y'(\psi_y - \phi_x) + z'\psi_z, \\ C &= \chi_x + y'\chi_y + z'(\chi_z - \phi_x), & D &= \phi + y'\psi + z'\chi. \end{aligned}$$

17. *The cubic curve* Γ. If the values of X, Y, Z are given, then, since the equations (31) are linear in the parameters, they will determine, in general, unique values of y', z'. Hence *the congruence is of order one.*

The equations (31) may be written in the form

$$(32) \qquad X + Yy' + Zz' = 0, \qquad \alpha + \beta y' + \gamma z' = 0,$$

where

$$\alpha = \psi_x Y + \chi_x Z - 3\phi,$$

$$\beta = \phi_y X + (\psi_y - \phi_x)Y + \chi_y Z - 3\psi,$$

$$\gamma = \phi_z X + \psi_z Y + (\chi_z - \phi_x) Z - 3\chi.$$

Hence the element corresponding to a general point X, Y, Z is given by the formulas

$$(33) \qquad dx : dy : dz = \Delta_1 : \Delta_2 : \Delta_3,$$

where

$$\Delta_1 = Y\gamma - Z\beta = \phi_z XY - \phi_y XZ + \psi_z Y^2 + (\chi_z - \psi_y) YZ - 3\chi Y - \chi_y Z^2 + 3\psi Z,$$

$$\Delta_2 = Z\alpha - X\gamma = -\phi_z X^2 - \psi_z XY + (\phi_z - \chi_z) XZ + 3\chi X + \psi_x YZ + \chi_x Z^2 + 3\phi Z,$$

$$\Delta_3 = X\beta - Y\alpha = \phi_y X^2 + (\psi_y - \phi_x) XY + \chi_y XZ - 3\psi X - \psi_x Y^2 - \chi_x YZ + 3\phi Y.$$

The point X, Y, Z will be singular if the equations (33) do not determine the element. This will be the case when

$$(34) \qquad \Delta_1 = \Delta_2 = \Delta_3 = 0.$$

From the identity

$$(35) \qquad X\Delta_1 + Y\Delta_2 + Z\Delta_3 \equiv 0,$$

which is readily verified, it is seen that (34) represents a cubic curve.

The singular points of the congruence Ω form a twisted cubic curve.

It will be convenient to have the equations of this curve Γ in parametric form. These may be found by expressing the fact that for a singular point the equations (32) differ only by a factor λ. Hence the required parametric equations are

$$(36) \qquad \begin{aligned} \lambda X + \psi_x Y + \chi_x Z &= 3\phi, \\ \phi_y X + (\psi_y - \phi_x + \lambda) Y + \chi_y Z &= 3\psi, \\ \phi_z X + \psi_z Y + (\chi_z - \phi_x + \lambda) Z &= 3\chi. \end{aligned}$$

The explicit expressions for X, Y, Z as cubic functions of the parameter λ will not be needed in the following discussion.

18. *Property* III_1. We now may prove that the congruence Ω is completely defined by the curve Γ, by showing that every straight line of Ω has two singular points, that is, is a secant of Γ.

Since any element may be made to take the direction of the axis of abscissas by revolving the axes, it will be sufficient to prove the statement for the single element $y' = 0$, $z' = 0$. The corresponding straight line, by (31), is

$$X = 0, \qquad \psi_x Y + \chi_x Z - 3\phi = 0.$$

It is easy to verify, by using either (34) or (36), that this line has two points in common with Γ.

PROPERTY III_1. *The straight lines which correspond, according to property* II, *to the ∞^2 elements at a given point, form a congruence which is composed of the secants of a twisted cubic curve.*

19. *Property* III_2. The preceding article associates with any given point a

definite cubic Γ. From the equations (34) it is seen that the curve passes through the given point, which is there taken as the origin of the X, Y, Z coördinates. In the parametric representation (36), the origin is given by the value $\lambda = \infty$. The direction of the curve at that point is found to be given by the ratios $\phi : \psi : \chi$; it is therefore the same as the direction of the force acting at the given point. This direction is defined geometrically in connection with property I. We may, therefore, state the new property as follows:

PROPERTY III$_2$. *The cubic curve* Γ, *associated by property* III$_1$ *with any point, passes through that point in the direction of the axis of the pencil of osculating planes described in property* I.

