FURTHER NOTE ON MACLAURIN'S SPHEROID*

BY

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In volume 4 (1903) of these Transactions a paper of mine was published on the approximate determination of Maclaurin's ellipsoid. It was there shown that it is possible to determine the form of the ellipsoid and its angular velocity by means of spherical harmonic analysis to a higher order of approximation than is usually supposed to be attainable by that method.

It appears that we ought to be able by means of the considerations adduced to determine the angular velocity corresponding to a given ellipticity as far as the cube of that ellipticity, and I was much puzzled to explain why I failed to obtain correctly the term involving the cube.

On page 117 of the paper I gave reasons for believing that the determination of this last term would prove to be illusory. Having had occasion recently to examine carefully another piece of work I saw that the reason given was in its turn fallacious, and that this last term should be attainable. Accordingly I looked carefully over the analysis to see whether there were any terms missing in my calculation, and having found the source of the discrepancy I propose in this note to supply the defect. No attempt will be made to make this new investigation complete, so that the present paper will only be intelligible in conjunction with the previous one.

The equation to the spheroid which is to be made a figure of equilibrium under rotation ω was written

$$\tau = -eS_2 - fS_4 - \sum f_i^e S_i^e,$$

where $\tau = (a^3 - r^3)/3a^3$, $S_2 = \frac{1}{3} - \mu^2$, and S_4 and S_i^e are the spherical surface harmonics of μ the cosine of colatitude from the axis of rotation.

On page 118 I showed that the potential of the sphere from which the ellipsoid is derived is

$$V = \tfrac{2}{3}\pi\rho(3a^2 - r^2),$$

and I put $r^2 = a^2(1 - 2\tau - \tau^2)$. Now one more term in the development of r^2 should have been included, and I ought to have written

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$$r^2 = a^2(1 - 2\tau - \tau^2 - \tfrac{4}{3}\tau^3 \cdots).$$

In order to evaluate the additional term in the part of the energy denoted $-SR$ another integral is required, besides those given on page 117.

It is, say,

$$\sigma_4 = \frac{3}{4\pi}\int (S_2)^4 d\sigma.$$

By the results given on page 117 for $(S_2)^2$ we have

$$\sigma_4 = \frac{3}{4\pi}\int \left[\frac{\omega_0}{\phi_0} S_0 + \frac{\omega_2}{\phi_2} S_2 + \frac{\omega_4}{\phi_4} S_4\right]^2 d\sigma$$

$$= \frac{(\omega_0)^2}{\phi_0} + \frac{(\omega_2)^2}{\phi_2} + \frac{(\omega_4)^2}{\phi_4}.$$

With the numerical values given on the same page, I find

$$\sigma_4 = \frac{16}{3^2.5.7}.$$

Returning to page 118 it is easy to see that the additional term in SR is $(M^2/a)\frac{1}{6}e^4\sigma_4$.

Hence the additional term in equation (5) which gives the value of $\frac{1}{2}SS - SR$ is $(M^2/a)(-\frac{1}{6}e^4\sigma_4)$.

On page 119 I avowedly dropped a term in CR, viz.:

$$-\frac{9M^2}{4\pi a}\int \frac{1}{3.5} e^4 (S_2)^4 d\sigma = \frac{M^2}{a}\left[-\frac{1}{5} e^4 \sigma_4\right].$$

Next on page 127 I avowedly dropped a term in the energy $\frac{1}{2}DD$, viz.:

$$\frac{1}{2}\frac{M\rho}{a}\int a^3 e^4 (S_2)^4 d\sigma = \frac{M^2}{a}\left[\frac{1}{2} e^4 \sigma_4\right].$$

Lastly on page 129 I omitted the whole of the energy given by the formula (12). It is

$$\frac{M^2}{a}\left[-\frac{9}{20}\frac{(\omega_2)^2}{\phi_2} - \frac{5}{6}\frac{(\omega_4)^2}{\phi_4}\right] e^4.$$

