PROJECTIVE DIFFERENTIAL GEOMETRY OF CURVED SURFACES*

(FIRST MEMOIR)

BY

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§ 1. *The simultaneous solutions of two linear homogeneous partial differential equations of the second order with two independent variables.*

Consider a system of partial differential equations of the form

$$(1)\qquad \begin{aligned} Ay_{uu} + By_{uv} + Cy_{vv} + Dy_u + Ey_v + Fy = 0, \\ A'y_{uu} + B'y_{uv} + C'y_{vv} + D'y_u + E'y_v + F'y = 0, \end{aligned}$$

where

$$(2)\qquad y_u = \frac{\partial y}{\partial u}, \qquad y_v = \frac{\partial y}{\partial v}, \qquad y_{uu} = \frac{\partial^2 y}{\partial u^2}, \quad \text{etc.},$$

and where $A, B, \cdots, F'$ are analytic functions of u and v.

If the determinant $AC' - A'C$ does not vanish identically, (1) may be written in the form

$$(3)\qquad \begin{aligned} y_{uu} = ay_{uv} + by_u + cy_v + dy, \\ y_{vv} = a'y_{uv} + b'y_u + c'y_v + d'y. \end{aligned}$$

If the determinant $AC' - A'C$ is equal to zero, while $BC' - B'C$ does not vanish, we may write (1) in the form

$$(4)\qquad \begin{aligned} y_{uu} = \alpha y_{vv} + \beta y_u + \gamma y_v + \delta y, \\ y_{uv} = \beta' y_u + \gamma' y_v + \delta' y, \end{aligned}$$

provided that A and A' do not vanish simultaneously. If they do, i. e., if $A = A' = 0$, while $BC' - B'C \neq 0$, we obtain from (1) a system of the same form as (4) except for an interchange of u and v, with the further specialization that $\alpha = 0$. Finally, if

$$AC' - A'C = BC' - B'C = 0,$$

one of the equations deducible from (1) reduces to the first order.

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Transform (4) by putting

$$u = r, \qquad v = r + s,$$

and let

$$y_r = \frac{\partial y}{\partial r}, \qquad y_s = \frac{\partial y}{\partial s}, \quad \text{etc.}$$

The system (4) becomes

$$y_{rr} - 2y_{rs} + y_{ss} = \alpha y_{ss} + \beta(y_r - y_s) + \gamma y_r + \delta y,$$
$$y_{rs} - y_{ss} = \beta'(y_r - y_s) + \gamma' y_s + \delta' y,$$

which equations may be solved for y_{rr} and y_{ss}. By such a transformation of the independent variables (4) may therefore be reduced to the form (3).

Unless, therefore, there exists a linear combination of the two equations (1) *which is of the first order only, system* (1) *may be reduced to the form* (3), *a reduction which may involve a linear transformation of the independent variables.*

Consider a system of form (3); y_{uu} and y_{vv} are expressed in terms of y, y_u, y_v, y_{uv}. Differentiation of the first of the two equations with respect to u will express y_{uuu} in terms of the same four quantities and of y_{uuv}. Similarly y_{vvv} will depend upon y_{uvv} as well as upon y, y_u, y_v, y_{uv}. In general, however, y_{uuv} and y_{uvv} may themselves be expressed in terms of y, y_u, y_v, y_{uv}. In fact differentiation of the first equation of (3) with respect to v, and of the second with respect to u will give

$$\begin{aligned} y_{uuv} - a y_{uvv} &= (a_v + b) y_{uv} + c y_{vv} + b_v y_u + (c_v + d) y_v + d_v y, \\ -a' y_{uuv} + y_{uvv} &= (a'_u + c') y_{uv} + b' y_{uu} + (b'_u + d') y_u + c'_u y_v + d'_u y, \end{aligned} \tag{5}$$

whence follows the truth of the above statement provided that

$$aa' - 1 \neq 0 \tag{6}$$

In this case, clearly all higher derivatives of y may be expressed in the form

$$\alpha y + \beta y_u + \gamma y_v + \delta y_{uv}, \tag{7}$$

where α, β, γ, δ are, for every such derivative, perfectly definite functions of u and v. It is to be noticed, however, that of the five derivatives of the fourth order, three may be computed in two different ways; for example

$$y_{uuuv} = \frac{\partial}{\partial u}(y_{uuv}) = \frac{\partial}{\partial v}(y_{uuu}.)$$

Therefore, in order that the eight expressions of form (7) for the five fourth derivatives may be consistent, three conditions must be satisfied. It is not difficult to see that, if these three so-called *integrability conditions* are satisfied, the expression of form (7) for a derivative of any higher order will be unique.

Let y be a solution of system (3), developable in powers of $u - u_0$ and $v - v_0$, and denote by $c_1, \cdots, c_4$ the values of y, y_u, y_v, y_{uv} for $u = u_0$, $v = v_0$. Then y may be expressed in the form

$$y = c_1 + c_2(u - u_0) + c_3(v - v_0) + c_4(u - u_0)(v - v_0)$$
$$+ \tfrac{1}{2}(d_0 c_1 + c_0 c_2 + b_0 c_3 + a_0 c_4)(u - u_0)^2 + \tfrac{1}{2}(d_0' c_1 + c_0' c_2 + b_0' c_3 + a_0' c_4)(v - v_0)^2 + \cdots,$$

where a_0, b_0, etc. are the values of $a, b, c, \cdots$ for $u = u_0$, $v = v_0$. All of the coefficients of this development will be linear combinations of $c_1, \cdots, c_4$. We may write, therefore,

$$y = c_1 y' + c_2 y'' + c_3 y^{(3)} + c_4 y^{(4)},$$

where $y', \cdots, y^{(4)}$ are holomorphic functions of $u - u_0$ and $v - v_0$. The most general analytic solution of (3) can, therefore, contain only four arbitrary constants which enter linearly.

The integrability conditions are of the form

$$\alpha y_{uv} + \beta y_u + \gamma y_v + \delta y = 0. \tag{8}$$

They may be satisfied *identically*, so that

$$\alpha = \beta = \gamma = \delta = 0.$$

In that case the solution just indicated in a formal manner actually contains four arbitrary constants. If the integrability conditions are not satisfied identically, they represent additional differential equations which y must verify. The number of arbitrary constants in the general solution of (3) is reduced at least by one for every independent non-identically satisfied integrability condition. The above condition (8), for example, requires that

$$\alpha_0 c_4 + \beta_0 c_2 + \gamma_0 c_3 + \delta_0 c_1$$

shall be equal to zero, where $\alpha_0 \cdots \delta_0$ are the values of $\alpha \cdots \delta$ for $u = u_0$, $v = v_0$. This clearly reduces the number of arbitrary constants to three.

If the integrability conditions are satisfied identically, the most general analytic solution of (3) can depend, therefore, upon only four arbitrary constants which enter linearly. Such an analytic solution actually exists if the coefficients of (3) are holomorphic in the vicinity of $u = u_0$, $v = v_0$. This may be proved by the method which, since the time of CAUCHY, has always been employed for such investigations.* Of course, it may seem desirable to prove the existence of just a four-fold infinity of solutions for a system of form (3), for which $aa' - 1$ does not vanish and whose integrability conditions are satisfied identically under more general assumptions. For our present purpose, it suffices to restrict our considerations to systems of form (3) with analytic coefficients.

* Cf. for example GOURSAT, *Leçons sur l'intégration des équations aux dérivées partielles du premier ordre*, Paris, 1891, chapter I.

We proceed to consider the case

$$aa' - 1 = 0. \tag{9}$$

If we multiply both members of the second equation of (5) by a and add to the members of the first, taking account moreover of (3), we find

$$\begin{aligned}[t] &[a_v + b + a(a'_u + c') + a^2 b' + ca']y_{uv} + [b_v + a(b'_u + d') + abb' + cb']y_u \\ &\quad + [c_v + d + ac'_u + acb' + cc']y_v + [d_v + ad'_u + adb' + cd']y = 0.\end{aligned} \tag{10}$$

Unless all of the coefficients of this equation are zero, it becomes possible, by means of it, to reduce every expression of the form $\alpha y + \beta y_u + \gamma y_v + \delta y_{uv}$ to one involving only three terms, so that the general analytic solution of (3) could depend upon no more than three arbitrary constants. If, on the other hand, all of the coefficients of (10) are equal to zero, one of the partial derivatives of each order will remain arbitrary, and the general solution will contain an infinite number of arbitrary constants. This solution actually exists. The two equations are then said to be in *involution*;* the system is *involutory*.

We may recapitulate as follows. 1. *The system* (3) *of simultaneous linear differential equations has precisely four linearly independent solutions if* $aa' - 1$ *is different from zero and if the integrability conditions are identically satisfied.*

2. *If* $aa' - 1$ *vanishes, and the integrability conditions are satisfied, while not all of the coefficients of* (10) *are zero, the system* (3) *has less than four linearly independent solutions.*

3. *If, however, the first two conditions of No.* 2 *are satisfied, and if besides all of the coefficients of* (10) *are equal to zero, the general solution of system* (3) *contains an infinite number of arbitrary constants. The system is an involutory one.*

§ 2. *Geometric interpretation. The integrating surface.*

If $aa' - 1$ is different from zero and if all of the integrability conditions are satisfied identically, system (3) has just four linearly independent solutions. Denote four such solutions by $y', y'', y^{(3)}, y^{(4)}$. We shall have

$$y^{(k)} = f^{(k)}(u, v) \qquad (k = 1, 2, 3, 4).$$

Interpret $y^{(k)}$ as the homogeneous coördinates of a point P_y in space. As u and v assume all of their values, P_y will describe a surface S_y an *integrating surface* of the system. *This surface cannot degenerate into a curve.* For if

*Cf. GOURSAT, *Leçons sur l'intégration des équations aux dérivées partielles du second ordre à deux variables indépendantes*, Paris, 1896–98, vol. 2, chap. 6, where a more general case is treated according to BIANCHI.

