ON CERTAIN ISOTHERMIC SURFACES*

BY

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I. *Introduction.*

§ 1. *Isothermic Surfaces.* CHRISTOFFEL proposed the problem of determining all pairs of surfaces, S and S_1, for which a one-to-one correspondence of their points can be established, such that the tangent planes at corresponding points will be parallel and the angles between corresponding lines will be equal. His discussion † of the problem shows that, aside from minimal and certain imaginary surfaces, the only solutions are surfaces which, when referred to lines of curvature, have as the expressions for their linear elements, respectively,

$$ds^2 = \lambda(du^2 + dv^2), \qquad ds_1^2 = \frac{1}{\lambda}(du^2 + dv^2),$$

where λ is a function of u and v. CHRISTOFFEL's problem, then, reduces to that of finding all possible isothermic surfaces.

No great progress has been made in the investigation of these surfaces, although WEINGARTEN ‡ and DARBOUX § have shown that they can be defined by a non-linear partial differential equation of the fourth order. The only solutions carried to completion have been obtained by imposing one or more additional conditions; the resulting surfaces have thus been very particular solutions of the general problem.||

§ 2. *Problem discussed in this paper.* The characteristic form for the linear element of many of the isothermic surfaces, when referred to lines of curvature, is well known.

Those of the quadrics and BONNET surfaces ¶ are respectively

$$ds^2 = (u+v)\left(\frac{du^2}{U} + \frac{dv^2}{V}\right), \qquad ds'^2 = \frac{1}{(u+v)^2}\left(\frac{du^2}{U_1} + \frac{dv^2}{V_1}\right),$$

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† Crelle, vol. 57 (1867), pp. 218–228.

‡ Sitzungsberichte der Akademie der Wissenschaften zu Berlin, vol. 2 (1883), p. 1163.

§ *Théorie des surfaces*, vol. 2, p. 248.

|| DARBOUX, loc. cit., vol. 2, p. 246.

¶ BONNET, Journal de L'Ecole Polytechnique, vol. 42, pp. 121 and 132–151.

where U and U_1 are certain functions of u, and V and V_1 of v. It is clear from CHRISTOFFEL's results that there must be other surfaces corresponding respectively to the expressions

$$ds_1^2 = \frac{1}{(u+v)}\left(\frac{du^2}{U} + \frac{dv^2}{V}\right), \qquad ds'^2_1 = (u+v)^2\left(\frac{du^2}{U_1} + \frac{dv^2}{V_1}\right).$$

Evidently the four preceding expressions are included under the general form

$$(1) \qquad ds^2 = (u+v)^K\left(\frac{du^2}{U} + \frac{dv^2}{V}\right),$$

where K is an arbitrary constant, and U and V arbitrary functions of u and v, respectively.

We have undertaken in this paper to determine the surfaces whose linear elements can be written in this form, the lines of reference being lines of curvature. The remainder of this first section is given up to the analytical discussion of the equations involved. It appears that the only surfaces, with the exception of the plane and cylindrical surfaces, that satisfy the conditions of the problem, are those associated with the BONNET surface, the quadrics, or the sphere referred to confocal spherical ellipses.

The associates of the BONNET surfaces are discussed in section II, and the others in section III of this paper. In the latter section, also, attention is called to an apparent oversight of BONNET, occurring in his discussion of the characteristic form for the linear element of the quadrics.

§ 3. *Problem reduced to the solution of a single equation.* The problem of determining all surfaces having their linear element in the form (1), when the lines of reference are lines of curvature, reduces to that of finding all solutions of the following system of equations: *

$$(2) \qquad \frac{\partial M}{\partial v} - \frac{\partial N}{\partial u} = PQ, \qquad \frac{\partial P}{\partial v} + MQ = 0, \qquad \frac{\partial Q}{\partial u} - NP = 0,$$

where P and Q are functions of u and v to be determined, and M and N, for the linear element (1), have the values:

$$M = \frac{K\sqrt{\frac{V}{U}}}{2(u+v)}, \qquad N = \frac{K\sqrt{\frac{U}{V}}}{2(u+v)}.$$

From (2) it follows easily that

$$\frac{\partial(P^2+M^2)}{\partial v} = 2M\frac{\partial N}{\partial u}, \qquad \frac{\partial(Q^2+N^2)}{\partial u} = 2N\frac{\partial M}{\partial v}.$$

* BONNET, loc. cit., p. 133.