20. *Quadrics through* Γ. Consider the ∞^2 trajectories passing through a given point and having initial directions in a given plane. The lineal elements thus satisfy a linear relation

$$a\,dx + b\,dy + c\,dz = 0. \tag{37}$$

It follows, from (33), that the corresponding centres X, Y, Z satisfy the equation

$$a\Delta_1 + b\Delta_2 + c\Delta_3 = 0. \tag{38}$$

The locus of the centers of the osculating spheres of the ∞^2 *trajectories touching a given plane at a given point is a quadric surface.*

The quadrics (38) are in fact the linear system of quadrics passing through the cubic curve Γ. The result is really a corollary of the previous properties and may be derived synthetically. The straight line corresponding to a given element is obtained by constructing the plane perpendicular to that element at the given point; this plane cuts Γ in two new points; the line connecting these is the required line. If then we take the pencil of elements determined by a plane through the given point, the corresponding normal planes form a pencil; hence the corresponding straight lines form a regulus. This determines a quadric of the linear system (38).

21. *Interpretation of the curl.* To each plane (37) through the given point there corresponds a definite quadric (38). For which planes will this quadric be a rectangular hyperboloid?

The condition for such a quadric is that the sum of the coefficients of X^2, Y^2, Z^2 shall vanish. This gives

$$a(\psi_z - \chi_y) + b(\chi_x - \phi_z) + c(\phi_y - \psi_x) = 0. \tag{39}$$

This means that the plane (37) must contain the fixed direction defined by the ratios

$$\psi_z - \chi_y : \chi_x - \phi_z : \phi_y - \psi_x. \tag{40}$$

These three quantities are the components of the curl of the force ϕ, ψ, χ. Hence:

The direction of the curl of the force at any given point is defined geometrically by the fact that the quadrics corresponding, according to article 20, *to the planes containing this direction are rectangular hyperboloids. This is not the case for any other planes through the given point.*

22. *Conservative force.* The force is conservative when, and only when, the curl vanishes identically. In this case condition (39) holds for all values of a, b, c. Hence the quadric corresponding to any plane is a rectangular hyperboloid. The cubic curve Γ is then of a particular species which may also be termed rectangular. Its definitive property may be expressed in simple form: *the three asymptotic directions of the cubic are mutually orthogonal.*

Our result is thus a purely geometric characterization of the conservative case:

When, and only when, the force is conservative are the cubic curves Γ, *corresponding to all points of space in accordance with property* III_1, *of the rectangular species.*

23. *Converse of* III. Our next problem is to determine all curve systems possessing properties III_1 and III_2 in addition to I and II. It will be convenient to refer to the joint statement of III_1 and III_2 as PROPERTY III. The discussion will show incidentally that III is independent of I and II.

The most general system for which I and II hold was proved in article 14 to be represented by equations of the form (27), namely,

$$z'' = Ky'', \qquad y''' = Gy'' + Hy''^2. \tag{41}$$

The straight line corresponding to a given element is here determined by the equations*

$$X + y'Y + z'Z = 0, \qquad MY + NZ + 1 = 0, \tag{42}$$

where

$$M,\, N = \frac{G,\; KG + K'}{H(1 + y'^2 + z'^2) - 3(y' + Kz')}. \tag{42'}$$

The problem is to determine the form of the functions G and H, or, what is equivalent, of the functions M and N, so that property III shall also hold, that is, so that the congruence (42) shall be of the particular type described in III_1 and III_2.

Let the equations of the cubic associated with the given point, which we take

* In the dynamical case these reduce, of course, to equations (30).

as origin of the homogeneous cartesian coördinates X, Y, Z, T, be, in parametric form,

$$(43)\qquad \begin{aligned} X &= a_4\lambda^3 + a_0\lambda^2 + a_1\lambda + a_2, \\ Y &= b_4\lambda^3 + b_0\lambda^2 + b_1\lambda + b_2, \\ Z &= c_4\lambda^3 + c_0\lambda^2 + c_1\lambda + c_2, \\ T &= d_4\lambda^3 + d_0\lambda^2 + d_1\lambda + d_2. \end{aligned}$$

