PROJECTIVE DIFFERENTIAL GEOMETRY OF CURVED SURFACES*

(SECOND MEMOIR)

BY

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§ 1. *Introduction.*

In the first memoir it was shown † that the projective differential geometry of a surface may be based upon a system of two linear partial differential equations of the second order, which may be reduced to the canonical form

$$(1)\qquad \begin{aligned} y_{uu} + 2by_v + fy &= 0, \\ y_{vv} + 2a'y_u + gy &= 0, \end{aligned}$$

the coefficients of which are seminvariants; the curves $u =$ const. and $v =$ const. will then be asymptotic lines of its integral surface S_y. The integrability conditions of system (1)

$$(2)\qquad \begin{aligned} a'_{uu} + g_u + 2ba'_v + 4a'b_v &= 0, \\ b_{vv} + f_v + 2a'b_u + 4ba'_u &= 0, \\ g_{uu} - f_{vv} - 4fa'_u - 2a'f_u + 4gb_v + 2bg_v &= 0, \end{aligned}$$

are supposed to be satisfied, so that (1) has just four linearly independent solutions, y', y'', $y^{(3)}$, $y^{(4)}$, which are interpreted as the homogeneous coordinates of a point P_y of the surface S_y. We shall usually denote the surface S_y by S whenever there is no chance for confusion arising from the suppression of the index.

The semi-covariants of (1) become

$$(3)\qquad y = y,\ z = y_u,\ \rho = y_v,\ \sigma = y_{uv}.$$

If four linearly independent solutions y', y'', $y^{(3)}$, $y^{(4)}$, of (1) be put for y into these expressions, there will be found three points P_z, P_ρ, P_σ semi-covariantly connected with the point P_y. These three points, together with P_y, form a

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† These Transactions, vol. 8 (1907), pp. 233–260.

non-degenerate tetrahedron, except for certain singular points. For, if they were coplanar $y', \cdots, y^{(4)}$ would satisfy an equation of the form

$$\alpha y_{uv} + \beta y_u + \gamma y_v + \delta y = 0,$$

which could have at most three linearly independent solutions in common with (1).

We shall make use of this fact very frequently in the following considerations for the purpose of making a detailed study of the surface in the vicinity of a point P_y. In fact we shall use the tetrahedron $P_y P_z P_\rho P_\sigma$ as a tetrahedron of reference, determining the coordinates of any point with respect to it in the following way. An expression of the form

$$\lambda = \alpha y + \beta z + \gamma \rho + \delta \sigma$$

assumes four values $\lambda', \cdots, \lambda^{(4)}$ corresponding to the four values of y. The coordinates of the point P_λ referred to the tetrahedron of reference $P_y P_z P_\rho P_\sigma$ shall be defined by the equations

$$x_1 = \alpha, \; x_2 = \beta, \; x_3 = \gamma, \; x_4 = \delta.$$

§ 2. *The two congruences of asymptotic tangents and their ruled surfaces.*

At each point P_y of the surface there are two asymptotic tangents, which join P_y to P_z and P_ρ respectively. Thus there are two congruences both of which have S as focal surface, the two families of asymptotic lines on S being the cuspidal edges of the developables of these two congruences. We shall speak of one of the two asymptotic tangents through P_y as being of the *first* or *second kind* according as it contains P_z or P_ρ. The corresponding congruences may be distinguished in a like manner.

Let us consider the congruence of asymptotic tangents of the first kind. The line $P_y P_z$ is one of its lines and, as u and v assume all of their values, this line describes the congruence. If u and v be chosen as functions of a single variable t,

$$u = \phi(t), \qquad v = \psi(t),$$

the line $P_y P_z$ will describe a ruled surface of this congruence, whose differential equations may be set up without great difficulty. For our purpose it suffices however to consider certain special ruled surfaces of this congruence. Let $u = u_0$ be the particular asymptotic curve of the second kind which passes through the point P_y. Let a point P move along this curve and construct the asymptotic tangent of the first kind of the point P in each of its positions. The ruled surface thus obtained, generated by $P_y P_z$ as u remains constant, while v assumes all of its values, shall be called *the osculating ruled surface of*

the first kind. It is the locus of the asymptotic tangents of the first kind along an asymptotic curve of the second kind. We shall denote it by R_1. The osculating ruled surface of the second kind R_2 is generated by $P_y P_\rho$ as v remains constant while u assumes all of its values.

We proceed to find the differential equations which characterize R_1. From equations (1) and (3) we find by differentiation

$$(4)\qquad \begin{aligned} y_v &= \rho, & y_{vv} &= -gy - 2a'z, \\ z_v &= \sigma, & z_{vv} &= (2a'f - g_u)y - (g + 2a'_u)z + 4a'b\rho, \end{aligned}$$

whence

$$(5)\qquad \begin{aligned} &y_{vv} + gy + 2a'z = 0, \\ &z_{vv} - 4a'by_v + (g_u - 2a'f)y + (g + 2a'_u)z = 0, \end{aligned}$$

the required equations.

These equations are of the form

$$(6)\qquad \begin{aligned} &y_{vv} + p_{11}y_v + p_{12}z_v + q_{11}y + q_{12}z = 0, \\ &z_{vv} + p_{21}y_v + p_{22}z_v + q_{21}y + q_{22}z = 0, \end{aligned}$$

where

$$(7)\qquad \begin{aligned} &p_{11} = 0, \quad p_{12} = 0, \quad q_{11} = g, \quad q_{12} = 2a', \\ &p_{21} = -4a'b, \quad p_{22} = 0, \quad q_{21} = g_u - 2a'f, \quad q_{22} = g + 2a'_u, \end{aligned}$$

and therefore serve to characterize the ruled surface R_1 generated by $P_y P_z$ when v alone varies.* The equation $p_{12} = 0$ shows that the curve C_y (the locus of P_y as v assumes its values), which is an asymptotic curve on the surface S, is also an asymptotic curve on R_1.† The curve C_z is a curve on R_1 whose exact geometrical significance will become apparent later.

The invariants of the ruled surface R_1 are expressible in terms of the coefficients (7). One of them, which will be needed in a later paragraph, is

$$(8)\qquad \theta = (u_{11} - u_{22})^2 + 4u_{12}u_{21}$$

where

$$(9)\qquad \begin{aligned} &u_{11} = -4g, \quad u_{12} = -8a', \quad u_{22} = -4g - 8a'_u \\ &u_{21} = -4g_u + 8a'f - 8a'_v b - 8a'b_v = 4a'_{uu} + 8a'f + 8a'b_v. \ddagger \end{aligned}$$

Its vanishing expresses the condition that the two branches of the flecnode curve of R_1 coincide. The equality of the two expressions for u_{21} is a consequence of (2).