S_y were to degenerate in that way, the ratios of y', y'', $y^{(3)}$, $y^{(4)}$, would become functions of a single variable $t = \phi(u, v)$. In that case $y', \cdots, y^{(4)}$ would be solutions of some linear partial differential equation of the first order, say

$$\alpha y_u + \beta y_v + \gamma y = 0, \tag{11}$$

besides satisfying (3). As a matter of fact let

$$\alpha w_u + \beta w_v = 0$$

be the partial differential equation satisfied by $t = \phi(u, v)$. Then if

$$\frac{y'}{y^{(4)}}, \quad \frac{y''}{y^{(4)}}, \quad \frac{y^{(3)}}{y^{(4)}},$$

were functions of t alone, they would satisfy the same equation, so that

$$\alpha(y_u^{(k)} y^{(4)} - y^{(k)} y_u^{(4)}) + \beta(y_v^{(k)} y^{(4)} - y^{(k)} y_v^{(4)}) = 0 \qquad (k = 1, 2, 3),$$

or

$$\alpha y_u^{(k)} + \beta y_v^{(k)} - \frac{\alpha y_u^{(4)} + \beta y_v^{(4)}}{y^{(4)}} y^{(k)} = 0 \qquad (k = 1, 2, 3).$$

Put

$$-\frac{\alpha y_u^{(4)} + \beta y_v^{(4)}}{y^{(4)}} = \gamma,$$

and it becomes clear that y', y'', $y^{(3)}$, $y^{(4)}$ would satisfy the same equation of form (11). Differentiate (11) with respect to u and v. If the two equations thus obtained coincide with the two equations of (3), $aa' - 1$ would be equal to zero, contrary to our assumption. (11) would be an intermediary integral of (3) whose general solution would depend upon an arbitrary function. If, on the other hand, the two equations obtained from (11) by differentiation were not identical with the two equations of (3), the most general function of u and v which satisfies all of these equations cannot depend upon more than two linearly independent functions. It is impossible, therefore, that the four linearly independent functions $y', \cdots, y^{(4)}$ should satisfy an equation of form (11), i. e., it is impossible that the integrating surface S_y should degenerate into a curve.

The integrating surface of system (3) is not unique. However, the most general system of linearly independent solutions of (3), in the case considered, is of the form

$$\eta^{(i)} = \sum_{k=1}^{4} c_{ik} y^{(k)} \qquad (i = 1, 2, 3, 4),$$

where the determinant of the constant coefficients,

$$|c_{ik}|,$$

does not vanish. *The most general integrating surface of* (3) *is, therefore, a projective transformation of any particular one.* It is for this reason that we shall make use of (3) as a basis for the projective theory of surfaces.

If $aa' - 1$ vanishes, while not all of the conditions for an involutory system are satisfied, the system (3) has only three or fewer linearly independent solutions. Its integrating surface, therefore, degenerates into a plane or a straight line. For the purposes of the theory of surfaces this case may, therefore, be left aside.

It remains to examine the case when the equations of (3) are in involution. That case must also be excluded from our considerations. For we wish to construct a *projective* theory of surfaces; but, if the equations (3) are in involution, they have more than four linearly independent solutions in common, so that the most general integrating surface of (3) will not be merely a projective transformation of any particular one. The question which requires investigation is this: what kind of surfaces are thus excluded from consideration?

If we put in (3)

$$\bar{y} = \lambda y$$

where λ is an arbitrary function of u and v, and if we introduce new independent variables by putting

$$\bar{u} = \mu(u, v), \qquad \bar{v} = \nu(u, v),$$

another system of differential equations of form (1) will be obtained from (3) with $\bar{y}$, $\bar{u}$, $\bar{v}$, in place of y, u, v. Moreover, it is clear that, if the equations (3) are in involution, the same must be true of the transformed equations and conversely. For if they were not, they could have only four linearly independent solutions and the same would have to be true of the original system of equations.

Let equations (3) be in involution. They have an infinite number of linearly independent solutions. Let $y', \cdots, y^{(4)}$ be four of these. Transform system (3) by putting $\bar{y} = \lambda y$ where $\lambda = 1/y^{(4)}$. The new system of differential equations is again an involutory system and has the four linearly independent solutions

$$1, \qquad x = \frac{y'}{y^{(4)}}, \qquad y = \frac{y''}{y^{(4)}}, \qquad z = \frac{y^{(3)}}{y^{(4)}},$$

where x, y and z may be considered as non-homogeneous coördinates of the point of the integral surface whose homogeneous coördinates were y', y'', $y^{(3)}$, $y^{(4)}$. Moreover x, y, z are functions of u and v. Introduce x and y as independent variables. This is possible unless x should happen to be a function of y alone. In that case we might introduce x and z as independent variables. For otherwise x, y and z would be functions of a single variable $t = \phi(u, v)$, so that $y', \cdots, y^{(4)}$, as we have seen above, would satisfy one and the same linear partial differential equation of the first order. In that case there would be no integral surface, but an integral curve; for our purpose that case must be excluded. Accordingly two of the three functions x, y, z of u and v will be independent, say x and y. We may then introduce x and y as independent variables in place of u and v. The transformed system of equations will be again an involutory system with the independent variables x and y, and of which $1, x, y$ and $z = f(x, y)$ are four linearly independent solutions. This system of equations will certainly be of form (1), and in most cases may at once be written in the form (3) by solving for θ_{xx} and θ_{yy}, denoting the dependent variable by θ. The system is then of the form

$$\theta_{xx} = a\theta_{xy} + b\theta_x + c\theta_y + d\theta,$$

$$\theta_{yy} = a'\theta_{xy} + b'\theta_x + c'\theta_y + d'\theta,$$

and it must be satisfied by $\theta = 1, x, y$ and $z = f(x, y)$, so that

$$b = c = d = b' = c' = d' = 0,$$

while $z = f(x, y)$ must satisfy the equations

$$\frac{\partial^2 z}{\partial x^2} = a\frac{\partial^2 z}{\partial x\, \partial y}, \qquad \frac{\partial^2 z}{\partial y^2} = a'\frac{\partial^2 z}{\partial x\, \partial y}, \tag{12}$$

which must, moreover, be in involution.

If, however, the system of form (1), which is obtained by the above process, cannot be solved for θ_{xx} and θ_{yy}, then as was shown in § 1, by a simple linear transformation of the independent variables x and y we may transform it into a system of form (3). This latter system must then have the solutions

$$1, \qquad \alpha x + \beta y, \qquad \gamma x + \delta y, \qquad z, \qquad \alpha\delta - \beta\gamma \neq 0,$$

where $\alpha, \beta, \gamma, \delta$ are constants, whence we conclude in the same way as before that z must satisfy an involutory system of form (12). But the equations (12) are in involution if, and only if,

$$aa' = 1, \qquad \frac{\partial a}{\partial y} + a\frac{\partial a}{\partial x} = 0, \tag{13}$$

so that

$$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x\, \partial y}\right)^2 = 0. \tag{14}$$

If x, y, z are cartesian coördinates it is well known that (14) is the condition for developable surfaces. But this condition is left invariant under projective transformations. Therefore, *the surfaces excluded from consideration are developables.* If the condition

$$\frac{\partial a}{\partial y} + a\frac{\partial a'}{\partial x} = 0 \tag{15}$$

were not satisfied, the surface would be a plane. All developables are thus excluded. For let $z = f(x, y)$ be the equation of a developable. We may determine a from the equation

$$\frac{\partial^2 z}{\partial x^2} = a \frac{\partial^2 z}{\partial x \partial y}$$

if $\partial^2 z/\partial x \partial y \neq 0$. Since the surface is a developable, (14) must be satisfied, which gives

$$\frac{\partial^2 z}{\partial y^2} = \frac{1}{a} \frac{\partial^2 z}{\partial x \partial y};$$

i. e., z satisfies a system of form (12) for which $aa' = 1$. Moreover (15) must then also be satisfied unless z is a linear function of x and y, i. e., unless the surface is a plane. If, however,

$$\frac{\partial^2 z}{\partial x \partial y} = 0,$$

equation (14) shows that either $\partial^2 z/\partial x^2$ or $\partial^2 z/\partial y^2$ must vanish. Suppose we have

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial x^2} = 0.$$

This system is not of form (3), but may be put into that form by a linear transformation of the independent variables. It will be an involutory system as one may see even without making the transformation. For the most general solution of the above system is

$$z = c_1 x + \phi(y)$$

where $\phi(y)$ is an arbitrary function of y, and therefore contains an infinite number of arbitrary constants.