The curve is to pass through the origin by III_2. We may take the parameter so that $\lambda = \infty$ shall correspond to this point. Therefore

$$(44)\qquad a_4 = b_4 = c_4 = 0.$$

The directional property involved in III_2 requires further that

$$(45)\qquad a_0 : b_0 : c_0 = 1 : \omega_1 : \omega_2.$$

We may therefore take our cubic curve in the form

$$(46)\qquad \begin{aligned} X &= a_0\lambda^2 + a_1\lambda + a_2, \\ Y &= b_0\lambda^2 + b_1\lambda + b_2, \\ Z &= c_0\lambda^2 + c_1\lambda + c_2, \\ T &= \lambda^3 + d_0\lambda^2 + d_1\lambda + d_2. \end{aligned} \qquad (a_0 : b_0 : c_0 = 1 : \omega_1 : \omega_2).$$

The coefficients a, b, c, d may involve x, y, z in any way.

If we substitute these values in (42), we find

$$(47)\qquad L_0\lambda^2 + L_1\lambda + L_2 = 0,$$

where

$$(47')\quad L_0 = a_0 + y'b_0 + z'c_0, \quad L_1 = a_1 + y'b_1 + z'c_1, \quad L_2 = a_2 + y'b_2 + z'c_2;$$

and

$$(48)\quad \lambda^3 + (Mb_0 + Nc_0 + d_0)\lambda^2 + (Mb_1 + Nc_1 + d_1)\lambda + (Mb_2 + Nc_2 + d_2) = 0.$$

The condition that the line (42) shall intersect the cubic (46) in two points is that the equations (47), (48) shall have two roots λ in common. This gives two relations

$$(49)\qquad \begin{aligned} L_0(b_0L_2)M + L_0(c_0L_2)N &= L_1L_2 - L_0(d_0L_2), \\ L_0(b_1L_2)M + L_0(c_1L_2)N &= L_2^2 - L_0(d_1L_2), \end{aligned}$$

where the parentheses represent determinants

$$(49')\qquad (b_0L_2) = b_0L_2 - b_2L_0, \text{ etc.}$$

in terms of their principal diagonals.

The solution of (49) gives, after some reduction,

$$(50)\qquad \begin{aligned} M &= \frac{L_1(c_1L_2) - L_2(c_0L_2) - L_0(d_1c_2L_3)}{(abc)L_0}, \\ N &= \frac{L_2(b_0L_2) - L_1(b_1L_2) + L_0(d_1b_2L_3)}{(abc)L_0}. \end{aligned}$$

Arranging these with respect to y', z', we find that the functions M and N are necessarily of the form

$$(51)\qquad \begin{aligned} M &= \frac{m_1 + m_2y' + m_3z' + y'(l_1y' + l_2z')}{3(1 + \omega_1y' + \omega_2z')}, \\ N &= \frac{n_1 + n_2y' + n_3z' + z'(l_1y' + l_2z')}{3(1 + \omega_1y' + \omega_2z')}, \end{aligned}$$

where the coefficients l, m, n involve only x, y, z.

Conversely, any functions of this form lead to a system (41) with the required properties. The congruence (42) may, in fact, for these values of M and N, be put into the form

$$(52)\qquad \begin{aligned} X + y'Y + z'Z &= 0, \\ -(l_1y' + l_2z')X + (m_1 + m_2y' + m_3z')Y + (n_1 + n_2y' + n_3z')Z &= 3(1 + \omega_1y' + \omega_2z'). \end{aligned}$$

A discussion entirely analogous to that given in articles 17 and 18 shows that the congruence is defined by the cubic curve

$$(53)\qquad \frac{m_1Y + n_1Z + 3}{X} = \frac{m_2Y + n_2Z - l_1X + 3\omega_1}{Y} = \frac{m_3Y + n_3Z - l_2X + 3\omega_2}{Z}.$$

This passes through the origin in the direction $1 : \omega_1 : \omega_2$, hence property III holds in its entirety. Our result may be stated as follows:

The most general quintuply infinite system of curves with properties I, II *and* III *is represented by equations of the form*

$$(54)\qquad z'' = Ky'', \qquad y''' = Gy'' + Hy''^2,$$

where

$$(54')\qquad K = \frac{z' - \omega_2}{y' - \omega_1},$$

and G and H are defined by the relations

$$(54'')\qquad \begin{aligned} G : KG + K' : H(1 + y'^2 + z'^2) - 3(y' + Kz') = m_1 + m_2y' + m_3z' \\ + y'(l_1y' + l_2z') : n_1 + n_2y' + n_3z' + z'(l_1y' + l_2z') : 3(1 + \omega_1y' + \omega_2z'). \end{aligned}$$

The system thus involves ten arbitrary functions of x, y, z, namely,

$$(55)\qquad \omega_1,\ \omega_2,\ l_1,\ l_2,\ m_1,\ m_2,\ m_3,\ n_1,\ n_2,\ n_3.$$

RELATIONS BETWEEN THE TEN COEFFICIENTS.