* E. J. WILCZYNSKI, *Projective differential geometry of curves and ruled surfaces.* Teubner, Leipzig, 1906, p. 130. We shall hereafter refer to this work as Proj. Diff. Geom.

† Proj. Diff. Geom., p. 142.

‡ Cf. Prof. Diff. Geom., p. 96, eq. (20) for definition of the quantities u_{ik}, and p. 110, eq. (75) for the invariant θ, there denoted by θ_4. Cf. p. 150 for the geometrical significance of θ_4.

The fundamental semi-covariants of (6) are

$$(10)\qquad \begin{aligned} \mathfrak{r} &= 2y_v + p_{11}y + p_{12}z = 2\rho, \\ \mathfrak{s} &= 2z_v + p_{21}y + p_{22}z = -4a'by + 2\sigma.* \end{aligned}$$

The lines $P_yP_\mathfrak{r}$ and $P_yP_\mathfrak{s}$ are asymptotic tangents to the ruled surface R_1 along its generator P_yP_z. The asymptotic tangent to R_1 at the point $\alpha y + \beta z$ of P_yP_z joins this point to $\alpha\mathfrak{r} + \beta\mathfrak{s}$.† The homogeneous coördinates of an arbitrary point on one of these asymptotic tangents, i. e., the coördinates of an arbitrary point on the hyperboloid H_1, which osculates R_1 along P_yP_z, will therefore be

$$\lambda(\alpha y + \beta z) + \mu(\alpha\mathfrak{r} + \beta\mathfrak{s}) = (\alpha\lambda - 4a'b\mu\beta)y + \lambda\beta z + 2\mu\alpha\rho + 2\mu\beta\sigma.$$

If we introduce the system of coördinates referred to in the introduction, which has $P_yP_zP_\rho P_\sigma$ as its fundamental tetrahedron, we find

$$x_1 = \alpha\lambda - 4a'b\beta\mu,\ x_2 = \beta\lambda,\ x_3 = 2\alpha\mu,\ x_4 = 2\beta\mu,$$

as the coördinates of an arbitrary point on H_1. The equation of H_1 with respect to this tetrahedron of reference is therefore,

$$(11)\qquad x_1x_4 - x_2x_3 + 2a'bx_4^2 = 0.$$

The osculating ruled surface of the second kind R_2 is characterized by the equations

$$(12)\qquad \begin{aligned} y_{uu} + p'_{11}y_u + p'_{12}\rho_u + q'_{11}y + q'_{12}\rho &= 0, \\ \rho_{uu} + p'_{21}y_u + p'_{22}\rho_u + q'_{21}y + q'_{22}\rho &= 0, \end{aligned}$$

where

$$(13)\qquad \begin{aligned} &p'_{11} = 0, \quad p'_{12} = 0,\ q'_{11} = f, \quad q'_{12} = 2b, \\ &p'_{21} = -4a'b,\ p'_{22} = 0,\ q'_{21} = f_v - 2bg,\ q'_{22} = f + 2b_v. \end{aligned}$$

The invariant θ' whose vanishing expresses the coincidence of the two branches of the flecnode curve of R_2, is

$$(14)\qquad \theta' = (u'_{11} - u'_{22})^2 + 4u'_{12}u'_{21},$$

where

$$(15)\qquad \begin{aligned} &u'_{11} = -4f, \quad u'_{12} = -8b, \quad u'_{22} = -4f - 8b_v, \\ &u'_{21} = -4f_v + 8bg - 8a'_ub - 8a'b_u = 4b_{vv} + 8bg + 8ba'_u. \end{aligned}$$

The fundamental semi-covariants of R_2 are

$$(16)\qquad \mathfrak{r}' = 2z, \qquad \mathfrak{s}' = -4a'by + 2\sigma = \mathfrak{s}.$$

* Proj. Diff. Geom., p. 124. The quantities $\mathfrak{r}$ and $\mathfrak{s}$ are there denoted by ρ and σ.

† For a proof of this statement cf. Proj. Diff. Geom., p. 146.

The equation of the hyperboloid H_2 which osculates R_2 along $P_y P_\rho$ is found to be (11). Therefore: *the same hyperboloid which osculates the osculating ruled surface of the first kind also osculates the osculating ruled surface of the second kind. The two sets of generators on this hyperboloid are the asymptotic tangents of the two osculating ruled surfaces.*

§ 3. *The osculating linear complexes of the osculating ruled surfaces.*

Consider the ruled surface R_1, which is determined by the system of equations (6). We proceed to determine its osculating linear complex. According to the general theory of ruled surfaces, this may be done as follows.* Let θ be different from zero. Then the flecnode curve of R_1 has two distinct branches C_η and C_ζ, where

$$y = \alpha\eta + \beta\zeta, \qquad z = \gamma\eta + \delta\zeta, \qquad \alpha\delta - \beta\gamma \neq 0, \tag{17}$$

and where δ/β and γ/α are the two (distinct) roots of the quadratic

$$u_{12}\lambda^2 + (u_{11} - u_{22})\lambda - u_{21} = 0. \tag{18}$$

Perform the transformation (17), (18). The surface R_1 is then referred to its flecnode curve. Let the result of this transformation upon (6) be the system

$$\begin{aligned} \eta_{vv} + \pi_{11}\eta_v + \pi_{12}\zeta_v + \kappa_{11}\eta + \kappa_{12}\zeta &= 0, \\ \zeta_{vv} + \pi_{21}\eta_v + \pi_{22}\zeta_v + \kappa_{21}\eta + \kappa_{22}\zeta &= 0. \end{aligned} \tag{19}$$

Let r and s be the fundamental semi-covariants of this system, i. e., let

$$r = 2\eta_v + \pi_{11}\eta + \pi_{12}\zeta, \qquad s = 2\zeta_v + \pi_{21}\eta + \pi_{22}\zeta. \tag{20}$$

Referred to the tetrahedron $P_\eta P_\zeta P_r P_s$, the equation of the osculating linear complex will be

$$\pi_{12}\bar{\omega}_{13} + \pi_{21}\bar{\omega}_{42} = 0, \tag{21}$$

$\bar{\omega}_{ik}$ being the line coordinates with reference to this special tetrahedron.†

Substituting the values (9) in (18) we find that we may take in (17)

$$\begin{aligned} \alpha &= 16a', \qquad \beta = 16a', \\ \lambda &= 8a'_u + \sqrt{\theta}, \quad \delta = 8a'_u - \sqrt{\theta}, \end{aligned} \tag{22}$$

whence

$$\Delta = \alpha\delta - \beta\gamma = -32a'\sqrt{\theta}. \tag{23}$$

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* Proj. Diff. Geom., p. 150. Another proof, independent of the general theory of ruled surfaces will also be given.