Substituting these values of M and N and integrating, we have

$$P^2 = \frac{K^2}{4}\left[\frac{U'}{(u+v)} - \frac{U+V}{(u+v)^2} + U_1\right], \qquad Q^2 = \frac{K^2}{4}\left[\frac{V'}{u+v} - \frac{(U+V)}{(u+v)^2} + V_1\right],$$

where the primes denote derivatives, and U_1 and V_1 are new functions of u and v respectively.

The first equation of (2) with these values substituted becomes

$$(3) \qquad \begin{aligned} K^2\left[U' - \frac{U+V}{u+v} + (u+v)U_1\right]\left[V' - \frac{U+V}{u+v} + (u+v)V_1\right] \\ = \left[U' + V' - 2\frac{U+V}{u+v}\right]^2, \end{aligned}$$

and the system (2) is solved as soon as all possible sets of values of K, U, U_1, V, V_1, which satisfy this equation, are found.

§ 4. *Discussion of equation* (3)*. In discussing equation (3) we shall assume that for some value, α, all the functions $U(\alpha)$, $U_1(\alpha)$, $V(-\alpha)$, and $V_1(-\alpha)$ are analytic.†

Multiplying both sides of (3) by $(u+v)^2$, and then replacing u by α and v by $-\alpha$, we have

$$(K^2 - 4)\{U(\alpha) + V(-\alpha)\} = 0.$$

Hence

$$1^\circ \quad K = \pm 2, \qquad \text{or} \qquad 2^\circ \quad U(\alpha) = -V(-\alpha).$$

Case 1°, when $K = -2$, has been discussed by BONNET.‡ The functions, as determined by him, which satisfy (3) when K has this value, will satisfy (3) evidently when K has the other value. We have § then, for $K = \pm 2$,

$$U = \alpha u^2 + \alpha' u + \alpha'', \qquad U_1 = -\tfrac{1}{2}U'',$$

$$V = \text{arbitrary function of } v, \qquad V_1 = \text{function of } v \text{ and } V.$$

Case 2° needs further consideration. Writing $U = f(u)$, we have

$$\begin{aligned} U(\alpha) &= f(\alpha), & V(-\alpha) &= -f(-\alpha), \\ U'(\alpha) &= f'(\alpha), & V'(-\alpha) &= f'(-\alpha), \\ U''(\alpha) &= f''(\alpha), & V''(-\alpha) &= -f''(-\alpha). \end{aligned}$$

* The following discussion of equation (3) was suggested by Professor J. E. WRIGHT.

† The question as to whether this assumption limits our results we shall not attempt to settle in this paper. The same point arises in connection with several other equations which the writer is studying and may be discussed by itself in another article.

‡ Loc. cit., p. 133–151.

§ This solution is not based upon the assumption at the beginning of this article with regard to the functions, but is the most general solution of (3) when $K^2 = 4$.

If u be replaced by α and v by $(-\alpha+h)$, then (3) may be written

$$(4)\qquad \begin{aligned} K^2[\tfrac{1}{2}f''h+U_1h-\tfrac{1}{6}f'''h^2+\cdots][-\tfrac{1}{2}f''h+\tfrac{1}{2}f'''h^2+\cdots \\ -\tfrac{1}{6}f'''h^2+\cdots+hV_1]=[\tfrac{1}{2}f'''h^2-\tfrac{1}{3}f'''h^2+\cdots]^2, \end{aligned}$$

where α is the argument of f, and the terms omitted contain cubes and higher powers of h.