24. The dynamical systems, defined by equations of the form (8) in connection with (8′) or (8″), involve only three arbitrary functions ϕ, ψ, χ and thus constitute a special case of the systems obtained in the preceding article. The question then arises, what is the special nature of the ten functions (55), which appear as coefficients in the equations (54″), if the system (54) is to be of the dynamical type?

Comparing the equations (52) with the corresponding equations (31), (31′) of the dynamical case, and introducing the functions ω_1, ω_2, Φ defined by (9), we find that

$$(56)\quad \begin{aligned} &l_1 = \Phi_y, && l_2 = \Phi_z, && \\ &m_1 = -\omega_1\Phi_x - \omega_{1x}, && m_2 = \Phi_x - \omega_{1y} - \omega_1\Phi_y, && m_3 = -\omega_{1z} - \omega_1\Phi_z, \\ &n_1 = -\omega_2\Phi_x - \omega_{2x}, && n_2 = -\omega_{2y} - \omega_2\Phi_y, && n_3 = \Phi_x - \omega_{2z} - \omega_2\Phi_z. \end{aligned}$$

In order that the ten functions (55) shall belong to a dynamical system it is necessary and sufficient that they shall be expressible in terms of three functions ω_1, ω_2, Φ according to (56).

25. The explicit conditions may be obtained by eliminating Φ from (56). The algebraic elimination of Φ_x, Φ_y, Φ_z yields five relations, namely:

$$(57)\quad \begin{gathered} m_3 + \omega_{1z} + \omega_1 l_2 = 0, \qquad n_2 + \omega_{2y} + \omega_2 l_1 = 0, \\ m_2 + \omega_{1y} + \omega_1 l_1 = n_3 + \omega_{2z} + \omega_2 l_2 = -\frac{m_1 + \omega_{1x}}{\omega_1} = -\frac{n_1 + \omega_{2x}}{\omega_2}. \end{gathered}$$

Furthermore, from (56) we obtain the equations

$$(58)\qquad \Phi_x = -\frac{m_1 + \omega_{1x}}{\omega_1} = -\frac{n_1 + \omega_{2x}}{\omega_2}, \qquad \Phi_y = l_1, \qquad \Phi_z = l_2,$$

for which the conditions of integrability may be written

$$(59)\quad l_{1z} - l_{2y} = 0, \qquad l_{2x} + \left(\frac{n_1 + \omega_{2x}}{\omega_2}\right)_z = 0, \qquad l_{1x} + \left(\frac{m_1 + \omega_{1x}}{\omega_1}\right)_y = 0.$$

26. We have thus obtained eight necessary relations (57), (59). We now prove that these are also sufficient. Suppose that any ten functions satisfying these relations are given. Then, on account of (59), the equations (58) will be consistent; hence a function Φ can be determined.* It is then easily verified that, on account of the relations (57), the equations (56) hold.

* The integration constant involved enters additively in Φ. Hence, from (9), the force ϕ, ψ, χ is determined except for a constant factor. This agrees with the result stated in article 3.

In order that a system of curves with properties I, II, *and* III, *that is, a system defined by equations of the form* (54), (54′), (54″), *shall be of the dynamical type, it is necessary and sufficient that the ten functions* (55) *shall satisfy the relations* (57) *and* (59).

27. *Preliminary interpretation.* It remains to interpret the relations (57), (59) geometrically. For any system of curves with properties I, II, III there is associated with each point x, y, z a definite cubic curve Γ, whose equations are given by (53). The association of point and corresponding cubic is fixed by the functions (55). Any relation between the functions is thus equivalent to a restriction on this association. It would in this way be possible to derive a direct geometrical interpretation of the relations (57), (59) which express the character of the association in the dynamical case. However, the result thus obtained is quite complicated and we shall not state it. A simple geometric interpretation will be obtained indirectly in the subsequent discussion of associated S-systems.