† Proj. Diff. Geom., p. 206.

We find, consequently, either by direct substitution or by making use of the general formulæ of the theory of ruled surfaces,*

$$
(24)\qquad
\begin{aligned}
\pi_{11} &= \frac{a'_v}{a'} + \frac{C}{\sqrt{\theta}} + \frac{\theta_v}{2\theta},\\
\pi_{12} &= \frac{a'_v}{a'} + \frac{C}{\sqrt{\theta}} - \frac{\theta_v}{2\theta},\\
\pi_{21} &= \frac{a'_v}{a'} - \frac{C}{\sqrt{\theta}} - \frac{\theta_v}{2\theta},\\
\pi_{22} &= \frac{a'_v}{a'} - \frac{C}{\sqrt{\theta}} + \frac{\theta_v}{2\theta},
\end{aligned}
$$

where

$$
(25)\qquad C = 8a'_{uv} - 8\frac{a'_u a'_v}{a'} - 32a'^2 b.
$$

It remains to find the relation between the original tetrahedron $P_y P_z P_\rho P_\sigma$ and the new one $P_\eta P_\zeta P_r P_s$.

We have, in the first place, from (17) and (22),

$$
(26)\qquad
\begin{aligned}
y &= 16a'\eta + 16a'\zeta,\\
z &= B\eta + A\zeta,
\end{aligned}
$$

where we have put

$$
(27)\qquad A = 8a'_u - \sqrt{\theta}, \qquad B = 8a'_u + \sqrt{\theta},
$$

for abbreviation. Therefore

$$
\begin{aligned}
-32a'\sqrt{\theta}\eta &= Ay - 16a'z,\\
-32a'\sqrt{\theta}\zeta &= -By + 16a'z.
\end{aligned}
$$

It is known moreover that a transformation of form (17) gives rise to a cogredient transformation of the fundamental semi-covariants $\mathfrak{r}$, $\mathfrak{s}$ into r, s.† We shall, therefore, have

$$
\begin{aligned}
-32a'\sqrt{\theta}r &= A\mathfrak{r} - 16a'\mathfrak{s},\\
-32a'\sqrt{\theta}s &= -B\mathfrak{r} + 16a'\mathfrak{s}.
\end{aligned}
$$

Consequently we find

$$
(28)\qquad
\begin{aligned}
-32a'\sqrt{\theta}\eta &= Ay - 16a'z,\\
-32a'\sqrt{\theta}\zeta &= -By + 16a'z,\\
-32a'\sqrt{\theta}r &= 64a'^2 by + 2A\rho - 32a'\sigma,\\
-32a'\sqrt{\theta}s &= -64a'^2 by - 2B\rho + 32a'\sigma.
\end{aligned}
$$

* Proj. Diff. Geom., p. 102.

† Proj. Diff. Geom., p. 124.

Let $x_1, \cdots, x_4$ be the coördinates of a point referred to the tetrahedron $P_y P_z P_\rho P_\sigma$ and let $\bar{x}_1, \cdots, \bar{x}_4$ be the coördinates of the same point with respect to the tetrahedron $P_\eta P_\zeta P_r P_s$. Then, the two expressions

$$x_1 y + x_2 z + x_3 \rho + x_4 \sigma \qquad \text{and} \qquad \bar{x}_1 \eta + \bar{x}_2 \zeta + \bar{x}_3 r + \bar{x}_4 s$$

can differ only by a factor. Let ω be a proportionality factor. We shall have

$$(29) \qquad \begin{aligned} \omega x_1 &= A\bar{x}_1 - B\bar{x}_2 + 64a'^2 b\bar{x}_3 - 64a'^2 b\bar{x}_4, \\ \omega x_2 &= -16a'\bar{x}_1 + 16a'\bar{x}_2, \\ \omega x_3 &= 2A\bar{x}_3 - 2B\bar{x}_4, \\ \omega x_4 &= -32a'\bar{x}_3 + 32a'\bar{x}_4, \end{aligned}$$

whence, if ω' denotes another factor of proportionality,

$$(30) \qquad \begin{aligned} \omega'\bar{x}_1 &= 32a'x_1 + 2Bx_2 + 64a'^2 bx_4, \\ \omega'\bar{x}_2 &= 32a'x_1 + 2Ax_2 + 64a'^2 bx_4, \\ \omega'\bar{x}_3 &= 16a'x_3 + Bx_4, \\ \omega'\bar{x}_4 &= 16a'x_3 + Ax_4. \end{aligned}$$

Let y_i, z_i be the coördinates of two points on a line l, and let $\bar{y}_i, \bar{z}_i$ be the coördinates of the same two points referred to the tetrahedron $P_\eta P_\zeta P_r P_s$. Let the Plückerian coördinates of the line with respect to the two systems be

$$\omega_{ik} = y_i z_k - y_k z_i \qquad \text{and} \qquad \bar{\omega}_{ik} = \bar{y}_i \bar{z}_k - \bar{y}_k \bar{z}_i$$

respectively. We shall have (omitting proportionality factors),

$$(31) \qquad \begin{aligned} \bar{\omega}_{13} &= 16^2 a'^2 \omega_{13} + 16Ba'\omega_{23} - 2\cdot 16^2 a'^3 b\omega_{34} + 16Ba'\omega_{14} - B^2\omega_{42}, \\ \bar{\omega}_{42} &= -16^2 a'^2 \omega_{13} - 16Aa'\omega_{23} + 2\cdot 16^2 a'^3 b\omega_{34} - 16Aa'\omega_{14} + A^2\omega_{42}. \end{aligned}$$

It only remains to substitute these expressions and the expressions (24) for π_{12} and π_{21} into (21) and to simplify. *The equation of the linear complex C_1, which osculates the osculating ruled surface of the first kind at the point P, referred to the tetrahedron $P_y P_z P_\rho P_\sigma$, will thus be found to be*

$$(32a) \qquad a_{12}\omega_{12} + a_{13}\omega_{13} + a_{14}\omega_{14} + a_{23}\omega_{23} + a_{34}\omega_{34} + a_{42}\omega_{42} = 0$$

where

$$(32b) \qquad \begin{aligned} &a_{12} = 0, \qquad a_{13} = 2^8 a'^2 C, \qquad a_{14} = -2^3(a'\theta_v - 2a'_v\theta - 2^4 a' a'_u C), \\ &a_{23} = a_{14}, \qquad a_{34} = -2^9 a'^3 bC, \\ &a_{42} = -\left[C(\theta + 64a'^2_u) + 16\theta\frac{a'_u a'_v}{a'} - 8a'_u\theta_v\right]. \end{aligned}$$

The invariant of this complex is

$$(33) \qquad \mathfrak{A} = a_{12}a_{34} + a_{13}a_{42} + a_{14}a_{23} = 2^6\left[(a'\theta_v - 2a'_v\theta)^2 - 4a'^2C^2\theta\right].$$

It may be noted that, although our investigation was made under the hypothesis $\theta \neq 0$, the equations (32) will be valid also if $\theta = 0$.* If $a' = 0$, however, not only do these equations become meaningless but the complex C_1 itself becomes indeterminate. In fact, in that case the original surface S is itself a ruled surface; it coincides with R_2, and the surface R_1 is the hyperboloid which osculates R_2 along that one of its generators which passes through P. Now a hyperboloid belongs to a double infinity of linear complexes, so that the osculating linear complex is necessarily indeterminate.