Equating coefficients of h^2, we find

$$[\tfrac{1}{2}f''(\alpha)+U_1(\alpha)]\,[-\tfrac{1}{2}f''(\alpha)+V_1(-\alpha)]=0.$$

Assuming, then, that $U_1(\alpha)=-\tfrac{1}{2}f''(\alpha)$, and equating coefficients of h^3, we have

$$f'''(\alpha)\,[-\tfrac{1}{2}f''(\alpha)+V_1(-\alpha)]=0,$$

so that if $f'''(\alpha)\neq 0$, then $V_1(-\alpha)=\tfrac{1}{2}f''(\alpha)$, which is the condition we passed over in the preceding equation. Finally, the coefficients of h^4 in (4) give

$$(K^2-1)\,f'''^2(\alpha)=0.$$

Hence two cases are left for consideration:

$$\text{(i)}\qquad f'''(\alpha)=0,$$

and therefore

$$U=\alpha u^2+\alpha' u+\alpha'',\qquad V=-\alpha v^2+\alpha' v-\alpha'',$$

$$\text{(ii)}\qquad K=\pm 1,\qquad U_1=-\tfrac{1}{2}U'',\qquad V_1=-\tfrac{1}{2}V''.$$

Case (i) will be considered later.

Case (ii). In order to determine the forms of the functions U and V we shall replace U_1 and V_1 by the values given in (ii), and then write (3) in the form

$$(5)\qquad \begin{aligned} [(U+V)-U'(u+v)+\tfrac{1}{2}U''(u+v)^2][(U+V)-V'(u+v)+\tfrac{1}{2}V''(u+v)^2] \\ =[(U'+V')(u+v)-2(U+V)]^2. \end{aligned}$$

Giving v some constant value, for which V is analytic, and putting

$$(U+V)-V'(u+v)+\tfrac{1}{2}V''(u+v)^2=F,$$

we may write (5) in the form

$$F[F-(u+v)F'+\tfrac{1}{2}F''(u+v)^2]=[F'(u+v)-2F]^2;$$

which, by the substitution

$$F=\phi\cdot(u+v)^2,$$

becomes

$$\tfrac{1}{2}\phi\phi''+\frac{\phi\phi'}{(u+v)}=\phi'^2,$$

where the primes denote derivatives with regard to u. Integrating, we find

$$\frac{1}{\phi} = \frac{\psi(v)}{(u+v)} + \chi(v),$$

where ψ and χ are arbitrary functions of v, and hence

$$F = (U+V) - V'(u+v) + \tfrac{1}{2}V''(u+v)^2 = \frac{(u+v)^3}{Au+B},$$

where A and B are functions of v. Finally, solving for U, we have

$$U = a + bu + cu^2 + \frac{(u+d)^3}{hu+k},$$

where a, b, c, etc., are constants, as U must be independent of v. This result is based on the assumption that the function ψ does not vanish.

If $\psi = 0$, then $f''' = 0$, a case already considered. If $\psi \neq 0$, the constants h and k cannot both be zero at the same time.

If $h \neq 0$, with a new set of variables $(u - k/h)$ and $(v + k/h)$, the new function can be written in the form

$$U = \alpha + \alpha' u + \alpha'' u^2 + \frac{\alpha'''}{u}.$$

If $h = 0$ and $k \neq 0$, the expression for the function may be written

$$U = \alpha + \alpha' u + \alpha'' u^2 + \alpha''' u^3.$$

The solutions of equation (3), which have just been found, are as follows:

1°. $K =$ arbitrary constant, $U = \alpha u^2 + \alpha' u + \alpha''$, $V = -\alpha v^2 + \alpha' v - \alpha''$.

2°. $K = \pm 2$, $U = \alpha u^2 + \alpha' u + \alpha''$, $V =$ arbitrary function of v.

3° (*a*). $K = \pm 1$, $U = \alpha u^3 + \alpha' u^2 + \alpha'' u + \alpha'''$, $V = \alpha v^3 - \alpha' v^2 + \alpha'' v - \alpha'''$.