28. We derive first a property which is equivalent to two of the eight relations (57). According to article 13 there corresponds to any element (y', z') at a given point a definite straight line. Consider now the particular element (ω_1, ω_2) whose direction is that of the force acting at the point. By taking the axis of abscissas in this direction, we find that the corresponding line is represented by

$$X = 0, \qquad \omega_{1x} Y + \omega_{2x} Z = 3. \tag{60}$$

The perpendicular distance from the given point, here taken as the origin of the X, Y, Z system, to this line is

$$\frac{3}{\sqrt{\omega_{1x}^2 + \omega_{2x}^2}}.$$

On the other hand the radius of curvature of the line of force (10) passing through the given point is, under the assumed conditions,

$$\frac{1}{\sqrt{\omega_{1x}^2 + \omega_{2x}^2}};$$

that is, just one third of the perpendicular distance.

The direction of the straight line (60) is also simply related to the line of force. The osculating plane of the latter is

$$\omega_{2x} Y - \omega_{1x} Z = 0.$$

It is thus perpendicular to the line (60). The results may be stated as follows:

The straight line which corresponds, in accordance with property II, *to a lineal element belonging to a line of force is parallel to the binormal of the* *line of force; and its perpendicular distance from the given point is three times the radius of curvature of the line of force.*

This may be stated in purely geometric form by replacing the direction of the force by that of the axis of the pencil considered in property I. It is found that the results hold for all systems in which the functions (55) satisfy the conditions

$$\{m_1 + m_2\omega_1 + m_3\omega_2 + \omega_1(l_1\omega_1 + l_2\omega_2)\}^2 + \{n_1 + n_2\omega_1 + n_3\omega_2 + \omega_2(l_1\omega_1 + l_2\omega_2)\}^2$$

$$(61)\qquad = (\omega_{1x} + \omega_1\omega_{1y} + \omega_2\omega_{1z})^2 + (\omega_{2x} + \omega_1\omega_{2y} + \omega_2\omega_{2z})^2 - \frac{1}{1+\omega_1^2+\omega_2^2}\begin{vmatrix} \omega_1 & \omega_{1x} + \omega_1\omega_{1y} + \omega_2\omega_{1z} \\ \omega_2 & \omega_{2x} + \omega_1\omega_{2y} + \omega_2\omega_{2z} \end{vmatrix},$$

$$(62)\qquad \begin{vmatrix} \omega_{1x} + \omega_1\omega_{1y} + \omega_2\omega_{1z} & m_1 + m_2\omega_1 + m_3\omega_2 + \omega_1(l_1\omega_1 + l_2\omega_2) \\ \omega_{1x} + \omega_1\omega_{2y} + \omega_2\omega_{2z} & n_1 + n_2\omega_1 + n_3\omega_2 + \omega_2(l_1\omega_1 + l_2\omega_2) \end{vmatrix} = 0.$$

These are, in fact, consequences of the relations (57), though of course the converse is not true. The property stated in this article thus holds not only for dynamical but for more general systems. Its addition to I, II, and III will therefore not yield a complete characterization of the dynamical type.

THE ASSOCIATED PLANE SYSTEMS S.

29. *Definition.* To reach the desired characterization we introduce certain plane systems of curves which may be associated with any quintuply infinite system of space curves.

Consider any plane π. Through each point of π there pass ∞^2 curves of the given system which are tangent to π. Project the differential elements of the third order belonging to these space curves orthogonally upon π, thus obtaining ∞^2 plane differential elements of the third order at the selected point. Applying this process to all points of π, we have a definite system of ∞^4 differential elements of the third order. These elements define a certain differential equation of the third order and thus determine ∞^3 integral curves. These form what we shall term the *associated S-system* in the plane π.

The system S is easily determined analytically. Let the given plane π be taken as the xy plane, and let the equations of the arbitrary quintuply infinite system be

$$(63)\qquad \begin{aligned} z'' &= g(x, y, z, y', z', y''), \\ y''' &= f(x, y, z, y', z', y''). \end{aligned}$$

Then for the lineal elements contained in π we have

$$z = 0, \qquad z' = 0.$$

Hence the differential equation of S is

$$(64) \qquad y''' = f(x, y, 0, y', 0, y'') = f_0(x, y, y', y'').$$

There will be in all ∞^3 of these triply infinite systems S, namely, one for each plane of space. Any geometric property of these systems is at the same time a property of the original space system; for the former are determined geometrically by the latter.