More generally we may say that, *whenever the complex C_1 is indeterminate, five consecutive generators of R_1 have a pair of straight line intersectors.* Excluding the case $a' = 0$ which we have already discussed, equations (32) show that C_1 *is indeterminate if*

$$(34) \qquad C = 0, \qquad a'\theta_v - 2a'_v\theta = 0.$$

Moreover, if the conditions (34) are satisfied identically the osculating ruled surface of S belongs to a linear congruence. This follows from the fact that if equations (34) are satisfied, and if $\theta \neq 0$, we find from (24) $\pi_{12} = \pi_{21} = 0$, so that C_η and C_ζ are at the same time flecnode curves and asymptotic curves on R_1; but this is possible only if they are straight lines.

If $\theta = 0$, $C = 0$, R_1 belongs to a linear congruence with coincident directrices. In that case, in fact, we make the substitution

$$y = \eta, \qquad z = \frac{a'_u}{2a'}\eta - \zeta,$$

in place of (17). We find a new system of form (6) for η and ζ whose coefficients π_{ik} are zero while the quantities $\bar{u}_{ik}$ corresponding to (9) become

$$\bar{u}_{11} - \bar{u}_{22} = 0, \qquad \bar{u}_{12} = 8a, \qquad \bar{u}_{21} = 0,$$

and this is sufficient to prove the above statement.†

If the conditions (34) *are not satisfied identically but only for certain values of u and v, five consecutive generators of the osculating ruled surface of the first kind which belongs to that point (u, v) of S_1 have a pair of straight line intersectors. These are distinct or not according as θ is or is not different from zero.*

If $\mathfrak{A} = 0$ the complex C_1 is special. Equation (33) therefore gives rise to the following theorem: *if the two branches of the flecnode curve of the oscu-*

* That (32) gives the osculating linear complex also in that case will be proved presently.

† Proj. Diff. Geom. p. 170.

lating ruled surface of the first kind coincide, its osculating linear complex is special.

It remains to prove equations (32) by another method, independent of the general theory of ruled surfaces, and not restricted by the assumption $\theta \neq 0$. In order to do this we shall develop the Plückerian coördinates of the line $P_y P_z$ according to powers of the increment δv, calculating the terms up to and including the fourth order. The elimination of the powers of δv will then give the equation of the osculating linear complex.

We have

$$
\begin{aligned}
&y = y, \qquad y_v = \rho, \qquad y_{vv} = -gy - 2a'z, \\
(35)\qquad &y_{vvv} = -g_v y - 2a'_v z - g\rho - 2a'\sigma, \\
&y_{vvvv} = \epsilon' y + \epsilon\, z + \epsilon^{(3)}\rho + \epsilon^{(4)}\sigma,
\end{aligned}
$$

where

$$
\begin{aligned}
&\epsilon' = -g_{vv} + g^2 - 4a'^2 f + 2a' g_u, \\
&\epsilon'' = 4a' a'_u + 4a' g - 2a'_{vv}, \\
(36)\qquad &\epsilon^{(3)} = -2g_v - 8a'^2 b, \\
&\epsilon^{(4)} = -4a'_v.
\end{aligned}
$$

These equations may be obtained from (1) by differentiation, remembering that

$$y_u = z, \qquad y_v = \rho, \qquad y_{uv} = \sigma.$$

Similarly, we find

$$
\begin{aligned}
&z = z, \qquad z_v = \sigma, \qquad z_{vv} = (2a'f - g_u)y - (g + 2a'_u)z + 4a'b\rho, \\
(37)\qquad &z_{vvv} = \delta' y + \delta'' z + \delta^{(3)}\rho + \delta^{(4)}\sigma, \\
&z_{vvvv} = \zeta' y + \zeta'' z + \zeta^{(3)}\rho + \zeta^{(4)}\sigma,
\end{aligned}
$$

where

$$
\begin{aligned}
&\delta' = -g_{uv} + 2a'_v f + 2a' f_v - 4a'bg, \\
&\delta'' = -g_v - 2a'_{uv} - 8a'^2 b, \\
(38)\qquad &\delta^{(3)} = -g_u + 4a'_v b + 4a' b_v + 2a' f, \\
&\delta^{(4)} = -g - 2a'_u,
\end{aligned}
$$

and

$$
\begin{aligned}
&\zeta' = \delta'_v - \delta^{(3)} g + \delta^{(4)}(2a'f - g_u). \\
&\zeta'' = \delta''_v - 2a'\delta^{(3)} - \delta^{(4)}(g + 2a'_u), \\
(39)\qquad &\zeta^{(3)} = \delta^{(3)}_v + \delta' + 4\delta^{(4)} a' b \\
&\zeta^{(4)} = \delta^{(4)}_v + \delta''.
\end{aligned}
$$

We shall have for a point Y, in the vicinity of the point (u, v) of the surface S, if u remains constant, while v alone varies,

$$Y = y + y_v \delta v + \tfrac{1}{2} y_{vv} (\delta v)^2 + \tfrac{1}{6} y_{vvv} (\delta v)^3 + \tfrac{1}{24} y_{vvvv} (\delta v)^4 + \cdots,$$

which may be written in the form

$$Y = y_1 y + y_2 z + y_3 \rho + y_4 \sigma,$$

where $y_1, \cdots, y_4$ will be the homogeneous coördinates of the point referred to the tetrahedron $P_y P_z P_\rho P_\sigma$, and where $y_1, \cdots, y_4$ will be power-series proceeding according to powers of δv. Similarly we denote by $z_1, \cdots, z_4$ the coördinates of the point Z. We find in this way, putting $\delta v = t$ for brevity:

$$(40) \qquad \begin{aligned} y_1 &= 1 - \tfrac{1}{2} g t^2 - \tfrac{1}{6} g_v t^3 + \tfrac{1}{24} \epsilon' t^4 + \cdots, \\ y_2 &= - a' t^2 - \tfrac{1}{3} a'_v t^3 + \tfrac{1}{24} \epsilon'' t^4 + \cdots, \\ y_3 &= t - \tfrac{1}{6} g t^3 + \tfrac{1}{24} \epsilon^{(3)} t^4 + \cdots, \\ y_4 &= - \tfrac{1}{3} a' t^3 + \tfrac{1}{24} \epsilon^{(4)} t^4 + \cdots, \end{aligned}$$

and

$$(41) \qquad \begin{aligned} z_1 &= \tfrac{1}{2}(2a'f - g_u) t^2 + \tfrac{1}{6} \delta' t^3 + \tfrac{1}{24} \zeta' t^4 + \cdots, \\ z_2 &= 1 - \tfrac{1}{2}(g + 2a'_u) t^2 + \tfrac{1}{6} \delta'' t^3 + \tfrac{1}{24} \zeta'' t^4 + \cdots, \\ z_3 &= 2a' b t^2 + \tfrac{1}{6} \delta^{(3)} t^3 + \tfrac{1}{24} \zeta^{(3)} t^4 + \cdots, \\ z_4 &= t + \tfrac{1}{6} \delta^{(4)} t^3 + \tfrac{1}{24} \zeta^{(4)} t^4 + \cdots. \end{aligned}$$

The Plückerian coördinates of the line joining the points (y_i) and (z_i) are

$$\omega_{ik} = y_i z_k - y_k z_i.$$

Consequently we find

$$(42) \qquad \begin{aligned} \omega_{13} &= 2a' b t^2 + \tfrac{1}{6}[\delta^{(3)} - 3(2a'f - g_u)] t^3 + \tfrac{1}{24}[\zeta^{(3)} - 4\delta' - 24a'bg] t^4 + \cdots, \\ \omega_{14} &= t + \tfrac{1}{6}(\delta^{(4)} - 3g) t^3 + \tfrac{1}{24}(\zeta^{(4)} - 4g_v) t^4 + \cdots, \\ \omega_{23} &= - t + \tfrac{1}{6}(4g + 6a'_u) t^3 - \tfrac{1}{24}(\epsilon^{(3)} + 4\delta'' + 48a'^2 b) t^4 + \cdots, \\ \omega_{34} &= t^2 + \tfrac{1}{6}(\delta^{(4)} - g) t^4 + \cdots, \\ \omega_{42} &= \tfrac{2}{3} a' t^3 + \tfrac{1}{24}(\epsilon^{(4)} + 8a'_v) t^4 + \cdots, \end{aligned}$$

whence

$$\begin{aligned} \omega_{14} + \omega_{23} &= \tfrac{1}{6}(\delta^{(4)} + g + 6a'_u) t^3 + \tfrac{1}{24}(\zeta^{(4)} - \epsilon^{(3)} - 4\delta'' - 4g_v - 48a'^2 b) t^4 + \cdots, \\ \omega_{13} - 2a'b\omega_{34} &= \tfrac{1}{6}(\delta^{(3)} - 6a'f + 3g_u) t^3 + \tfrac{1}{24}(\zeta^{(3)} - 4\delta' - 16a'bg - 8a'b\delta^{(4)}) t^4 + \cdots, \\ \omega_{42} &= \tfrac{2}{3} a' t^3 + \tfrac{1}{24}(\epsilon^{(4)} + 8a'_v) t^4 + \cdots. \end{aligned}$$

These equations may be written more simply:

$$
\begin{aligned}
\omega_{14} + \omega_{23} &= lt^3 + mt^4 + \cdots, \\
\omega_{13} - 2a'b\omega_{34} &= nt^3 + pt^4 + \cdots, \\
\omega_{42} &= qt^3 + rt^4 + \cdots,
\end{aligned}
\tag{43}
$$

where

$$
\begin{aligned}
&l = \tfrac{2}{3}a'_u, \qquad m = \tfrac{1}{6}(a'_{uv} - 4a'^2 b), \\
&n = \tfrac{1}{6}(2g_u + 4a'_v b + 4a'b_v - 4a'f) = -\tfrac{1}{12}u_{21}, \\
&p = \tfrac{1}{12}(g_{uv} + 2a'_{vv}b + 4a'_v b_v + 2a'b_{vv} - 2a'_v f - 2a'f_v + 4a'ba'_u) \\
&\quad = \tfrac{1}{12}\left(-\tfrac{1}{4}\frac{\partial u_{21}}{\partial v} + 4a'ba'_u\right), \\
&q = \tfrac{2}{3}a', \qquad r = \tfrac{1}{6}a'_v.
\end{aligned}
\tag{44}
$$

If t^3 and t^4 are eliminated from (43), the equation of the osculating linear complex is obtained in the form

$$(pq - nr)(\omega_{14} + \omega_{23}) + (lr - mq)(\omega_{13} - 2a'b\omega_{34}) + (mn - lp)\omega_{42} = 0,$$

and this becomes identical with (32) since

$$pq - nr = \frac{a_{14}}{-2^{11}\cdot 3^2 a'}, \qquad lr - mq = \frac{a_{13}}{-2^{11}\cdot 3^2 a'}, \qquad mn - lp = \frac{a_{42}}{-2^{11}\cdot 3^2 a'}.$$

In the same way we may find the equation of the linear complex which osculates the osculating ruled surface of the second kind, R_2. It is

$$b_{12}\omega_{12} + b_{13}\omega_{13} + b_{14}\omega_{14} + b_{23}\omega_{23} + b_{34}\omega_{34} + b_{42}\omega_{42} = 0 \tag{45a}$$

where

$$
\begin{aligned}
&b_{12} = 2^8 b^2 C', \; b_{13} = 0, \; b_{14} = -b_{23} = -2^3(b\theta'_u - 2b_u\theta' - 2^4 bb_v C'), \\
&b_{34} = C'(\theta' + 64b_v^2) + 16\theta'\frac{b_u b_v}{b} - 8b_v\theta'_u, \; b_{42} = 2^9 a'b^3 C',
\end{aligned}
\tag{45b}
$$

the quantities C' and θ' being defined by the equations

$$
\begin{aligned}
C' &= 8b_{uv} - 8\frac{b_u b_v}{b} - 32a'b^2, \\
\theta' &= (u'_{11} - u'_{22})^2 + 4u'_{12}u'_{21},
\end{aligned}
\tag{46}
$$

where u'_{ik} are the quantities (15).