We may recapitulate as follows: 1. *Given a system of partial differential equations of form* (3), *whose integrability conditions are satisfied identically and for which* $aa' - 1$ *is not equal to zero. Its most general integrating surface is a projective transformation of any particular one; it does not degenerate into a curve and is not a developable.*

2. *If for such a system of equations, which is not involutory,* $aa' - 1$ *is equal to zero, its integrating surface degenerates into a plane or line.*

3. *If the system is involutory, its integral surfaces are either curves or developables and the most general integral curve or developable is not a mere projective transformation of any particular one.*

For the purpose of constructing a projective differential geometry of non-developable surfaces we may therefore confine ourselves to the first case. Moreover, *any* non-developable surface may be studied by means of such a system of equations. For let x, y, z be the non-homogeneous coördinates of a point on any non-developable surface, and take x and y as independent variables. Then z will be a certain function of x and y, say

$$z = f(x, y).$$

We can determine two functions a and a' of x and y so that z will satisfy equations (12), provided that

$$\frac{\partial^2 z}{\partial x \partial y} \neq 0.$$

If, however,

$$\frac{\partial^2 z}{\partial x \partial y} = 0,$$

we may transform the independent variables by putting

$$\bar{x} = \alpha x + \beta y, \qquad \bar{y} = \gamma x + \delta y, \qquad \alpha\delta - \beta\gamma \neq 0,$$

where α, β, γ, δ are constants. These constants may always be chosen in such a way that $\partial^2 z / \partial \bar{x} \partial \bar{y}$ will not be zero, unless not only $\partial^2 z / \partial x \partial y$, but also $\partial^2 z / \partial x^2$ and $\partial^2 z / \partial y^2$ are equal to zero. In that case the surface would be a plane. Excepting this case it is, therefore, always possible to construct a system of form (12) for every surface. Since, moreover, the surface is not developable $aa' - 1$ will not vanish. If homogeneous coördinates be employed, and an arbitrary set of independent variables, a system of form (3) will be obtained for which $aa' - 1$ is different from zero, and which is not involutory.

Therefore, *the theory of a completely integrable non-involutory system of two partial differential equations of the second order, with two independent and one dependent variable, is identical with the projective differential geometry of non-degenerate, non-developable surfaces in three-dimensional space.*

§ 3. *General notions on invariants and covariants.*

Consider a system of form (3) whose integrability conditions are satisfied identically and for which $aa' - 1$ does not vanish. Let

$$(16) \qquad y^{(k)} = f^{(k)}(u, v) \qquad (k = 1, 2, 3, 4),$$

be four linearly independent solutions of the system. Equations (16) determine an integral surface S_y of (3), and the most general integral surface of (3) will be a projective transformation of S_y.

But there are some, so to speak, accidental elements in the representation (16) of the surface S_y. In the first place, since the coördinates are homogeneous, multiplication of the four functions $y^{(k)}$ by an arbitrary function λ of u and v will not alter the surface. In the second place the independent variables may be transformed arbitrarily. If, therefore, we make any transformation of the form

(17) $$\bar{y} = \lambda(u, v)y, \qquad \bar{u} = \phi(u, v), \qquad \bar{v} = \psi(u, v),$$

$$\lambda \neq 0, \qquad \frac{\partial(\phi, \psi)}{\partial(u, v)} \neq 0,$$

the transformed system of differential equations will have the same integral surfaces as (3).

Those combinations of the coefficients of (3) which are unaltered in value by a transformation of form (17), will be called *invariants*. Such invariant functions as depend also upon y and its derivatives will be called *covariants*. Any projective property of a surface will be expressed by an invariant equation, or system of equations, i. e., by one which is left unchanged by transformations of the form (17). The covariants will determine other surfaces and other geometrical configurations which have a projective relation to the given one.

The determination of the invariants and covariants of system (3) under the transformations (17) constitutes, therefore, nothing more or less than a projective differential theory of surfaces. In this general form, however, it is exceedingly difficult. The calculations involved are so long and complicated as to become tedious and uninstructive. We have, therefore, decided to reduce (3) to a canonical form, a form which exists for every surface, so that there is no sacrifice in the generality of our considerations, in so far as they refer to any non-developable surface. It remains desirable, nevertheless, to construct directly an invariant theory for a system of form (3), or more generally still, for a system of form (1) which is not in the canonical form. This, however, is a question which we shall, for the present, leave untouched.

§ 4. *The intermediate form. The surface referred to its asymptotic curves.*

Transform the independent variables in (3) by putting

$$\bar{u} = \phi(u, v), \qquad \bar{v} = \psi(u, v).$$

Let $\bar{y}_u$, $\bar{y}_v$, etc., denote $\partial y/\partial\bar{u}$, $\partial y/\partial\bar{v}$, etc., and put

$$\phi_u = \frac{\partial\phi}{\partial u}, \qquad \phi_v = \frac{\partial\phi}{\partial v}, \qquad \phi_{uu} = \frac{\partial^2\phi}{\partial u^2},$$

etc. Then we shall find

(18) $$y_u = \phi_u\bar{y}_u + \psi_u\bar{y}_v, \qquad y_v = \phi_v\bar{y}_u + \psi_v\bar{y}_v,$$

and

$$y_{uu} = \phi_u^2\bar{y}_{uu} + 2\phi_u\psi_u\bar{y}_{uv} + \psi_u^2\bar{y}_{vv} + \phi_{uu}\bar{y}_u + \psi_{uu}\bar{y}_v,$$

(19) $$y_{uv} = \phi_u\phi_v\bar{y}_{uu} + (\phi_u\psi_v + \psi_u\phi_v)\bar{y}_{uv} + \psi_u\psi_v\bar{y}_{vv} + \phi_{uv}\bar{y}_u + \psi_{uv}\bar{y}_v,$$

$$y_{vv} = \phi_v^2\bar{y}_{uu} + 2\phi_v\psi_v\bar{y}_{uv} + \psi_v^2\bar{y}_{vv} + \phi_{vv}\bar{y}_u + \psi_{vv}\bar{y}_v.$$

If these values be substituted into (3), if the resulting system of equations be solved for $\bar{y}_{uu}$ and $\bar{y}_{vv}$, and if the new coefficients be denoted by $\bar{a}, \bar{b}, \cdots, \bar{a}', \cdots, \bar{d}'$, we shall find

$$(20)\qquad \begin{aligned} \bar{a} &= \frac{a\psi_v^2 - 2\psi_u\psi_v + a'\psi_u^2}{\phi_u\psi_v + \phi_v\psi_u - a\phi_v\psi_v - a'\phi_u\psi_u}, \\ \bar{a}' &= \frac{a\phi_v^2 - 2\phi_u\phi_v + a'\phi_u^2}{\phi_u\psi_v + \phi_v\psi_u - a\phi_v\psi_v - a'\phi_u\psi_u}, \end{aligned}$$

the other coefficients being fractions with the common denominator

$$(21)\qquad \Delta = (\phi_u\psi_v - \phi_v\psi_u)(\phi_u\psi_v + \phi_v\psi_u - a\phi_v\psi_v - a'\phi_u\psi_u),$$

the first factor of which will not be equal to zero since ϕ and ψ are independent functions of u and v.

Let $\phi_v : \phi_u$ and $\psi_v : \psi_u$ be the two roots of the quadratic

$$at^2 - 2t + a' = 0,$$

so that, for example,

$$\frac{\phi_v}{\phi_u} = \frac{1 + \sqrt{1 - aa'}}{a}, \qquad \frac{\psi_v}{\psi_u} = \frac{1 - \sqrt{1 - aa'}}{a},$$

or

$$(22)\qquad (1 + \sqrt{1 - aa'})\phi_u - a\phi_v = 0, \qquad (1 - \sqrt{1 - aa'})\psi_u - a\psi_v = 0.$$

These equations are distinct since $1 - aa'$ does not vanish, so that the Jacobian of the functions ϕ and ψ will not vanish as a consequence of (22). Assume $a \neq 0$. Then neither ϕ_u nor ψ_u can vanish since, as a consequence of (22), ϕ_v or ψ_v would then also vanish, and this would reduce the Jacobian of ϕ and ψ to zero. The denominator of the expressions (20), or the second factor of Δ, becomes

$$2\,\frac{\phi_u\psi_u(1 - aa')}{a},$$

which is different from zero. The transformation indicated, therefore, makes $\bar{a} = \bar{a}' = 0$. We have proved this under the assumption that $a \neq 0$. If $a = 0$ and $a' \neq 0$ we may still make $\bar{a} = \bar{a}' = 0$ by a very slight modification of the above argument. If $a = a' = 0$ the original system has the required form without transformation.

Suppose this to be the case. Equations (20) show that this form will not be altered if we make either the transformation

$$\bar{u} = \phi(u), \qquad \bar{v} = \psi(v),$$

or

$$\bar{u} = \phi(v), \qquad \bar{v} = \psi(u),$$

where ϕ and ψ are functions of a single argument.

We may recapitulate as follows. *A system of form* (3), *whose integrating* *surfaces are not developables, may always be reduced to the form*

$$
(23) \qquad \begin{aligned} y_{uu} &= by_u + cy_v + dy, \\ y_{vv} &= b'y_u + c'y_v + d'y, \end{aligned}
$$

which shall be called its intermediate form. The reduction requires the integration of the partial differential equation

$$
(24) \qquad a'\left(\frac{\partial\chi}{\partial u}\right)^2 - 2\frac{\partial\chi}{\partial u}\frac{\partial\chi}{\partial v} + a\left(\frac{\partial\chi}{\partial v}\right)^2 = 0.
$$

If the independent variables have been selected in such a way as to cause the system to assume this intermediate form, the most general transformation of the independent variables which does not disturb this form is either

$$
(25) \qquad \bar{u} = \phi(u), \qquad \bar{v} = \psi(v),
$$

or

$$
(26) \qquad \bar{u} = \phi(v), \qquad \bar{v} = \psi(u),
$$

where ϕ and ψ are arbitrary functions of their arguments.