30. We apply this notion to the dynamical systems and obtain:

PROPERTY IV. *The plane systems S associated with any quintuply infinite system of dynamical trajectories are of the (two-dimensional) dynamical type.*

To prove this it will be sufficient to consider the S-system associated with the xy plane. Applying the method given in the preceding article to the equations (8), or, as is equivalent, to the equations (5) and (6), we find that the system is defined by

$$(65) \qquad (\bar{\psi} - y'\bar{\phi})y''' = \begin{vmatrix} 1 & \bar{\phi}_x + y'\bar{\phi}_y \\ y' & \bar{\psi}_x + y'\bar{\psi}_y \end{vmatrix} - 3\bar{\phi}y''^2,$$

where $\bar{\phi}$, $\bar{\psi}$ are functions of x, y obtained from ϕ, ψ by substituting 0 for z This is precisely the equation of the trajectories generated by the plane field of force whose components are $\bar{\phi}$, $\bar{\psi}$.* The force acting in the plane is seen to be derivable by orthogonal projection from the given force (1).

31. The geometrical properties of plane dynamical systems are given in the paper already cited. We shall number the properties as in that paper, but distinguish them from the spatial properties already obtained by attaching a subscript p. As the characteristic set we may take †

$$(67) \qquad \mathrm{I}_p,\ \mathrm{II}_p,\ \mathrm{III}_p,\ \mathrm{V}_p,\ \mathrm{VI}_p.$$

Property IV, given in article 30, may thus be restated in purely geometric form as follows:

PROPERTY IV. *The systems S associated with any quintuply infinite system of dynamical trajectories possess the plane properties* I_p, II_p, III_p, V_p, VI_p.

It thus may be broken up into five distinct statements, of which the first, for example, would be: Every S-system is such that the osculating parabolas (constructed at the common point) of the ∞^1 curves passing through a given point in a given direction have their foci located on a circle passing through the given point.

* Cf. *Trajectories*, p. 403, formula (5).

† Loc. cit., p. 417.

32. *Converse of* I_p. The object of the remaining discussion is to show that the addition of property IV to properties I, II, III gives a set which completely characterizes the dynamical type in space. We must therefore examine the effect of adding successively the properties (67) to I, II, III. The first result is this:

If any quintuply infinite system of space curves possesses properties I, II, *then the associated S-systems necessarily possess property* I_p.

This is proved by noting that the condition for I_p is that the differential equation of the plane system shall be of the form *

$$y''' = G(x, y, y')y'' + \bar{H}(x, y, y',)y''^2. \tag{68}$$

That the S-system is of this type is seen from the equations (27), which represent the most general system with properties I and II.

It is thus shown that I_p is redundant, in the sense that it is a consequence of preceding spatial properties.

33. *Converse of* II_p, III_p. In order that the system (68) shall have properties II_p, III_p, it is necessary that the $\bar{G}$ and $\bar{H}$ shall have the special forms †

$$G = \frac{\lambda y'^2 + \mu y' + \nu}{y' - \omega}, \qquad \bar{H} = \frac{3}{y' - \omega}, \tag{69}$$

where λ, μ, ν, ω involve x, y only.

We now determine under what conditions a system with properties I, II, and III, that is, a system whose differential equations are of the type defined by (54), (54′), (54″), will have associated S-systems possessing properties II_p, III_p. This is to be the case, of course, for the S-system in every plane.

Consider first the planes parallel to the xy plane. The systems S are then found by substituting 0 for z' and an arbitrary constant for z. By making use of forms (69) in connection with (54″), we find

$$\begin{aligned} m_1 + m_2 y' + l_1 y'^2 &= \lambda y'^2 + \mu y' + \nu, \\ n_1 + n_2 y' &= -\frac{\omega_2(\lambda y'^2 + \mu y' + \nu) + (y' - \omega_1)\omega_2' + \omega_2\omega_1'}{y' - \omega_1}. \end{aligned} \tag{70}$$

The elimination of λ, μ, ν then gives

$$\begin{vmatrix} y' - \omega_1 & m_1 + \omega_{1x} + (m_2 + \omega_{1y})y' + l_1 y'^2 \\ -\omega_2 & n_1 + \omega_{2x} + (n_2 + \omega_{2y})y' \end{vmatrix} = 0.$$

* *Trajectories*, article 13.