3° (*b*). $K = \pm 1$, $U = \alpha u^2 + \alpha' u + \alpha'' + \dfrac{\alpha'''}{u}$, $V = -\alpha v^2 + \alpha' v - \alpha'' + \dfrac{\alpha'''}{v}$.

II. *Associates of the* BONNET *surfaces.*

§ 6. *Solution* 1° *of equation* (3). The surfaces arising from the first solution of equation (3), § 4, are the plane and cylindrical surfaces.*

The plane corresponds to the linear element.

$$ds^2 = (u+v)^K \left(\frac{du^2}{\alpha u^2 + \alpha' u + \alpha''} + \frac{dv^2}{-\alpha v^2 + \alpha' v - \alpha''} \right)$$

for any value of K, the form of the lines of reference depending of course upon the value of this constant.

* The proof of these statements follows from equations (2) and (3) and the values of the direction cosines of the normals to the surface found in the usual way.

For $K=2$, one set is composed of trochoids, and the other of logarithmic curves, which include the catenary as a particular case.* Cylindrical surfaces correspond to the linear element above when $K=0$.

§ 7. *Solution* 2° *of equation* (3). In discussing the surfaces arising from the solution of equation (3) when $K=-2$, BONNET found that two cases should be considered, depending upon whether the constant α which appears in the expression for U vanishes or not.

The expressions † for the linear elements and the cartesian coördinates of the surfaces in the two cases, as found by him, are respectively

$$(6)\qquad ds^2=\frac{1}{(u+v)^2}\left[\frac{du^2}{\alpha' u-u^2}+\frac{dv^2}{V}\right],$$

$$x=\frac{2\sqrt{\alpha' u-u^2}}{\alpha'(u+v)},\qquad z=-\frac{2\sqrt{v^2+\alpha' v}}{\alpha'(u+v)}\sin V_3,$$

$$y=-\frac{2\sqrt{v^2+\alpha' v}}{\alpha'(u+v)}\cos V_3,\qquad V_3=\int\frac{4\sqrt{v^2+\alpha' v-V}}{2(v^2+\alpha' v)\sqrt{V}}dv;$$

$$(7)\qquad ds^2=\frac{1}{(u+v)^2}\left(\frac{du^2}{d'u}+\frac{dv^2}{V}\right),$$

$$x=2\sqrt{\frac{u}{\alpha'}}\frac{1}{(u+v)},\qquad z=-2\sqrt{\frac{v}{\alpha'}}\frac{\sin V_3}{(u+v)},$$

$$y=-2\sqrt{\frac{v}{\alpha'}}\frac{\cos V_3}{(u+v)},\qquad V_3=\int\frac{\sqrt{\alpha' v-V}}{2v\sqrt{V}}dv.$$

BONNET shows that the surfaces defined by (6) and (7) are characterized by having lines of curvature whose geodesic curvature is constant.‡ According to him, the surfaces defined by (6) are the envelopes of a variable sphere, whose center describes a plane curve, while its surface always passes through two fixed points, real or imaginary, which lie symmetrically with regard to the plane of the centers. In the case of the surfaces defined by (7), the two fixed points are coincident and lie in the plane of centers.

The cartesian coördinates (x_1, y_1, z_1) of the surfaces associated with (6) and (7) can be obtained by solving the system of equations,§

*See § 8.

† BONNET, loc. cit., p. 150. He gives only the expressions for the first case, but the others are easily derived from them.

‡ RIBAUCOUR has shown that the only surfaces having this property are cones, cylinders, and surfaces of revolution, together with surfaces obtained from these by inversion. (See DARBOUX, loc. cit., vol. 3, p. 122.) The surfaces (6) are the inverse of cones and surfaces of revolution, and (7) the inverse of cylindrical surfaces.

§ DARBOUX, loc. cit., vol. 2, p. 244.

$$\lambda \frac{\partial \theta_1}{\partial u} = \frac{\partial \theta}{\partial u}, \qquad \lambda \frac{\partial \theta_1}{\partial v} = -\frac{\partial \theta}{\partial v},$$

where λ, which has its usual meaning, is obtained in the one case, from the expression for the linear element in (6), and in the other (7), and θ and θ_1 are to be replaced by x, x_1, y, y_1 and z, z_1 successively, the values of x, y and z being taken from (6) or (7), as the case may be.