It is easy to find the geometrical significance of this intermediate form. Consider a curve $v =$ const. on the surface S_y. Let λ and μ be arbitrary functions of u and v. Then the four values of $\lambda y^{(k)} + \mu y_u^{(k)}$ will be the coördinates of an arbitrary point on the tangent of this curve constructed at P_y; similarly $\lambda y^{(k)} + \mu y_u^{(k)} + \nu y_{uu}^{(k)}$ will represent the coördinates of an arbitrary point of the osculating plane of the same curve at P_y. An aggregate of the form $\rho y^{(k)} + \sigma y_u^{(k)} + \tau y_v^{(k)}$ represents a point in the plane tangent to the surface S_y at P_y. Equations (23) show that any arbitrary point of the plane which osculates the curve $v =$ const. at P_y, is a point of the plane which is tangent to the surface at P_y. The curve $v =$ const., therefore, has the property that its osculating planes are at the same time tangent planes of the surface. Such curves are called *asymptotic* curves. From our considerations one easily deduces that there are just two families of ∞^1 asymptotic curves upon the surface. They are the curves $u =$ const. and $v =$ const. determined by the above process.

Reduction of a system of form (3) *to its intermediate form is, therefore, equivalent to the determination of the asymptotic curves of its integrating surfaces.*

It is also clear why the transformations (25) or (26) leave the intermediate form unchanged. They merely transform the asymptotic curves of each family into themselves, or else interchange the two families.

§ 5. *The integrability conditions for a system in the intermediate form. The canonical form.*

Consider a system of the intermediate form

$$
(27) \qquad \begin{aligned} y_{uu} + 2ay_u + 2by_v + cy &= 0, \\ y_{vv} + 2a'y_u + 2b'y_v + c'y &= 0. \end{aligned}
$$

We proceed to develop its integrability conditions explicitly. By differentiation, we find from (27),

$$(28)\qquad \begin{aligned} y_{uuu} &= p_1 y_{uv} + p_2 y_u + p_3 y_v + p_4 y, \\ y_{uuv} &= q_1 y_{uv} + q_2 y_u + q_3 y_v + q_4 y, \\ y_{uvv} &= r_1 y_{uv} + r_2 y_u + r_3 y_v + r_4 y, \\ y_{vvv} &= s_1 y_{uv} + s_2 y_u + s_3 y_v + s_4 y, \end{aligned}$$

where

$$(29)\qquad \begin{array}{llll} p_1 = -2b, & p_2 = 4a^2 - 2a_u - c, & p_3 = 4ab - 2b_u, & p_4 = 2ac - c_u, \\ q_1 = -2a, & q_2 = 4a'b - 2a_v, & q_3 = 4bb' - 2b_v - c, & q_4 = 2bc' - c_v, \\ r_1 = -2b', & r_2 = 4aa' - 2a'_u - c', & r_3 = 4a'b - 2b'_u, & r_4 = 2a'c - c'_u, \\ s_1 = -2a', & s_2 = 4a'b' - 2a'_v, & s_3 = 4b'^2 - 2b'_v - c', & s_4 = 2b'c' - c'_v. \end{array}$$

The integrability conditions are obtained by writing

$$\frac{\partial y_{uuu}}{\partial v} = \frac{\partial y_{uuv}}{\partial u}, \qquad \frac{\partial y_{uuv}}{\partial v} = \frac{\partial y_{uvv}}{\partial u}, \qquad \frac{\partial y_{uvv}}{\partial v} = \frac{\partial y_{vvv}}{\partial u}.$$

It will be found that the first and last of these three conditions are fulfilled identically. We find further

$$\begin{aligned} \frac{\partial y_{uuv}}{\partial v} = {} & \left(\frac{\partial q_1}{\partial v} + q_1 r_1 + q_2 \right) y_{uv} + \left(\frac{\partial q_2}{\partial v} + q_1 r_2 - 2a' q_3 \right) y_u \\ & + \left(\frac{\partial q_3}{\partial v} + q_1 r_3 - 2b' q_3 + q_4 \right) y_v + \left(\frac{\partial q_4}{\partial v} + q_1 r_4 - c' q_3 \right) y, \end{aligned}$$

$$\begin{aligned} \frac{\partial y_{uvv}}{\partial u} = {} & \left(\frac{\partial r_1}{\partial u} + r_1 q_1 + r_3 \right) y_{uv} + \left(\frac{\partial r_2}{\partial u} + r_1 q_2 - 2a r_2 + r_4 \right) y_u \\ & + \left(\frac{\partial r_3}{\partial u} + r_1 q_3 - 2b r_2 \right) y_v + \left(\frac{\partial r_4}{\partial u} + r_1 q_4 - c r_2 \right) y. \end{aligned}$$

In order that these two expressions may be equal for all values of y, y_u, y_v, y_{uv}, the corresponding coefficients in the two expressions must be identical. We thus find the following four conditions:

$$(30)\qquad \begin{aligned} & a_v - b'_u = 0, \\ & a'_{uu} + c'_u - 2a' a_u - 2aa'_u - (a_{vv} + 2b' a_v - 2ba'_v - 4a' b_v) = 0, \\ & b_{vv} + c_v - 2bb'_v - 2b' b_v - (b'_{uu} + 2ab'_u - 2a' b_u - 4ba'_u) = 0, \\ & c'_{uu} - 4ca'_u - 2a' c_u + 2ac'_u - (c_{vv} - 4c' b_v - 2bc'_v + 2b' c_v) = 0. \end{aligned}$$

The conditions (30) *must be satisfied in order that the general solution of* (27) *may depend upon just four arbitrary constants, for this is the only case of* *interest from the point of view of the theory of surfaces*, as was explained in § 2. These integrability conditions (30) are symmetrically constructed from the coefficients of (27) in such a way that the operation which consists in interchanging simultaneously the letters in the two rows of the symbol

$$\begin{pmatrix} a & a' & c & u \\ b' & b & c' & v \end{pmatrix},$$

converts the first and fourth equations into themselves and interchanges the other two.

From the significance of the conditions (30), it is clear that they must form an invariant system of equations. In other words, if the variables in (27) are changed arbitrarily, but in such a way as not to disturb the form of the system, the integrability conditions for the transformed system of equations will be fulfilled if they are fulfilled for the original system, and conversely.

On account of the first condition of (30) we may write

$$a = p_u, \qquad b' = p_v. \tag{31}$$

Transform (27) by putting

$$y = \lambda \bar{y},$$

where λ is a function of u and v. We find a new system of the same form for $\bar{y}$ whose coefficients are

$$\bar{a} = a + \frac{\lambda_u}{\lambda}, \qquad \bar{b} = b, \qquad \bar{c} = c + \frac{\lambda_{uu} + 2a\lambda_u + 2b\lambda_v}{\lambda},$$

$$\bar{a}' = a', \qquad \bar{b}' = b' + \frac{\lambda_v}{\lambda}, \qquad \bar{c}' = c' + \frac{\lambda_{vv} + 2a'\lambda_u + 2b'\lambda_v}{\lambda}.$$

Owing to (31) we may make $\bar{a} = \bar{b}' = 0$ by putting

$$\lambda = e^{-p},$$

so that

$$\frac{\lambda_u}{\lambda} = -p_u, \qquad \frac{\lambda_v}{\lambda} = -p_v,$$

$$\frac{\lambda_{uu}}{\lambda} = -p_{uu} + p_u^2, \qquad \frac{\lambda_{vv}}{\lambda} = -p_{vv} + p_v^2,$$

whence

$$\begin{aligned} \bar{c} &= c - a_u - a^2 - 2bb' = f, \\ \bar{c}' &= c' - b'_v - b'^2 - 2aa' = g. \end{aligned} \tag{32}$$

It becomes possible, therefore, to put the system (27) *into the form*

$$\begin{aligned} y_{uu} + 2by_v + fy &= 0, \\ y_{vv} + 2a'y_u + gy &= 0, \end{aligned} \tag{33}$$

which is characterized by the conditions $a = b' = 0$. We shall speak of (33) as the *canonical form* of the given system of differential equations. It may be obtained from the intermediate form in the way just indicated. *The first of the four integrability conditions* (30) *is satisfied identically. The other three assume the simplified form:*

$$
\begin{aligned}
a'_{uu} + g_u + 2ba'_v + 4a'b_v &= 0, \\
b_{vv} + f_v + 2a'b_u + 4ba'_u &= 0, \\
g_{uu} - f_{vv} - 4ga'_u - 2a'f_u + 4fb_v + 2bg_v &= 0.
\end{aligned} \tag{34}
$$

§ 6. *Seminvariants and semi-covariants.*

Consider again the system (27) of differential equations, and let it be transformed by putting

$$y = \lambda \bar{y}, \tag{35}$$

where λ is an arbitrary function of u and v. Those functions of the coefficients of (27), of y, and of the derivatives of these quantities, which are left unchanged by every such transformation, shall be called *seminvariants* or *semi-covariants* according as they do not, or do contain y and its derivatives.