† Loc. cit., articles 14, 15.

This is to be satisfied identically; hence

$$
\begin{gathered}
n_2 + \omega_{2y} + l_1\omega_2 = 0, \\
\omega_1(n_1 + \omega_{2x}) - \omega_2(m_1 + \omega_{1x}) = 0, \\
n_1 + \omega_{2x} - \omega_1(n_2 + \omega_{2y}) + \omega_2(m_2 + \omega_{1y}) = 0.
\end{gathered} \tag{71}
$$

An analogous discussion for the systems S in the planes parallel to the xz plane yields the relations

$$
\begin{gathered}
m_3 + \omega_{1z} + l_2\omega_1 = 0, \\
\omega_1(n_1 + \omega_{2x}) - \omega_2(m_1 + \omega_{1x}) = 0, \\
m_1 + \omega_{1x} + \omega_1(n_3 + \omega_{2z}) - \omega_2(m_3 + \omega_{1z}) = 0.
\end{gathered} \tag{72}
$$

No new relations are found by considering other planes. The relations (71), (72) are seen to reduce to five and to be equivalent to the relations (57).

The only systems of curves with properties I, II, III, I_p, II_p, III_p *are those systems of type* (54), (54′), (54″) *in which the functions* (55) *satisfy the relations* (57).

34. It has been proved incidentally that when II_p, III_p hold for the systems S in two series of parallel planes, say the planes parallel to two of the coördinate planes, they then necessarily hold for all planes of space. Of course it is assumed here, as in the following, that I, II, III hold for the original space system.

35. *Converse of* V_p. It will now be shown that property V_p imposes no additional restrictions. The condition that a plane system defined by the forms (69) shall have this property is*

$$\lambda\omega^2 + \mu\omega + \nu + \omega_x + \omega\omega_y = 0. \tag{73}$$

Applying this to the S-systems in the planes parallel to the xy plane, for which, according to (70),

$$\lambda = l_1, \qquad \mu = m_2, \qquad \nu = m_1, \qquad \omega = \omega_1, \tag{74}$$

we find

$$l_1\omega_1^2 + m_2\omega_1 + m_1 + \omega_{1x} + \omega_1\omega_{1y} = 0. \tag{75}$$

This, however, is seen to be a consequence of the relations (57).

Property V_p *is a consequence of* II_p, III_p *in connection with* I, II, III.

36. *Converse of* VI_p. It will now be shown that property VI_p does impose additional restrictions and that these are precisely equivalent to the relations (59) obtained in article (25).

* *Trajectories*, article 22.

The condition for VI_p is *

$$\lambda_x + \left(\frac{\nu + \omega_x}{\omega}\right)_y = 0. \tag{76}$$

We begin by applying this to the S-systems in the planes parallel to the xy plane. In this case we may use the values given in (74); hence

$$l_{1x} + \left(\frac{m_1 + \omega_{1x}}{\omega_1}\right)_y = 0;$$

which is the third of the equations (59). The corresponding condition for the planes parallel to the xz plane is obtained by replacing y by z, ω_1 by ω_2, l_1 by l_2 and m_1 by n_1. This gives the second of the equations (59).

The discussion for the S-systems in planes parallel to the yz plane is not so simple; for here we can no longer employ x as the independent variable. It is necessary to transform the second of the equations (54) so that say y shall be the independent variable. The result is

$$\frac{d^3z}{dy^3} = \frac{\lambda\left(\dfrac{dz}{dy}\right)^2 + \mu\dfrac{dz}{dy} + \nu}{\dfrac{dz}{dy} - \omega}\,\frac{d^2z}{dy^2} + \frac{3}{\dfrac{dz}{dy} - \omega}\left(\frac{d^2z}{dy^2}\right)^2, \tag{77}$$

where

$$\lambda = -\frac{m_3}{\omega_1}, \qquad \mu = \frac{n_3 - m_2 - m_1\dfrac{dx}{dy}}{\omega_1}, \qquad \nu = \frac{n_2 + n_1\dfrac{dx}{dy}}{\omega_1}, \qquad \omega = \frac{\omega_2}{\omega_1}. \tag{77'}$$

For the S-systems considered, x is an arbitrary constant and $dx/dy = 0$. Thus the condition (76) gives

$$\frac{\partial}{\partial y}\left(-\frac{m_3}{\omega_1}\right) + \frac{\partial}{\partial z}\left(\frac{\dfrac{n_2}{\omega_1} + \left(\dfrac{\omega_2}{\omega_1}\right)_y}{\dfrac{\omega_2}{\omega_1}}\right) = 0.$$

By using the relations (57), it may be shown that this takes the simple form

$$l_{1z} - l_{2y} = 0,$$

which is the first of the relations (59).