We have, then, for the surfaces associated with (6),

$$ds_1^2 = (u+v)^2 \left[\frac{du^2}{\alpha' u + u^2} + \frac{dv^2}{V} \right],$$

and

$$x_1 = \frac{2}{\alpha'} \sqrt{\alpha' u - u^2} \left(v + \frac{\alpha'}{2} \right) - \frac{\alpha'}{2} \operatorname{vers}^{-1} \frac{2u}{\alpha'},$$

$$(8) \qquad y_1 = -\frac{2}{\alpha'} \sqrt{v^2 + \alpha' v}\,(u+v) \cos V_3 + \frac{4}{\alpha'} \int \sqrt{v^2 + \alpha' v} \cos V_3 \, dv,$$

$$z_1 = \frac{2}{\alpha'} \sqrt{v^2 + \alpha' v}\,(u+v) \sin V_3 - \frac{4}{\alpha'} \int \sqrt{v^2 + \alpha' v} \sin V_3 \, dv,$$

where

$$V_3 = \frac{\alpha'}{2} \int \frac{\sqrt{v^2 + \alpha' v - V}}{(v^2 + \alpha' v)\sqrt{V}} dv.$$

Likewise for those associated with (7), we have

$$ds_1^2 = (u+v)^2 \left(\frac{du^2}{\alpha' u} + \frac{dv^2}{V} \right),$$

$$x_1 = 2 \sqrt{\frac{u}{\alpha'}} \left(\frac{u}{3} - v \right),$$

$$(9) \qquad y_1 = -2 \sqrt{\frac{v}{\alpha'}}\,(u+v) \cos V_3 + \frac{4}{\sqrt{\alpha'}} \int \sqrt{v} \cos V_3 \, dv,$$

$$z_1 = 2 \sqrt{\frac{v}{\alpha'}}\,(u+v) \sin V_3 - \frac{4}{\sqrt{\alpha'}} \int \sqrt{v} \sin V_3 \, dv,$$

where

$$V_3 = \int \frac{\sqrt{\alpha' v - V}}{2v\sqrt{V}} dv.$$

It follows from a theorem due to DARBOUX,* that any surface, S_1, defined by (8) or (9), can be so placed with regard to its associate surface, S, defined by (6) or (7), that at corresponding points the normals to the surfaces and the tangents to the lines of curvature will be respectively parallel.

* Loc. cit., vol. 2, p. 243.

The lines $v = \text{const.}$ on the BONNET surfaces are circles, therefore the corresponding lines on (8) and (9), namely, one set of the lines of curvature, must be plane.

The problem of determining isothermic surfaces having one set of the lines of curvature plane has been solved in its most general form by DARBOUX.* Of three possible exceptional or limiting cases,† which DARBOUX did not consider, one has been treated by ADAM,‡ another leads to the BONNET surfaces, and the third to their associates. This problem, then, is completely solved by the addition of the surfaces defined by (8) and (9).

§ 8. *Discussion of the associates of the Bonnet surfaces.* The principle of inversion applied to the surfaces (8) and (9) gives little information with regard to them, but we do know of course that the resulting surfaces will be isothermic and will have spherical lines for one set of the lines of curvature.

A discussion of equations (8) and (9) shows that the plane lines of curvature on the surfaces defined by them lie in a system of planes which envelope a cylindrical surface, the form of the latter depending entirely upon the form of the arbitrary function V. These lines are in fact the intersections of the planes and a system of cylindrical surfaces depending upon v as a parameter, whose elements are perpendicular to the elements of the cylinder enveloped by the planes.

For the surfaces defined by (8), the cross-section of these cylinders are either trochoids with their base lines parallel to the elements of the enveloped cylinder or curves whose equations are of the form

$$x = a\sqrt{2by + y^2} - \frac{\alpha_1}{2}\log\left(y + b + \sqrt{2by + y^2}\right),$$

according as the constant α' appearing in (8) is real or imaginary. The plane lines on the surfaces (9) are nodal cubic curves.