We find from (35)

$$
\begin{aligned}
y_u &= \lambda \bar{y}_u + \lambda_u \bar{y}, \qquad y_v = \lambda \bar{y}_v + \lambda_v \bar{y}, \\
y_{uu} &= \lambda \bar{y}_{uu} + 2\lambda_u \bar{y}_u + \lambda_{uu} \bar{y}, \\
y_{uv} &= \lambda \bar{y}_{uv} + \lambda_u \bar{y}_v + \lambda_v \bar{y}_u + \lambda_{uv} \bar{y}, \\
y_{vv} &= \lambda \bar{y}_{vv} + 2\lambda_v \bar{y}_v + \lambda_{vv} \bar{y},
\end{aligned} \tag{36}
$$

which gives, upon substitution into (27),

$$
\begin{aligned}
\bar{y}_{uu} + 2\bar{a}\bar{y}_u + 2\bar{b}\bar{y}_v + \bar{c}\bar{y} &= 0, \\
\bar{y}_{vv} + 2\bar{a}'\bar{y}_u + 2\bar{b}'\bar{y}_v + \bar{c}'\bar{y} &= 0,
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{a} &= a + \frac{\lambda_u}{\lambda}, & \bar{b} &= b, & \bar{c} &= c + \frac{\lambda_{uu} + 2a\lambda_u + 2b\lambda_v}{\lambda}, \\
\bar{a}' &= a', & \bar{b}' &= b' + \frac{\lambda_v}{\lambda}, & \bar{c}' &= c' + \frac{\lambda_{vv} + 2a'\lambda_u + 2b'\lambda_v}{\lambda}.
\end{aligned} \tag{37}
$$

We find the following seminvariants

$$
\begin{gathered}
a', \qquad b, \\
f = c - a_u - a^2 - 2bb', \qquad g = c' - b'_v - b'^2 - 2aa'.
\end{gathered} \tag{38}
$$

The derivatives of these four quantities are also seminvariants. *Every semin-*

variant is a function of a', b, f, g and of their derivatives. That this is true may be, most simply, seen as follows: A seminvariant has the same value for system (27) as for any system obtained from it by a transformation of form (35). Make the transformation of form (35), which reduces (27) to its canonical form according to the method of § 5. The resulting system (33) has precisely a', b, f and g as its coefficients. Every seminvariant can, therefore, contain only these quantities and their derivatives, as was to be shown.

Semi-covariants contain y, y_u, y_v, y_{uv}, etc., in addition to the coefficients of (27). Since all higher derivatives of y may be expressed in terms of the four quantities

$$y, \; y_u, \; y_v, \; y_{uv},$$

we may confine our search for semi-covariants to such as involve them. We find, from (36),

$$\begin{aligned} y &= \frac{1}{\lambda}\bar{y}, \\ \bar{y}_u &= \frac{1}{\lambda}\left(y_u - \frac{\lambda_u}{\lambda}y\right), \qquad \bar{y}_v = \frac{1}{\lambda}\left(y_v - \frac{\lambda_v}{\lambda}y\right), \\ \bar{y}_{uv} &= \frac{1}{\lambda}\left[y_{uv} - \frac{\lambda_u}{\lambda}y_v - \frac{\lambda_v}{\lambda}y_u + \left(\frac{2\lambda_u\lambda_v}{\lambda^2} - \frac{\lambda_{uv}}{\lambda}\right)y\right], \end{aligned} \tag{39}$$

so that y itself is a *relative* semi-covariant. By making use of (37) we find three other relative semi-covariants

$$\begin{aligned} z &= y_u + ay, \qquad \rho = y_v + b'y, \\ \sigma &= y_{uv} + b'y_u + ay_v + \tfrac{1}{2}(a_v + b'_u + 2ab')y; \end{aligned} \tag{40}$$

in fact these quantities satisfy the equations

$$\bar{y} = \lambda y, \qquad \bar{z} = \lambda z, \qquad \bar{\rho} = \lambda\rho, \qquad \bar{\sigma} = \lambda\sigma,$$

so that their ratios are absolute semi-covariants. The four semi-covariants y, z, ρ, σ are clearly independent. Moreover equations (40) show that y, y_u, y_v, y_{uv} may be expressed in terms of y, z, ρ, σ. Since every relative semicovariant can be expressed in terms of y, y_u, y_v, y_{uv}, it may be also expressed in terms of y, z, ρ, σ. By reducing to the canonical form it becomes obvious that *all semi-covariants can be expressed as functions of y, z, ρ, σ and of seminvariants.*

§ 7. *Invariants and covariants.*

In order to find invariants we introduce transformations of the independent variables which leave the intermediate form of the system of partial differential equations unaltered. These are either of form (25) or (26). Since (26) may be obtained from (25) by the simple transformation

$$u = \bar{v}, \qquad v = \bar{u},$$

which merely interchanges the two families of asymptotic lines, it becomes an easy matter to find the invariants under the general transformation if those under transformation (25) have been found. It is desirable, moreover, to keep the two families of asymptotic curves separate. For these reasons we confine ourselves to the transformation (25) or as we shall now write it

$$\bar{u} = \alpha(u), \qquad \bar{v} = \beta(v), \tag{41}$$

denoting the new independent variables by $\bar{u}$ and $\bar{v}$ where α is a function of u alone, and β is a function of the single variable v. Upon making the transformation (41), we find

$$\begin{aligned} y_u &= \alpha_u \frac{\partial y}{\partial \bar{u}}, \qquad y_{uu} = \alpha_u^2 \frac{\partial^2 y}{\partial \bar{u}^2} + \alpha_{uu} \frac{\partial y}{\partial \bar{u}}, \\ y_v &= \beta_v \frac{\partial y}{\partial \bar{v}}, \qquad y_{vv} = \beta_v^2 \frac{\partial^2 y}{\partial \bar{v}^2} + \beta_{vv} \frac{\partial y}{\partial \bar{v}}, \\ y_{uv} &= \alpha_u \beta_v \frac{\partial^2 y}{\partial \bar{u}\, \partial \bar{v}}. \end{aligned} \tag{42}$$

Equations (27) are transformed into a similar system, whose coefficients are

$$\begin{aligned} \bar{a} &= \frac{1}{\alpha_u}(a + \tfrac{1}{2}\eta), \qquad \bar{b} = \frac{\beta_v b}{\alpha_u^2}, \qquad \bar{c} = \frac{c}{\alpha_u^2}, \\ \bar{a}' &= \frac{\alpha_u a'}{\beta_v^2}, \qquad \bar{b}' = \frac{1}{\beta_v}(b' + \tfrac{1}{2}\zeta), \qquad \bar{c}' = \frac{c'}{\beta_v^2}, \end{aligned} \tag{43}$$

where

$$\eta = \frac{\alpha_{uu}}{\alpha_u}, \qquad \zeta = \frac{\beta_{vv}}{\beta_v}. \tag{44}$$

These equations show that a' and b are (relative) invariants as well as seminvariants.

Since invariants are also seminvariants, they can only be functions of a', b, f, g and of their derivatives. It becomes necessary, therefore, to investigate how the transformation (41) affects these seminvariants. Invariants are such functions of seminvariants as are left unaltered by this transformation.

We find, from (43),

$$\begin{aligned} \frac{\partial \bar{a}}{\partial \bar{u}} &= \frac{1}{\alpha_u^2}[a_u - \eta a + \tfrac{1}{2}(\eta_u - \eta^2)], & \frac{\partial \bar{a}}{\partial \bar{v}} &= \frac{a_v}{\alpha_u \beta_v}, \\ \frac{\partial \bar{b}'}{\partial \bar{u}} &= \frac{b'_u}{\alpha_u \beta_v}, & \frac{\partial \bar{b}'}{\partial \bar{v}} &= \frac{1}{\beta_v^2}[b'_v - \zeta b' + \tfrac{1}{2}(\zeta_v - \zeta^2)], \\ \frac{\partial \bar{a}'}{\partial \bar{u}} &= \frac{1}{\beta_v^2}(a'_u + \eta a'), & \frac{\partial \bar{a}'}{\partial \bar{v}} &= \frac{\alpha_u}{\beta_v^3}(a'_v - 2\zeta a'), \\ \frac{\partial \bar{b}}{\partial \bar{u}} &= \frac{\beta_v}{\alpha_u^3}(b_u - 2\eta b), & \frac{\partial \bar{b}}{\partial \bar{v}} &= \frac{1}{\alpha_u^2}(b_v + \zeta b). \end{aligned} \tag{45}$$

From the expressions (38) for f and g, and the above equations, we find

$$(46) \qquad \bar{f} = \frac{1}{\alpha_u^2}(f - b\zeta - \tfrac{1}{2}\mu), \qquad \bar{g} = \frac{1}{\beta_v^2}(g - a'\eta - \tfrac{1}{2}\nu),$$

where

$$(47) \qquad \mu = \eta_u - \tfrac{1}{2}\eta^2, \qquad \nu = \zeta_v - \tfrac{1}{2}\zeta^2.$$