Thus all the omitted portions together are

$$\frac{M^2}{a}\left[\frac{2}{3.5}\sigma_4 - \frac{9}{20}\frac{(\omega_2)^2}{\phi_2} - \frac{5}{6}\frac{(\omega_4)^2}{\phi_4}\right] e^4.$$

With the numerical values given above the additional term in the lost energy

of the system (omitted because I thought terms in e^4 would lead to no further accuracy) is found to be

$$\frac{M^2}{a}\left[-\frac{16}{3^3.7^2}e^4\right].$$

In calculating the moment of inertia C_r on page 129, I omitted purposely

$$-\frac{3M^2}{4\pi a}\cdot\frac{3}{4\pi\rho}\int\left[\tfrac{2}{3}+S_2\right]\left[-\tfrac{1}{3}e^3(S_2)^2\right]d\sigma^* = \frac{3M^2e^3}{4\pi\rho a}\left(\tfrac{2}{9}\omega_2+\tfrac{1}{3}\sigma_4\right).$$

Hence this with its sign changed and multiplied by $\frac{1}{2}\omega^2$ is an additional term in (14) giving $\frac{1}{2}C\omega^2$. Thus (14) should run

$$\tfrac{1}{2}C\omega^2 = \frac{M^2}{a}\frac{\omega^2}{4\pi\rho}\left[\tfrac{3}{5}+\tfrac{3}{2}e\phi_2+e^2\left(\phi_2+\tfrac{3}{2}\omega_2\right)-e^3\left(\tfrac{1}{3}\omega_2+\tfrac{1}{2}\sigma_4\right)+3ef\omega_4\right].$$

It is convenient to insert numerical values throughout, instead of retaining symbols as I did before.

I find then that the first formula of § 7 will run

$$E\div\frac{M^2}{a} = -\frac{4}{3.5^2}e^2-\frac{64}{3^3.5^2.7}e^3-\frac{16}{3^3.7^2}e^4+\frac{104}{3^2.5^2.7}e^2f-\frac{1}{3^2}f^2-\Sigma\frac{i-1}{2i+1}(f_i^s)^2\phi_i^s$$
$$+\frac{\omega^2}{4\pi\rho}\left[\frac{3}{5}+\frac{2}{5}e+\frac{4}{3.7}e^2-\frac{8}{3^3.5.7}e^3+\frac{8}{5.7}ef\right].$$

In this the terms in e^4 in the first part and in e^3 in the second part are the additional terms now included.

Since $\partial E/\partial f_i^s = 0$ is one of the conditions for the figure of equilibrium, it follows at once that $f_i^s = 0$, for all values of i and s.

By neglecting all terms above those of the second order, and putting $\partial E/\partial e = 0$, we see that as a first approximation $\omega^2/4\pi\rho = 4e/3.5$ On putting $\partial E/\partial f = 0$ we obtain

$$\frac{104}{3^2.5^2.7}e^2-\frac{2}{3^2}f+\frac{\omega^2}{4\pi\rho}\frac{8}{5.7}e = 0.$$

Then by means of the first approximation for $\omega^2/4\pi\rho$, we find

$$f = \frac{4}{7}e^2.$$

On substituting this value for f in the expression for the energy, we have

$$E\div\frac{M^2}{a} = -\frac{4}{3.5^2}e^2-\frac{64}{3^3.5^2.7}e^3-\frac{352}{3^3.5^2.7^2}e^4+\frac{\omega^2}{4\pi\rho}\left[\frac{3}{5}+\frac{2}{5}e+\frac{4}{3.7}e^2+\frac{808}{3^3.5.7^2}e^3\right].$$

* In line 7 from foot of page 129 there is a misprint; the double integral should obviously be a single one, and $d\tau$ should be deleted.