We shall speak of this complex as the complex C_2. *Its invariant is*

$$\mathfrak{B} = -2^6[(b\theta'_u - 2b_u\theta')^2 - 4b^2 C'^2\theta'].\tag{47}$$

The simultaneous invariant of the two complexes C_1 and C_2 is

$$(\mathfrak{A}, \mathfrak{B}) = 0; \tag{48}$$

i. e., *the linear complexes C_1 and C_2, which osculate the two osculating ruled surfaces, are in involution.* The congruence common to the two complexes has distinct directrices unless one of them is special. If both are special, their axes intersect.

A linear complex always determines a null-system, i. e., a point-plane correspondence with corresponding elements in united position. Let P' be a point on the generator g_1 of R_1 which passes through P. The plane which corresponds to it in the linear complex C_1 contains g_1; that which corresponds to it in C_2 contains the tangent of the asymptotic curve of R_1 which passes through that point. This may be verified without any difficulty by setting up the equations of the null-systems concerned. In general, a linear complex associates with each point of a surface a direction through that point, namely the intersection of its tangent plane with its null-plane. Let us, with LIE, speak of the configuration of a point with a line through it, as a *line-element.* The ∞^2 line-elements which a linear complex determines on a surface may be collected so as to give rise to a single infinity of curves. Thus there is upon every surface a single infinity of curves whose tangents belong to a given linear complex. These may be called the *complex-curves* of the surface. We extend this idea of complex-curve by considering, in connection with every generator of R_1, a different linear complex, namely the complex C_2 which is associated with that generator. We may then express our above result as follows.

The complex curves, determined upon the osculating ruled surface of one kind by the osculating linear complex of the osculating ruled surface of the other kind, are its asymptotic lines.

§ 4. *The asymptotic lines of the surface and their osculating linear complexes.*

Consider an asymptotic curve Γ' of the first kind. It will be generated by the motion of P_y as v remains constant while u passes through all of its values. The line $P_y P_z$ will generate the developable of which the asymptotic curve is the cuspidal edge.

The linear differential equation of the fourth order which is characteristic of Γ' will be obtained by eliminating z, ρ, σ from the four equations

$$\begin{aligned} y_u &= z, \qquad y_{uu} = -fy - 2b\rho, \\ y_{uuu} &= -f_u y - fz - 2b_u\rho - 2b\sigma, \\ y_{uuuu} &= (f^2 - 4b^2 g + 2bf_v - f_{uu})y - 2(f_u + 4a'b^2)z \\ &\qquad + 2(2bf + 2bb_v - b_{uu})\rho - 4b_u\sigma. \end{aligned} \tag{49}$$

The result is

$$(50a) \qquad y_{uuuu} + 4p_1' y_{uuu} + 6p_2' y_{uu} + 4p_3' y_u + p_4' y = 0,$$

where

$$p_1' = -\frac{b_u}{2b}, \qquad p_2' = \tfrac{1}{3}(f + b_v) - \tfrac{1}{6}\left(\frac{b_{uu}}{b} - 2\frac{b_u^2}{b^2}\right),$$

$$(50b) \quad p_3' = \tfrac{1}{2}\left(f_u + 4a'b^2 - f\frac{b_u}{b}\right),$$

$$p_4' = f_{uu} - f^2 + 4b^2 g - 2bf_v + \frac{f}{b}(2bf + 2bb_v - b_{uu}) - 2\frac{b_u}{b^2}(bf_u - b_u f).$$

The seminvariants of (50a) may be computed without difficulty.* For our present purpose we need only

$$P_2' = p_2' - \frac{\partial p_1'}{\partial u} - p_1'^2,$$

and this becomes

$$(51) \qquad P_2' = \tfrac{1}{3}(f + b_v) + \tfrac{1}{3}\frac{b_{uu}}{b} - \tfrac{5}{12}\frac{b_u^2}{b^2}.$$

We shall denote the fundamental semi-covariants of (50) by y', z', ρ', σ', so that

$$y' = y, \qquad z' = y_u + p_1' y,$$

$$\rho' = y_{uu} + 2p_1' y_u + p_2' y,$$

$$\sigma' = y_{uuu} + 3p_1' y_{uu} + 3p_2' y_u + p_3' y.\dagger$$

Express these in terms of the fundamental semi-covariants of the surface S. We find

$$y' = y, \qquad z' = -\frac{b_u}{2b} y + z,$$

$$(52) \quad \rho' = \left[\tfrac{1}{3}(b_v - 2f) - \tfrac{1}{6}\left(\frac{b_{uu}}{b} - 2\frac{b_u^2}{b^2}\right)\right] y - \frac{b_u}{b} z - 2b\rho,$$

$$\sigma' = \left[\tfrac{1}{2}(-f_u + 4a'b^2) + \frac{b_u}{b} f\right] y + \left[b_v - \tfrac{1}{2}\left(\frac{b_{uu}}{b} - 2\frac{b_u^2}{b^2}\right)\right] z + b_u \rho - 2b\sigma.$$

Consequently, if $x_1, \cdots, x_4$ denote the coördinates of a point with respect to the tetrahedron of reference $P_y P_z P_\rho P_\sigma$, and if $x_1', \cdots, x_4'$ denote the coördinates of the same point referred to the tetrahedron $P_{y'} P_{z'} P_{\rho'} P_{\sigma'}$, we shall have

* Proj. Diff. Geom., p. 239.

† Ibid., p. 239.

$$
(53)\qquad
\begin{aligned}
\omega x_1 &= x_1' - \frac{b_u}{2b}x_2' + \left[\tfrac{1}{3}(b_v - 2f) - \tfrac{1}{6}\left(\frac{b_{uu}}{b} - 2\frac{b_u^2}{b^2}\right)\right]x_3' \\
&\qquad + \left(-\tfrac{1}{2}f_u + 2a'b^2 + \frac{b_u}{b}f\right)x_4', \\
\omega x_2 &= x_2' - \frac{b_u}{b}x_3' + \left[b_v - \tfrac{1}{2}\left(\frac{b_{uu}}{b} - 2\frac{b_u^2}{b^2}\right)\right]x_4', \\
\omega x_3 &= -2bx_3' + b_u x_4', \\
\omega x_4 &= -2bx_4',
\end{aligned}
$$

where ω is a proportionality factor.