We have further

$$(48a) \qquad \begin{aligned} \frac{\partial^2 \bar{a}'}{\partial \bar{u}^2} &= \frac{1}{\alpha_u \beta_v^2}[a'_{uu} + \eta a'_u + \eta_u a'], \\ \frac{\partial^2 \bar{a}'}{\partial \bar{u}\,\partial \bar{v}} &= \frac{1}{\beta_v^3}[a'_{uv} + \eta a'_v - 2\zeta a'_u - 2\eta\zeta a'], \\ \frac{\partial^2 \bar{a}'}{\partial \bar{v}^2} &= \frac{\alpha_u}{\beta_v^4}[a'_{vv} - 5\zeta a'_v - 2\zeta_v a' + 6\zeta^2 a'], \end{aligned}$$

and

$$(48b) \qquad \begin{aligned} \frac{\partial^2 \bar{b}}{\partial \bar{u}^2} &= \frac{\beta_v}{\alpha_u^4}[b_{uu} - 5\eta b_u - 2\eta_u b + 6\eta^2 b], \\ \frac{\partial^2 \bar{b}}{\partial \bar{u}\,\partial \bar{v}} &= \frac{1}{\alpha_u^3}[b_{uv} + \zeta b_u - 2\eta b_v - 2\eta\zeta b], \\ \frac{\partial^2 \bar{b}}{\partial \bar{v}^2} &= \frac{1}{\alpha_u^2 \beta_v}[b_{vv} + \zeta b_v + \zeta_v b]. \end{aligned}$$

By means of these equations it is easy to verify that

$$(49) \qquad \bar{h} = \frac{\beta_v^2 h}{\alpha_u^6}, \qquad \bar{k} = \frac{\alpha_u^2 k}{\beta_v^6},$$

where

$$(50) \quad h = b^2(f + b_v) - \tfrac{1}{4}bb_{uu} + \tfrac{5}{16}b_u^2, \qquad k = a'^2(g + a'_u) - \tfrac{1}{4}a'a'_{vv} + \tfrac{5}{16}a'^2_v,$$

so that we have found four invariants a', b, h and k. We shall show how other invariants may be derived from these four by certain differentiation processes. Put

$$(51) \qquad A = a'b^2, \qquad B = a'^2 b, \qquad H = a'h, \qquad K = bk$$

so that

$$(52) \qquad \bar{A} = \frac{A}{\alpha_u^3}, \qquad \bar{B} = \frac{B}{\beta_v^3}, \qquad \bar{H} = \frac{H}{\alpha_u^5}, \qquad \bar{K} = \frac{K}{\beta_v^5},$$

it being assumed that a' and b are different from zero.

Introduce two symbols of operation

$$(53) \qquad U = a'\frac{\partial}{\partial u}, \qquad V = b\frac{\partial}{\partial v}.$$

Then clearly

$$(54) \qquad U(B), \qquad V(A), \qquad U(K), \qquad V(H)$$

are also invariants. In fact

$$(55)\quad \bar{V}(\bar{A})=\frac{V(A)}{\alpha_u^5},\quad \bar{U}(\bar{B})=\frac{U(B)}{\beta_v^5},\quad \bar{V}(\bar{H})=\frac{V(H)}{\alpha_u^7},\quad \bar{U}(\bar{K})=\frac{U(K)}{\beta_v^7},$$

so that it becomes clear that every repetition of these two processes gives rise to an invariant.

Further, from two invariants λ and μ for which

$$\bar{\lambda}=\frac{1}{\alpha_u^l}\lambda,\qquad \bar{\mu}=\frac{1}{\alpha_u^m}\mu,$$

we can always form another, its Wronskian. In fact we find

$$\overline{\lambda^m\mu^{-l}}=\lambda^m\mu^{-l},$$

whence, by logarithmic differentiation,

$$\overline{m\frac{\lambda_u}{\lambda}-l\frac{\mu_u}{\mu}}=\frac{1}{\alpha_u}\left(m\frac{\lambda_u}{\lambda}-l\frac{\mu_u}{\mu}\right),$$

so that the function

$$(56)\qquad (\mu,\lambda_u)=m\mu\lambda_u-l\lambda\mu_u$$

is an invariant for which

$$(57)\qquad \overline{(\mu\lambda_u)}=\frac{(\mu\lambda_u)}{\alpha_u^{l+m+1}}.$$

We shall speak of $(\mu\lambda_u)$ as the Wronskian of μ and λ with respect to u.

By combining the Wronskian process with the operations U and V, it clearly becomes possible to deduce an infinity of invariants from A, B, H and K. We proceed to show that any invariant is a function of those obtained in this manner.

In the first place, the equations which define A, B, H and K, show that a', b, f and g may be expressed in terms of A, B, H, K and of their derivatives. Since all invariants are functions of a', b, f, g, and of their derivatives, any invariant may be expressed as a function of A, B, H, K and of the derivatives of these quantities.

Introduce infinitesimal transformations by putting

$$(58)\qquad \alpha(u)=u+\phi(u)\delta t\qquad \beta(v)=v+\psi(v)\delta t,$$

where δt is an infinitesimal. Then

$$(59)\qquad \begin{aligned} &\alpha_u=1+\phi_u\delta t, && \alpha_v=0,\\ &\beta_u=0, && \beta_v=1+\psi_v\delta t,\\ &\alpha_u\beta_v-\alpha_v\beta_u=1+(\phi_u+\psi_v)\delta t. \end{aligned}$$

The infinitesimal transformations of A, B, H and K become

$$(6)\quad \delta A = -3\phi_u A\delta t,\ \delta B = -3\psi_v B\delta t,\ \delta H = -5\phi_u H\delta t,\ \delta K = -5\psi_v K\delta t.$$

The infinitesimal transformation, δF, of any function of A, B, etc., being known, we may easily find the infinitesimal transformations δF_u and δF_v of its first derivatives. In fact we shall have

$$(61)\quad \begin{aligned} \delta F_u &= \frac{\partial \bar{F}}{\partial \bar{u}} - \frac{\partial F}{\partial u} = \frac{\partial(F+\delta F)}{\partial u}\,\frac{1}{1+\phi_u\delta t} - \frac{\partial F}{\partial u} = \frac{\partial(\delta F)}{\partial u} - \phi_u F_u \delta t, \\ \delta F_v &= \frac{\partial \bar{F}}{\partial \bar{v}} - \frac{\partial F}{\partial v} = \frac{\partial(F+\delta F)}{\partial v}\,\frac{1}{1+\psi_v\delta t} - \frac{\partial F}{\partial v} = \frac{\partial(\delta F)}{\partial v} - \psi_v F_v \delta t. \end{aligned}$$

Thus we find

$$(62)\quad \begin{aligned} \delta A_u &= -(3\phi_{uu}A + 4\phi_u A_u)\delta t, & \delta A_v &= -(3\phi_u + \psi_v)A_v\delta t, \\ \delta B_u &= -(\phi_u + 3\psi_v)B_u\delta t, & \delta B_v &= -(3\psi_{vv}B + 4\psi_v B_v)\delta t, \\ \delta H_u &= -(5\phi_{uu}H + 6\phi_u H_u)\delta t, & \delta H_v &= -(5\phi_u + \psi_v)H_v\delta t, \\ \delta K_u &= -(\phi_u + 5\psi_v)K_u\delta t, & \delta K_v &= -(5\psi_{vv}K + 6\psi_v K_v)\delta t. \end{aligned}$$

All absolute invariants θ which involve only A, B, H, K and their first derivatives must, therefore, satisfy the system of partial differential equations obtained by forming the infinitesimal transformation $\delta\theta$ of an arbitrary function of these arguments and equating to zero the coefficients of ϕ_{uu}, ψ_{vv}, ϕ_u and ψ_v. If we are looking for relative invariants we may dispense with the latter two equations, as the following consideration will show. Consider the special transformation for which

$$(63)\qquad \alpha(u) = cu, \qquad \beta(v) = c'v,$$

so that

$$\alpha_u = c, \qquad \beta_v = c',$$

where c and c' are arbitrary constants. Then

$$\bar{A} = c^{-3}A, \qquad \bar{B} = (c')^{-3}B, \quad \text{etc.},$$

$$\bar{A}_u = c^{-4}A_u, \quad \text{etc.}$$

In general, let P be a combination of the quantities A, B, H, K and of their derivatives, and suppose that the effect of the transformation (63) is to convert it into $\bar{P}$, where

$$\bar{P} = c^{-p}c'^{-p'}P.$$

We shall then say that P has an α-weight equal to p and a β-weight equal to p'. Thus the weights of A and B are $(+3, 0)$ and $(0, +3)$ respectively. It is clear that differentiation with respect to u and v diminishes the α- or β-

weight respectively by unity. The weights of a product are the sums of the respective weights of the factors.