§ 9. *Surfaces having both sets of the lines of curvature plane.* The surfaces corresponding to the linear element

$$ds^2 = (u + v)^2\left(\frac{du^2}{\alpha u^2 + \alpha' u + \alpha''} + \frac{dv^2}{\beta v^2 + \beta' v + \beta''}\right),$$

when

* Excluding surfaces of revolution, DARBOUX reduced the general problem to the simultaneous solution of the equations

$$\frac{\partial h}{\partial u} = Ue^h + U_1 e^{-h}, \qquad \frac{\partial^2 h}{\partial u^2} + \frac{\partial^2 h}{\partial v^2} = 0,$$

where $e^h = \sqrt{\lambda}$ and U and U_1 are independent functions of u. He solved the problem when $U \neq U_1 \neq 0$, and ADAM considered the case $U = U_1$. The case $U_1 = 0$ leads to the BONNET surfaces, and $U = 0$ to their associates (DARBOUX, loc. cit., vol. 4, p. 222).

† See preceding note.

‡ Annales scientifiques de l'Ecole Normal supérieure, series 3, vol. 10 (1893), p. 319.

$$4(\alpha+\beta)(\alpha''+\beta'')-(\alpha'-\beta')^2=0,$$

have plane lines for both sets of the lines of curvature, since on the corresponding BONNET surfaces both sets are circles.* It is clear that surfaces obtained from these by inversion will be isothermic and that the lines of curvature will be spherical. The condition imposed upon the constants is satisfied when

1°. $\alpha=-1,\ \beta\neq 1,\ \alpha'\neq\beta',\ \beta''\neq 0,$
2°. $\alpha=-1,\ \beta=1,\ \alpha'=\beta',\ \beta''\neq 0,$
3°. $\alpha=-1,\ \beta\neq 1,\ \alpha'=\beta',\ \beta''=0,$
4°. $\alpha=0,\ \beta\neq 0,\ \alpha'\neq\beta',\ \beta''\neq 0,$
5°. $\alpha=0,\ \beta=0,\ \alpha'=\beta'.$

All other cases lead to imaginary surfaces.

In each of these cases, the integration indicated in (8) or (9) can be performed, the resulting functions being trigonometric or logarithmic. Case 1° is the general case. The lines $v=$ const. are the same as in the general case for (8), and the lines $u=$ const. are transcendental plane curves.

Case 2° leads to a minimal surface, which was first derived by DARBOUX † in connection with the general subject of minimal surfaces.

In case 3° the expression for the linear element becomes

$$ds^2=(u+v)^2\left(\frac{du^2}{\alpha' u-u^2}+\frac{dv^2}{\beta v^2+\alpha' v}\right),$$

and the cartesian coördinates are

$$\begin{aligned} x&=\frac{2}{\alpha'}\sqrt{\alpha' u-u^2}\cdot v-\int\frac{u\,du}{\sqrt{\alpha' u-u^2}},\\ y&=\frac{2}{\alpha'}\sqrt{\beta v^2+\beta' v}\cdot u-\int\frac{v\,dv}{\sqrt{\beta v^2+\beta' v}},\\ z&=\frac{2}{\alpha'}\sqrt{1-\beta}\cdot uv. \end{aligned}\tag{10}$$

The two systems of planes in which the lines of curvature lie envelope cylindrical surfaces whose elements are perpendicular to one another.

(α) If, however, the constant β is greater than unity, the only real points on the surface are along the lines $u=0$ and $v=0$.

(β) If $\beta=1$ the surface is a plane and the lines of reference are like those on the DARBOUX minimal surface of the preceding case.

* BONNET, loc. cit., pp. 151–2.
Loc. cit., vol. 1, p. 315.

(γ) If $\beta = 0$, the lines $u =$ const. are nodal cubic curves and the lines $v =$ const. are trochoids.

(δ) If $0 < \beta < 1$, the lines of curvature are similar to those on the DARBOUX minimal surface.