We find, therefore, if ω' denotes another factor of proportionality,

$$
(54)\qquad
\begin{aligned}
\omega' x_1' &= 2bx_1 + b_u x_2 + \left[\tfrac{1}{3}(b_v - 2f) - \tfrac{1}{6}\frac{b_{uu}}{b} - \tfrac{1}{6}\frac{b_u^2}{b^2}\right]x_3 \\
&\qquad + \left[-\tfrac{1}{2}f_u + 2a'b^2 + \tfrac{1}{3}\frac{b_u}{b}\left(2f + 2b_v - \frac{b_{uu}}{b}\right) + \tfrac{5}{12}\frac{b_u^3}{b^3}\right]x_4, \\
\omega' x_2' &= 2bx_2 - \frac{b_u}{b}x_3 + \left(b_v - \tfrac{1}{2}\frac{b_{uu}}{b} + \tfrac{1}{2}\frac{b_u^2}{b^2}\right)x_4, \\
\omega' x_3' &= -x_3 - \frac{b_u}{2b}x_4, \\
\omega' x_4' &= -x_4.
\end{aligned}
$$

The equation of the osculating linear complex of Γ' referred to the tetrahedron $P_{y'}P_{z'}P_{\rho'}P_{\sigma'}$ is

$$
(55)\qquad \omega_{14}' - 2P_2'\omega_{34}' - \omega_{23}' = 0,
$$

where ω_{ik}' are the Plückerian coördinates of a line with respect to the tetrahedron just mentioned.*

Let ω_{ik} denote the Plückerian coördinates of a line with respect to the tetrahedron $P_yP_zP_\rho P_\sigma$. Then we shall find from (54), (omitting proportionality factors),

$$
(56)\qquad
\begin{aligned}
\omega_{14}' &= -2b\omega_{14} + b_u\omega_{42} - \left[\tfrac{1}{3}(b_v - 2f) - \tfrac{1}{6}\frac{b_{uu}}{b} - \tfrac{1}{6}\frac{b_u^2}{b^2}\right]\omega_{34}, \\
\omega_{34}' &= \omega_{34}, \qquad \omega_{23}' = -2b\omega_{23} + \left(b_v - \tfrac{1}{2}\frac{b_{uu}}{b} + \frac{b_u^2}{b^2}\right)\omega_{34} + b_u\omega_{42}.
\end{aligned}
$$

Substitution of these values and of (51) into (55) gives

$$
(57)\qquad -b_v\omega_{34} - b\omega_{14} + b\omega_{23} = 0,
$$

* Proj. Diff. Geom., p. 253.

as the equation of the linear complex C' which osculates the asymptotic curve Γ' of the first kind. Its invariant is

$$\mathfrak{A}' = -b^2. \tag{58}$$

Therefore, the complex C' can be special only if the asymptotic curve Γ' is a straight line.

The differential equation of the asymptotic curve of the second kind Γ'' which passes through P_y is

$$y_{vvvv} + 4p_1'' y_{vvv} + 6p_2'' y_{vv} + 4p_3'' y_v + p_4'' y = 0, \tag{59a}$$

where

$$p_1'' = -\frac{a_v'}{2a'} \qquad p_2'' = \tfrac{1}{3}(g + a_u') - \tfrac{1}{6}\left(\frac{a_{vv}'}{a'} - 2\frac{a_v'^2}{a'^2}\right),$$

$$p_3'' = \tfrac{1}{2}(g_v + 4a'^2 b) - \tfrac{1}{2} g \frac{a_v'}{a'}, \tag{59b}$$

$$p_4'' = g_{vv} - g^2 + 4a'^2 f - 2a' g_u + \frac{g}{a'}(2a'g + 2a'a_u' - a_{vv}') - 2\frac{a_v'}{a'^2}(a'g_v - a_v' g).$$

Its simplest semi-invariant is

$$P_2'' = \tfrac{1}{3}(g + a_u') + \tfrac{1}{3}\frac{a_{vv}'}{a'} - \tfrac{5}{12}\frac{a_v'^2}{a'^2}. \tag{60}$$

Its fundamental semi-covariants are

$$y'' = y, \qquad z'' = -\frac{a_v'}{2a'} y + \rho,$$

$$\rho'' = \left[\tfrac{1}{3}(a_u' - 2g) - \tfrac{1}{6}\left(\frac{a_{vv}'}{a'} - 2\frac{a_v'^2}{a'^2}\right)\right] y - 2a' z - \frac{a_v'}{a'}\rho, \tag{61}$$

$$\sigma'' = \left[\tfrac{1}{2}(-g_v + 4a'^2 b) + \frac{a_v'}{a'} g\right] y + a_v' z + \left[a_u' - \tfrac{1}{2}\left(\frac{a_{vv}'}{a'} - 2\frac{a_v'^2}{a'^2}\right)\right]\rho - 2a'\sigma.$$

If $x_1, \cdots, x_4$ are the coördinates of a point referred to the tetrahedron $P_y P_z P_\rho P_\sigma$ and $x_1'', \cdots, x_4''$ are the coördinates of the same point with reference to the tetrahedron $P_{y''} P_{z''} P_{\rho''} P_{\sigma''}$, we find, therefore,

$$\omega x_1 = x_1'' - \frac{a_v'}{2a'} x_2'' + \left[\tfrac{1}{3}(a_u' - 2g) - \tfrac{1}{6}\left(\frac{a_{vv}'}{a'} - 2\frac{a_v'^2}{a'^2}\right)\right] x_3''$$
$$+ \left(-\tfrac{1}{2} g_v + 2a'^2 b + \frac{a_v'}{a'} g\right) x_4'',$$

$$\omega x_2 = -2a' x_3'' + a_v' x_4'', \tag{62}$$

$$\omega x_3 = x_2'' - \frac{a_v'}{a'} x_3'' + \left[a_u' - \tfrac{1}{2}\left(\frac{a_{vv}'}{a'} - 2\frac{a_v'^2}{a'^2}\right)\right] x_4'',$$

$$\omega x_4 = -2a' x_4'',$$

where ω is a factor of proportionality. Consequently, if ω'' denotes a second proportionality factor,

$$\omega'' x_1'' = 2a' x_1 + \left[\tfrac{1}{3}(a_u' - 2g) - \tfrac{1}{6}\frac{a_{vv}'}{a'} - \tfrac{1}{6}\frac{a_v'^2}{a'^2} \right] x_2 + a_v' x_3$$

$$+ \left[-\tfrac{1}{2} g_v + 2a'^2 b + \tfrac{1}{3}\frac{a_v'}{a'}\left(2g + 2a_u' - \frac{a_{vv}'}{a'} \right) + \tfrac{5}{12}\frac{a_v'^3}{a'^3} \right] x_4,$$

$$(63) \qquad \omega'' x_2'' = -\frac{a_v'}{a'} x_2 + 2a' x_3 + \left(a_u' - \tfrac{1}{2}\frac{a_{vv}'}{a'} + \tfrac{1}{2}\frac{a_v'^2}{a'^2} \right) x_4,$$

$$\omega'' x_3'' = -x_2 - \frac{a_v'}{2a'} x_4,$$

$$\omega'' x_4'' = -x_4.$$

The equation of the osculating linear complex C'' of Γ'' referred to the tetrahedron $P_{y''} P_{z''} P_{\rho''} P_{\sigma''}$ is