We may avoid the calculations that would be introduced by the direct consideration of other planes as follows. It has been shown that when the S-systems in planes parallel to the coördinate planes possess property VI_p, then relations (59) are necessarily fulfilled. The relations (57) are already fulfilled on account of properties II_p, III_p. From the theorem of article 25 it then follows that the

* *Trajectories*, p. 415.

original system of space curves is of dynamical type. Hence, by article 30, the systems S for all the planes of space are of dynamical type and hence possess VI_p.

In order that a system with properties I, II, III, II_p, III_p *shall also possess property* VI_p *it is necessary and sufficient that the relations* (59) *shall be fulfilled.*

37. It has been shown incidentally that when VI_p holds for the S-systems constructed in a triply orthogonal series of planes, it then holds for all S-systems.

38. *Restatement of* IV. It will be recalled that the plane systems S were obtained, not by the orthogonal projection of the space curves, but by the orthogonal projection of the differential elements of the third order belonging to these curves. If we consider the ∞^4 space curves of the original quintuply infinite system which touch a given plane π, their orthogonal projections upon that plane would give rise to a system of ∞^4 plane curves. The relation between these curves and those belonging to the system S in the same plane is as follows: the projections of the ∞^2 space curves touching π at a common point O have, at that point, contact of the third order with the ∞^2 S-curves passing through O.

Properties I_p, II_p, III_p have reference to the curves passing through a given point; properties V_p, VI_p relate to certain variations produced by changing that point. Neither set involves differential elements of higher than the third order. Hence it is possible to replace property IV by the following equivalent:

If the ∞^2 trajectories touching a given plane π at a common point O are projected orthogonally upon π, the plane curves thus obtained possess the properties I_p, II_p, III_p *and hence also the derived property* IV_p *as stated in the two-dimensional theory.* When the point O varies in the plane π, the direction associated with it by* II_p *and the conic associated with it by* IV_p *vary in accordance with the restrictions expressed in* V_p *and* VI_p.*

Complete characterization.

39. The main result of our entire discussion may be stated as follows:

In order that a system of ∞^5 space curves, of which ∞^1 pass through each point in each direction, shall be identifiable with the system of trajectories due to a (positional) field of force

$$\frac{d^2x}{dt^2} = \phi(x, y, z), \qquad \frac{d^2y}{dt^2} = \psi(x, y, z), \qquad \frac{d^2z}{dt^2} = \chi(x, y, z),$$

it is necessary and sufficient that it shall have the following purely geometric properties:

I. *The osculating planes of the ∞^2 curves passing through a given point form a pencil; that is, all the planes pass through a fixed direction.*

* See *Trajectories*

II. *The osculating spheres of the* ∞^1 *curves passing through a given point in a given direction form a pencil; their centers thus lie on a straight line.*

III. *The straight lines which correspond, in accordance with* II, *to all the directions at a given point, form a congruence, of order one and class three, consisting of the secants of a twisted cubic curve; which cubic passes through the given point in the direction fixed by property* I.

IV. *The associated systems* S, *constructed as in article* 29, *have the characteristic properties of plane dynamical systems.*

These four properties are independent in the sense that no one can be derived from those which precede it.* The last one, IV, may be replaced by weaker requirements: of the component parts it is sufficient to retain II_p, III_p, VI_p, and of all S-systems it is sufficient to consider those constructed in a triply orthogonal series of planes.

40. From article 22 it is seen that a complete geometric characterization of the trajectories produced by a *conservative force* is obtained by adding to III in the preceding set the requirement that, for every point of space, the corresponding cubic curve shall be of the rectangular species.

COLUMBIA UNIVERSITY, NEW YORK.

* The question of absolute independence is left open. It may be that the other properties are consequences of IV.