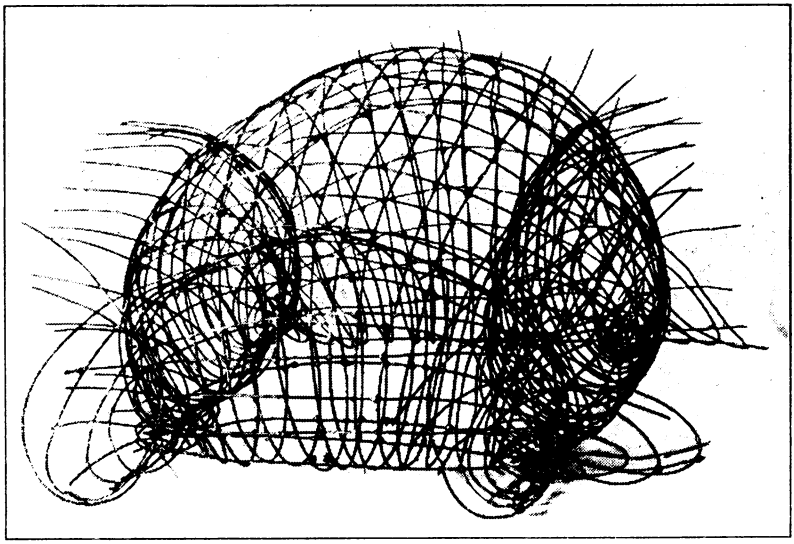
(ϵ) Finally, if β is negative, both sets of the lines of curvature are trochoids which have loops.

A model of one of these surfaces, the one corresponding to

$$ds^2 = (u + v)^2 \left(\frac{du^2}{2u - u^2} + \frac{dv^2}{2v - v^2} \right),$$

has been made by the writer. This surface is divided into segments corresponding to the repeating parts of the lines of curvature. On the four sides of each segment to which others are joined are "rolls," formed by the loops of the lines of curvature, which taper uniformly from the middle of the side of the section, and reduce to points at the ends.

The whole surface is of such a form that it might be placed between two parallel planes, in such a way that one plane would touch the surface in a system



of isolated points (circular points on the surface, each being the crest of a segment) and so that the other would touch it along a double system of straight lines, which divide the plane into a net-work of squares, the sides of each being the locus of the points common to the plane and one segment of the surface.

In case 4° one set of the lines of curvature on the surface are nodal cubic curves, and the other consists of trochoids, or logarithmic curves. Finally, case 5° leads to the well known minimal surface of ENNEPER, an algebraic surface

of the ninth order, on which both sets of the lines of curvature are nodal cubic curves.

III. *Surfaces associated with the quadrics and with the sphere referred to confocal ellipses.*

§ 10. *Solution* 3° (a) *of equation* (3). BONNET * undertook to prove that the only surfaces having as their linear element the expression

$$ds^2 = (u+v)\left(\frac{du^2}{U} + \frac{dv^2}{V}\right),$$

when referred to lines of curvature, were the quadrics; but in solving (3), for the case $K=1$, he overlooked † solution 3° (a) for this value of K, and considered merely the solution 3°(b). Since however the solution overlooked leads to the sphere (as may be shown by computing the total curvature), his final results are not affected.

The lines of reference for the sphere are confocal spherical ellipses.‡

The expressions for the cartesian coördinates of the minimal § surface associated with the sphere may be obtained from its coördinates by a simple integration. They are as follows:

$$(11)\qquad \begin{aligned} x_1 &= \frac{1}{\sqrt{(a-b)(a-c)}} \log\left(\frac{\sqrt{a-v}+\sqrt{a-u}}{\sqrt{u-v}}\right), \\ y_1 &= \frac{1}{\sqrt{(b-a)(b-c)}} \log\left(\frac{\sqrt{b-v}+\sqrt{b-u}}{\sqrt{u-v}}\right), \\ z_1 &= \frac{1}{\sqrt{(c-a)(c-b)}} \log\left(\frac{\sqrt{v-c}+\sqrt{u-c}}{\sqrt{u-v}}\right), \end{aligned}$$

where $c < v < b < u < a$, and where the corresponding linear element is

$$ds_1^2 = \frac{1}{(u-v)}\left(\frac{du^2}{(u-a)(u-b)(u-c)} + \frac{dv^2}{-(v-a)(v-b)(v-c)}\right).$$

§ 11. *Solutions* 3° (b) *of equation* (3). BONNET has shown that the surfaces corresponding to this solution of (3) when $K=1$ are the quadrics.