We shall say of a sum that it is *isobaric* of weights w, w' if each of its terms has the weights w, w'. It is clear that every absolute invariant is isobaric of weights $0, 0$. *Every relative invariant must be isobaric.*

We obtain an infinitesimal transformation of the form (63) if we let ϕ_u and ψ_v in (60) and (62) represent constants. Under this assumption, if F is any isobaric function of weights w and w', the infinitesimal transformation of F will be of the form

$$\delta F = -(w\phi_u + w'\psi_v)F\delta t. \tag{64}$$

If ϕ_u and ψ_v are not constants, the expression for δF will contain other terms which will have ϕ_{uu}, ψ_{vv}, etc. as factors. If, however F is an irreducible (relative or absolute) invariant, the equation $\delta F = 0$ must be a consequence of $F = 0$, which implies that if F is an invariant of weights w, w', the equation (64) must be satisfied even if ϕ_u and ψ_v are not constants.*

Suppose then that we form the general expression for δF, where F depends upon A, B, H, K and the derivatives of these quantities up to any order n, and equate to zero the coefficients of all of the derivatives of the arbitrary functions ϕ and ψ, including ϕ_u and ψ_v. We thus obtain a complete system S of partial differential equations, whose solutions are the *absolute* invariants up to the nth order. If we omit the two equations of this system obtained by equating to zero the coefficients of ϕ_u and ψ_v, we obtain a system S' of partial differential equations satisfied by all relative invariants. Moreover, only such solutions of this system S' will be relative invariants for which (64) is satisfied, i. e., only isobaric solutions of the system are admissible as invariants. The relative invariants are certainly not all of weight zero. We know in fact that A, B, H and K are independent relative invariants, both of whose weights are not zero. Any relative invariant can, however, by multiplication with properly chosen powers of A and B be converted into an absolute invariant. Consequently there are two more independent relative invariants than absolute invariants. If the system S' were not a complete system, the difference between the number of relative and absolute invariants could not be equal to two. *The system S' is, therefore, a complete system and every isobaric solution of S' is a relative invariant.*

All relative invariants which involve only A, B, H, K and their first derivatives are, therefore, isobaric solutions of the complete system of two equations

$$3A\frac{\partial\theta}{\partial A_u} + 5H\frac{\partial\theta}{\partial H_u} = 0, \qquad 3B\frac{\partial\theta}{\partial B_v} + 5K\frac{\partial\theta}{\partial K_v} = 0,$$

* For the coefficients of ϕ_{uu}, ψ_{vv}, etc. cannot vanish in consequence of $F = 0$, if we assume F to be irreducible, since their coefficients are of weights lower than w, w'. The complete discussion would be the same as that in the author's treatise *Projective Differential Geometry*, etc., p. 22.

which involves (in part with zero coefficients) twelve independent variables. It has therefore ten independent solutions. These may be selected in the form

$$(65a)\qquad A,\ B,\ H,\ K,\ V(A),\ U(B),\ V(H),\ U(K),$$

together with the two Wronskians

$$(65b)\qquad (A,H_u),\qquad (B,K_v).$$

For they are clearly independent. In fact, taken in this order, each of these functions involves one variable which does not occur in any of the preceding ones.

All relative invariants which involve also the second derivatives of A, B, H, K will be isobaric solutions of a complete system of four independent equations with twenty-four independent variables. They are, therefore, twenty in number. Ten of these we have already found. The other ten may be taken in the form

$$(66)\qquad \begin{array}{lllll} V^2(A), & V^2(H), & V(A,H_u), & \big(A,V(A)_u\big), & \big(A,(A,H_u)_u\big), \\ U^2(B), & U^2(K), & U(B,K_v), & \big(B,U(B)_v\big), & \big(B,(B,K_v)_v\big); \end{array}$$

for it is apparent that these are independent.

We have now determined all invariants up to and including those of the second order; they are isobaric functions of the twenty invariants (65) and (66). We make the following remarks. Let an invariant be called an α- or a β-invariant if only one of its two weights is different from zero, so that the β-weight of an α-invariant is zero, etc. Our invariants have been so chosen that half of them are α- and half of them β-invariants. Consider the five α-invariants of the second order. They are those in the first row of (66). Two of them, the first and the fourth do not contain H, and contain only two of the three derivatives of the second order of A, viz., A_{vv} and A_{uv}. The other three α-invariants contain each a different one of the three second derivatives of H. A similar classification may be made of the β-invariants. Suppose we had determined all of the invariants up to and including those of the $n-1$th order. To determine those of the nth order we should have to solve a system of partial differential equations containing two more equations and $4(n+1)$ more variables than the system for the invariants of order $n-1$. There will be, therefore, $4n+2$ invariants which involve the nth derivatives of A, B, H, K. Similarly there will be just $4n-2$ invariants of the $n-1$th order.

Let us suppose that what we have found to be true for the invariants of the second order is also true for those of order $n-1$, i. e., let us suppose that $2n-1$ of them are α- and $2n-1$ of them β-invariants. Denote these α-invariants of the $n-1$th order by

$$I_1,\ I_2,\ \cdots,\ I_{n-1};\qquad I'_1,\ I'_2,\ \cdots,\ I'_n.$$

The notation indicates a division of these α-invariants into two classes; those of the first class do not depend upon H and its derivatives, those of the second class do. Let

$$P^{(n-1)}_{u_r,\,v_s} \qquad (r+s=n-1),$$

denote the derivative of P taken r times with respect to u and s times with respect to v. Then we shall assume that $I_1, I_2, \cdots, I_{n-1}$ contain in this order $A^{(n-1)}_{u_0,\,v_{n-1}}, A^{(n-1)}_{u_1,\,v_{n-2}}, \cdots, A^{(n-1)}_{u_{n-2},\,v_1}$, and that no function I_k contains $A^{(n-1)}_{u_{n-1},\,v_0}$.

We assume further that the functions $I'_1, I'_2, \cdots, I'_n$ taken in this order contain respectively $H^{(n-1)}_{u_0,\,v_{n-1}}, H^{(n-1)}_{u_1,\,v_{n-2}}, \cdots, H^{(n-1)}_{u_{n-1},\,v_0}$.

The $2n-1$ functions

$$V(I_k), \qquad V(I'_k) \tag{67}$$

will then be α-invariants of the nth order. Those of the first set contain all of the derivatives of A of the nth order except $A^{(n)}_{u_{n-1},\,v_1}, A^{(n)}_{u_n,\,v_0}$. Those of the second set contain all of the derivatives of H of the nth order except $H^{(n)}_{u_n,\,v_0}$.

Consider the Wronskian of A and I_{n-1} with respect to u. It will be an α-invariant which will contain the derivative $A^{(n)}_{u_{n-1},\,v_1}$. Similarly the Wronskian of A and I'_n will be an α-invariant which contains $H^{(n)}_{u_n,\,v_0}$. If we add these two α-invariants to the $2n-1$ of (67), we shall have $2n+1$ independent α-invariants of order n which can be divided into two classes precisely in the same way as was assumed to be the case for those of the $n-1$th order. Since this classification was actually possible for the invariants of 2, it follows that this same classification will actually hold for any order n.

We have seen, therefore, that the Wronskian process and the process V enables us to obtain $2n+1$ independent α-invariants of order n from A and H. Similarly, by means of the Wronskian process and the process U we may obtain $2n+1$ independent β-invariants of order n from B and K. It only remains to show that these $4n+2$ invariants are also independent of each other. There can be no question about the independence of those which contain H and K. But the α-invariants of the first class contain derivatives of B as well as of A, and the β-invariants of the first class contain derivatives of A as well as of B. However, the derivatives of B in the α-invariants of order n are only of order $n-1$, so that no relation is possible between the α- and β-invariants.

The Wronskian, the U-process and the V-process suffice, therefore, to deduce all invariants from the four fundamental ones A, B, H and K. There are $2(n^2+2n+2)$ independent invariants of order equal to or less than n.

In order to find a system of covariants we investigate the result of the transformation

$$\bar{u}=\alpha(u), \qquad \bar{v}=\beta(v),$$

upon the four independent semi-covariants (40). We find

$$(68)\qquad \bar{y} = y, \qquad \bar{z} = \frac{1}{\alpha_u}(z + \tfrac{1}{2}\eta y), \qquad \bar{\rho} = \frac{1}{\beta_v}(\rho + \tfrac{1}{2}\zeta y),$$
$$\bar{\sigma} = \frac{1}{\alpha_u\beta_v}(\sigma + \tfrac{1}{2}\eta\rho + \tfrac{1}{2}\zeta z + \tfrac{1}{4}\eta\zeta y).$$

It now becomes an easy matter to verify that the following four functions are covariants

$$(69)\qquad y, \qquad 6Az + A_u y, \qquad 6B\rho + B_v y, \qquad \sigma - a\rho - b'z + ab'y.$$

They are clearly independent. *Every other covariant may be expressed in terms of the covariants* (69) *and of invariants.*

The most general transformation which leaves the canonical form invariant may be found as follows. We have shown that a system of differential equations of form (3) may be reduced to the canonical form

$$y_{uu} + 2by_v + fy = 0,$$
$$y_{vv} + 2a'y_u + gy = 0.$$

Make successively the transformations

$$y = \lambda\bar{y}, \qquad \bar{u} = \alpha(u), \qquad \bar{v} = \beta(v).$$

The resulting system will again be in the canonical form if, and only if,

$$\frac{\lambda_u}{\lambda} + \frac{1}{2}\frac{\alpha_{uu}}{\alpha_u} = 0, \qquad \frac{\lambda_v}{\lambda} + \frac{1}{2}\frac{\beta_{vv}}{\beta_v} = 0,$$

i. e., if

$$\lambda = \frac{g(v)}{\sqrt{\alpha_u}} = \frac{h(u)}{\sqrt{\beta_v}},$$

where $g(v)$ and $h(u)$ are functions of the single arguments indicated. We find, therefore,

$$\lambda = \frac{C}{\sqrt{\alpha_u\beta_v}},$$

where C is an arbitrary constant.

Therefore, *the most general transformation which leaves the canonical form invariant is*

$$(70)\qquad \bar{y} = C\sqrt{\alpha_u\beta_v}\,y, \qquad \bar{u} = \alpha(u), \qquad \bar{v} = \beta(v),$$

where α and β are arbitrary functions of u and v respectively, and where C is an arbitrary constant.