On equating $\partial E/\partial e$ to zero we find

$$\frac{4}{3.5}e\left(1+\frac{8}{3.7}e+\frac{176}{3^2.7^2}e^2\right)=\frac{\omega^2}{4\pi\rho}\left[1+\frac{20}{3.7}e+\frac{404}{3^2.7^2}e^2\right].$$

Whence

$$\frac{\omega^2}{4\pi\rho}=\frac{4}{3.5}e\left(1-\frac{4}{7}e+\frac{4}{3.7^2}e^2\right).$$

Only the first two terms were evaluated in the former paper, and this is the required new result.

The equation to the ellipsoid, as calculated in this way, is

$$\frac{r^3}{a^3}=1+3\left(eS_2+\tfrac{4}{7}e^2S_4\right).$$

The method then will give the coefficient of S_4 only as far as e^2, and fails to give the coefficient of S_6. Nevertheless it gives the square of the angular velocity as far as e^3.

It remains to verify the correctness of our result.

It may be proved, as in the former paper, that the equation to an ellipsoid whose equatorial and polar semi-axes are a_1 and $a_1(1-e_1)$ may be written in the form

$$r^3=a_1^3(1-e_1)\left[1+3\left(e_1+\tfrac{5}{14}e_1^2+\tfrac{2}{21}e_1^3\right)S_2+\tfrac{12}{7}e_1^2\left(1+\tfrac{9}{11}e_1\right)S_4-\tfrac{40}{33}e_1^3S_6\right].$$

Now a denotes the mean radius of our spheroid, so that

$$a^3=a_1^3(1-e_1).$$

Since $3e$, $3f$, and say $3g$ are to be the coefficients S_2, S_4, S_6, in the equation to the ellipsoid

$$r^3=a^3\left[1+3eS_2+3fS_4+3gS_6\right],$$

we have

$$e=e_1+\tfrac{5}{14}e_1^2+\tfrac{2}{21}e_1^3,$$

$$f=\tfrac{4}{7}e_1\left(1+\tfrac{9}{11}e_1\right),$$

$$g=-\tfrac{40}{99}e_1^3.$$

By inverting the expression for e, we find

$$e_1=e-\frac{5}{14}e^2+\frac{47}{2.3.7^2}e^3,$$

$$f=\frac{4}{7}e^2\left(1+\frac{8}{77}e\right),$$

$$g=-\frac{40}{99}e^3.$$

If the eccentricity of the ellipsoid be denoted by $\sin\gamma$, we have

$$\cos\gamma = 1 - e_1, \qquad \sin^2\gamma = 2e_1\left(1 - \frac{1}{2}e_1\right).$$

Hence

$$\cos\gamma\sin^2\gamma = 2e_1\left(1 - \frac{3}{2}e_1 + \frac{1}{2}e_1^2\right) = 2e\left(1 - \frac{13}{7}e + \frac{509}{2.3.7^2}e^2\right),$$

$$\cos\gamma\sin^4\gamma = 4e_2^1(1 - 2e_1) = 4e^2\left(1 - \frac{19}{7}e\right),$$

$$\cos\gamma\sin^6\gamma = 8e_1^3 = 8e^3.$$

Now the rigorous solution for the angular velocity of Maclaurin's ellipsoid is

$$\frac{\omega^2}{4\pi\rho} = \cos\gamma\sum_1^\infty \frac{(2n-1)!\sin^{2n}\gamma}{(2n+1)(2n+3)[(n-1)!]^2 2^{2n-3}}$$

$$= \frac{2}{3.5}\cos\gamma\sin^2\gamma + \frac{3}{5.7}\cos\gamma\sin^4\gamma + \frac{5}{2^2.3.7}\cos\gamma\sin^6\gamma\cdots.$$

Substituting for γ in terms of e, we find

$$\frac{\omega^2}{4\pi\rho} = \frac{4}{3.5}e\left(1 - \frac{4}{7}e + \frac{4}{3.7^2}e^2\right).$$

This agrees, as it should, with the result found by means of spherical harmonic analysis.

I do not think that the additional term found suffices to assign a limit to the stability of the ellipsoid, as I conjectured that it would do.

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