We obtain the expressions for the cartesian coördinates of the associated surfaces by integration, from those obtained by him for the quadrics,‖ in the usual

* Loc. cit., p. 118–132.

† Loc. cit., p. 126.

‡ DARBOUX, loc. cit., vol. 1, p. 242.

§ DARBOUX, loc. cit., vol. 2, p. 244.

‖ Loc. cit., p. 115.

way. Corresponding to the linear element written in the form

$$ds_1^2 = \frac{1}{\mu^2 - \nu^2}\left(\frac{(\rho^2 - \mu^2)\,d\mu^2}{(\mu^2 - b^2)(c^2 - \mu^2)} + \frac{(\rho^2 - \nu^2)\,d\nu^2}{(b^2 - \nu^2)(c^2 - \nu^2)}\right),$$

we find the following systems of equations:

$$x_1 = \frac{\rho}{2bc}\log\left(\frac{\mu - \nu}{\mu + \nu}\right),$$

$$y_1 = \frac{1}{b}\sqrt{\frac{\rho^2 - b^2}{c^2 - b^2}}\sin^{-1}\sqrt{\frac{\mu^2 - b^2}{\mu^2 - \nu^2}},$$

$$z_1 = \frac{1}{2c}\sqrt{\frac{\rho^2 - c^2}{c^2 - b^2}}\log\left(\frac{\sqrt{c^2 - \nu^2} + \sqrt{c^2 - \mu^2}}{\sqrt{c^2 - \nu^2} - \sqrt{c^2 - \mu^2}}\right),$$

$$\nu^2 < b^2 < \mu^2 < c^2 < \rho^2;$$

$$x_1 = \frac{\rho}{2bc}\log\left(\frac{\mu - \nu}{\mu + \nu}\right),$$

$$y_1 = \frac{1}{b}\sqrt{\frac{\rho^2 - b^2}{c^2 - b^2}}\sin^{-1}\sqrt{\frac{\mu^2 - b^2}{b^2 - \nu^2}},$$

$$z_1 = \frac{1}{c}\sqrt{\frac{c^2 - \rho^2}{c^2 - b^2}}\sin^{-1}\sqrt{\frac{\mu^2 - c^2}{\mu^2 - \nu^2}},$$

$$\nu^2 < b^2 < \rho^2 < c^2 < \mu^2;$$

$$x_1 = \frac{\rho}{2bc}\log\left(\frac{\nu - \mu}{\nu + \mu}\right),$$

$$y_1 = \frac{1}{2b}\sqrt{\frac{\rho^2 - b^2}{c^2 - b^2}}\log\left(\frac{\sqrt{\nu^2 - b^2} + \sqrt{\mu^2 - b^2}}{\sqrt{\nu^2 - b^2} - \sqrt{\mu^2 - b^2}}\right),$$

$$z_1 = \frac{1}{c}\sqrt{\frac{c^2 - \rho^2}{c^2 - b^2}}\sin^{-1}\sqrt{\frac{c^2 - \mu^2}{\nu^2 - \mu^2}},$$

$$\rho^2 < b^2 < \mu^2 < c^2 < \nu^2.$$

The surfaces defined by the last three sets of equations correspond respectively to the ellipsoid, the hyperboloid of one sheet, and the hyperboloid of two sheets. The parameters can be eliminated in each of the three cases, giving an equation of the form $f(x_1, y_1, z_1) = 0$, where the non-algebraic functions involved are trigonometric or logarithmic.

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