§ 8. *Principle of duality. Adjoined systems. The fundamental theorem.*

Given a system of partial differential equations in the intermediate form

$$(71)\qquad \begin{aligned} y_{uu} + 2ay_u + 2by_v + cy = 0, \\ y_{vv} + 2a'y_u + 2b'y_v + c'y = 0, \end{aligned}$$

whose integrability conditions are fulfilled identically. If

$$y^{(k)} = f^{(k)}(u, v) \qquad (k = 1, 2, 3, 4),$$

are four independent solutions of (71), we interpret them as the homogeneous coördinates of a point P_y of a surface S. Let p_y be the plane which is tangent to S at P_y. Its equation will be

$$(72)\qquad \begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y' & y'' & y^{(3)} & y^{(4)} \\ y'_u & y''_u & y^{(3)}_u & y^{(4)}_u \\ y'_v & y''_v & y^{(3)}_v & y^{(4)}_v \end{vmatrix} = 0.$$

The coördinates of the plane p_y will be proportional to the minors of x_1, x_2, x_3, x_4 in this determinant. Denote these minors by $\lambda', \lambda'', \lambda^{(3)}, \lambda^{(4)}$. We shall have

$$\lambda' = \begin{vmatrix} y'' & y^{(3)} & y^{(4)} \\ y''_u & y^{(3)}_u & y^{(4)}_u \\ y''_v & y^{(3)}_v & y^{(4)}_v \end{vmatrix}, \qquad \lambda'' = -\begin{vmatrix} y' & y^{(3)} & y^{(4)} \\ y'_u & y^{(3)}_u & y^{(4)}_u \\ y'_v & y^{(3)}_v & y^{(4)}_v \end{vmatrix}, \text{ etc.}$$

Let us convert the columns into rows and write only a representative row. We may then write for each of the four λ's, disregarding signs, an expression of the form

$$(73)\qquad \lambda = D(y, y_u, y_v),$$

which stands for a determinant of the third order of the matrix

$$|y^{(k)}, y^{(k)}_u, y^{(k)}_v| \qquad (k = 1, 2, 3, 4).$$

We find from (73), making use of (71),

$$(74)\qquad \begin{aligned} \lambda_u = D(y, y_u, y_{uv}) - 2a\lambda, \\ \lambda_v = D(y, y_{uv}, y_v) - 2b'\lambda. \end{aligned}$$

Therefore, making use of (28) and (29),

$$(75)\qquad \begin{aligned} \lambda_{uu} &= -2a\lambda_u + (4bb' - 2b_v - c - 2a_u)\lambda - 4aD(y, y_u, y_{uv}) - 2bD(y, y_v, y_{uv}), \\ \lambda_{vv} &= -2b'\lambda_v + (4aa' - 2a'_u - c' - 2b'_v)\lambda - 2a'D(y, y_{uv}, y_u) - 4b'D(y, y_{uv}, y_v). \end{aligned}$$

But from (74),

$$D(y,\ y_u,\ y_{uv}) = \lambda_u + 2a\lambda,$$
$$D(y,\ y_{uv},\ y_v) = \lambda_v + 2b'\lambda.$$

Substituting these values in (75) we find

$$(76)\qquad \begin{aligned} &\lambda_{uu} + 6a\lambda_u - 2b\lambda_v + (2a_u + 2b_v + c + 8a^2 - 8bb')\lambda = 0, \\ &\lambda_{vv} - 2a'\lambda_u + 6b'\lambda_v + (2b'_v + 2a'_u + c' + 8b'^2 - 8aa')\lambda = 0. \end{aligned}$$

The coördinates of the plane p_y, *therefore, satisfy the equations* (76) *which are of the same form as* (71).

This system of equations (76) may be simplified. Consider the determinant,

$$(77)\qquad \Delta = \begin{vmatrix} y'_{uv} & y'_u & y'_v & y' \\ y''_{uv} & y''_u & y''_v & y'' \\ y^{(3)}_{uv} & y^{(3)}_u & y^{(3)}_v & y^{(3)} \\ y^{(4)}_{uv} & y^{(4)}_u & y^{(4)}_v & y^{(4)} \end{vmatrix} = D(y_{uv},\, y_u,\, y_v,\, y).$$

We find

$$(78)\qquad \frac{\partial\Delta}{\partial u} = -4a\Delta, \qquad \frac{\partial\Delta}{\partial v} = -4b'\Delta.$$

But we have also

$$\frac{\partial a}{\partial v} = \frac{\partial b'}{\partial u},$$

according to the first equation of (30). We may put, therefore,

$$a = \frac{\partial p}{\partial u}, \qquad b' = \frac{\partial p}{\partial v},$$

so that (78) shows Δ to have the value

$$(79)\qquad \Delta = Ce^{-4p},$$

where C is a non-vanishing constant. In fact, if Δ were equal to zero there would be relations possible of the form

$$\alpha y^{(k)}_{uv} + \beta y^{(k)}_u + \gamma y^{(k)}_v + \delta y^{(k)} = 0 \qquad (k=1,2,3,4),$$

with non-vanishing coefficients. But this is impossible in the case under consideration in which the general solution of system (71) depends upon four arbitrary constants.

The coördinates of the plane p_y were proportional to λ', λ'', $\lambda^{(3)}$, $\lambda^{(4)}$. Let us define them as being equal to

$$Y^{(k)} = \frac{\lambda^{(k)}}{\sqrt{\Delta}} \qquad (k=1,2,3,4).$$

These four functions $Y^{(k)}$ will satisfy a system of differential equations obtained from (76) by the transformation

$$\lambda = \sqrt{\Delta}\, Y = e^{-2p} Y. \tag{80}$$

The resulting system of equations is

$$\begin{aligned} &Y_{uu} + 2aY_u - 2bY_v + (c + 2b_v - 4bb')Y = 0, \\ &Y_{vv} - 2a'Y_u + 2b'Y_v + (c' + 2a'_u - 4aa')Y = 0. \end{aligned} \tag{81}$$

This shall be called *the system adjoined to* (71). Denote its coefficients by $2\bar{a}$, $2\bar{b}$, etc. We have

$$\bar{a} = a, \qquad \bar{b} = -b, \qquad \bar{c} = c + 2b_v - 4bb',$$

$$\bar{a}' = -a', \qquad \bar{b}' = b', \qquad \bar{c}' = c' + 2a'_u - 4aa',$$

whence

$$a = \bar{a}, \qquad b = -\bar{b}, \qquad c = \bar{c} + 2\bar{b}_v - 4\bar{b}\bar{b}',$$

$$a' = -\bar{a}', \qquad b' = \bar{b}', \qquad c' = \bar{c}' + 2\bar{a}'_u - 4\bar{a}\bar{a}',$$

so that *the relation between* (71) *and* (81) *is reciprocal, i. e., each is the adjoint of the other.* The seminvariants of (81) are

$$\bar{a}', \bar{b}, \bar{f}, \bar{g},$$

which are respectively equal to the seminvariants

$$-a', -b, f, g$$

of (81).

System (81) *represents the same surface as* (71) *in plane- instead of in point-coördinates. But we may also interpret* Y', Y'', $Y^{(3)}$, $Y^{(4)}$ *as point-coördinates. In that case every integral surface of* (81) *would be dualistic to every integral surface of* (71).

It is clear, therefore, that *dualistic properties of any integral surface of* (71) *are analytically characterized by an invariant system of equations which remains unaltered if the signs of* a' *and* b *are changed.*

If we think of a surface as being at the same time described by a point and enveloped by its tangent planes we must consider the systems (71) and (81) simultaneously. A surface S may be *identically* self-dual; i. e., there may exist a dualistic transformation which converts it into itself in the particular way that every point P_y is converted into the plane p_y tangent to S at P_y and conversely. In that case (71) and (81) will be identical, so that

$$a' = b = 0.$$

The surface is then a quadric. In fact, we always have

$$y' Y' + y'' Y'' + y^{(3)} Y^{(3)} + y^{(4)} Y^{(4)} = 0,$$

and in this case $Y^{(k)}$ would have to be proportional to $y^{(k)}$ so that there will be a quadratic relation between $y' \cdots, y^{(4)}$. We may also see this in another way. If $a' = 0$, equations (71) show that the curves $u =$ const. are straight lines. If $b = 0$, the curves $v =$ const. are straight lines. But a quadric is the only surface which has two separate families of straight lines upon it.

We have, therefore, the following further result. *If either a' or b is equal to zero, the surface S is a ruled surface. If both a' and b vanish it is a quadric. Quadrics, moreover, are the only identically self-dual surfaces.*

We shall close this paper with the enunciation of a theorem which may be regarded as the *fundamental theorem* of the projective differential geometry of curved surfaces. Any non-developable surface being given

$$y^{(k)} = f^{(k)}(u, v) \qquad (k = 1, 2, 3, 4),$$

it determines a system of form (3) which may always be reduced to the intermediate or even the canonical form, from which the invariants a', b, h and k may be computed as functions of u and v. If on the other hand a', b, h and k are given as arbitrary functions of u and v, subject of course to the integrability conditions (34), the coefficients of the canonical form will be uniquely determined if a' and b are different from zero. In other words,

If the four invariants a', b, h, k are given as arbitrary functions of two variables u and v, subject to the integrability conditions (34), *and if moreover a' and b are not equal to zero, they determine a non-developable, non-ruled surface except for a projective transformation.*

It is not necessary to attempt to remove the restrictions of this theorem, as the author has already constructed a projective differential geometry of ruled and developable surfaces upon a different basis.

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