$$\omega_{14}'' - 2P_2'' \omega_{34}'' - \omega_{23}'' = 0.$$

We find

$$\omega_{14}'' = -2a'\omega_{14} + \left[\tfrac{1}{3}(a_u' - 2g) - \tfrac{1}{6}\frac{a_{vv}'}{a'} - \tfrac{1}{6}\frac{a_v'^2}{a'^2} \right] \omega_{42} - a_v' \omega_{34},$$

$$\omega_{34}'' = -\omega_{42},$$

$$\omega_{23}'' = +2a'\omega_{23} - \left(a_u' - \tfrac{1}{2}\frac{a_{vv}'}{a'} + \tfrac{1}{2}\frac{a_v'^2}{a'^2} \right) \omega_{42} - \tfrac{1}{2}\frac{a_v'^2}{a'^2}\omega_{42},$$

so that the above equation becomes

$$(64) \qquad -a_u' \omega_{42} + a' \omega_{14} + a' \omega_{23} = 0.$$

Equation (64) *characterizes the linear complex C'' which osculates the asymptotic curve Γ'' of the second kind*, the tetrahedron of reference being $P_y P_z P_\rho P_\sigma$. *The invariant of C'' is*

$$(65) \qquad \mathfrak{A}'' = a'^2.$$

The complexes C' and C'' are in involution. In fact their simultaneous invariant vanishes,

$$(66) \qquad (\mathfrak{A}', \mathfrak{A}'') = 0.$$

Therefore, *the osculating linear complexes of the two asymptotic curves which intersect in a point of a surface are always in involution.*

Let

$$(67) \qquad \sum a_{ik}' \omega_{ik} = 0, \qquad \sum a_{ik}'' \omega_{ik} = 0,$$

be two linear complexes. Let $\mathfrak{A}'$ and $\mathfrak{A}''$ be their respective invariants, and let $(\mathfrak{A}', \mathfrak{A}'')$ denote their simultaneous invariant. Then the equations of the

directrices of the congruence which the two complexes have in common may be obtained by writing down the conditions which must be satisfied by a point in order that the same plane may correspond to it in both complexes. We thus find

$$
(68)\qquad
\begin{aligned}
&* + (a'_{12} - \omega_k a''_{12})x_2 + (a'_{13} - \omega_k a''_{13})x_3 + (a'_{14} - \omega_k a''_{14})x_4 = 0,\\
&(-a'_{12} + \omega_k a''_{12})x_1 + * + (a'_{23} - \omega_k a''_{23})x_3 + (a'_{24} - \omega_k a''_{24})x_4 = 0,\\
&(-a'_{13} + \omega_k a''_{13})x_1 + (-a'_{23} + \omega_k a''_{23})x_2 + * + (a'_{34} - \omega_k a''_{34})x_4 = 0,\\
&(-a'_{14} + \omega_k a''_{14})x_1 + (-a'_{24} + \omega_k a''_{24})x_2 + (-a'_{34} + \omega_k a''_{34})x_3 + * = 0,
\end{aligned}
$$

where $a'_{24} = -a'_{42}$, $a''_{24} = -a''_{42}$, and where ω_1, ω_2 are the two roots of the quadratic

$$
(69)\qquad \mathfrak{A}'\omega^2 - (\mathfrak{A}', \mathfrak{A}'')\omega + \mathfrak{A}'' = 0.
$$

In the present case we may put

$$
\omega_1 = \frac{a'}{b}, \qquad \omega_2 = -\frac{a'}{b},
$$

so that the equations of the two directrices become

$$
(70a)\qquad x_4 = 0, \qquad 2a'bx_1 + ba'_u x_2 + a'b_v x_3 = 0,
$$

and

$$
(70b)\qquad 2bx_2 + b_v x_4 = 0, \qquad 2a'x_3 + a'_u x_4 = 0.
$$

We shall denote them by the letters, d and d', and shall speak of them as being of the first or second kind respectively. From equations (70) we deduce the following result. *One of the directrices of the congruence common to the osculating linear complexes of the two asymptotic curves of a point lies in the tangent plane of this point, while the other one passes through the point itself. They are called directrices of the first and second kinds respectively.*

It is easy to verify, moreover, that the two directrices are reciprocal polars with respect to the hyperboloid which osculates the two osculating ruled surfaces.

§ 5. *Relations between the four linear complexes.*

If we compute the simultaneous invariants of the four complexes C_1, C_2, C', C'', we find

$$
(71)\qquad
\begin{aligned}
&(\mathfrak{A}, \mathfrak{B}) = 0,\ (\mathfrak{A}, \mathfrak{A}') = 0,\ (\mathfrak{A}, \mathfrak{A}'') = -2^4 a'(a'\theta_v - 2a'_v\theta),\\
&(\mathfrak{B}, \mathfrak{A}') = -2^4 b(b\theta'_u - 2b_u\theta'),\ (\mathfrak{B}, \mathfrak{A}'') = 0,\\
&(\mathfrak{A}', \mathfrak{A}'') = 0,
\end{aligned}
$$

so that *each of these four complexes is in involution with two of the others.* Those pairs which are not in involution consist of one complex which osculates

an osculating ruled surface of one kind, and of one complex which osculates an asymptotic curve of the other kind. We notice, moreover, that an asymptotic curve of one kind lies on the osculating ruled surface of the other kind, and we may express this relation by saying that they are united in position. *Those of the four osculating complexes which correspond to an asymptotic curve and to an osculating ruled surface in united position are the ones which are not involutory in general.* If the surface is not a ruled surface, one or both of these pairs of complexes are also in involution only if one or both of the two invariants

$$a'\theta_v - 2a'_v\theta \quad \text{or} \quad b\theta'_u - 2b_u\theta'$$

are equal to zero.

We notice that

$$(\mathfrak{A}, \mathfrak{A}'')^2 - 4\mathfrak{A}\mathfrak{A}'' = 2^{10}a'^4 C^2\theta. \tag{72}$$

If this invariant vanishes the complexes C_1 and C'' will have in common a congruence with coincident directrices. However, if a' is equal to zero, or if C vanishes, this congruence is indeterminate. In fact, we have already noticed that if $a' = 0$, the complex C_1 is indeterminate. We now notice further that if $C = 0$ the complexes C_1 and C'' become identical, unless $a'\theta_v - 2a'_v\theta$ is also equal to zero, in which case the complex C_1 is indeterminate. Therefore we may state the following theorem.

Let the osculating ruled surface R_1, of a surface S, have a determinate osculating linear complex C_1, i. e., *let it not be included in a linear congruence. Then not both of the invariants C and $a'\theta_v - 2a'_v\theta$ will vanish. If $C = 0$, this complex C_1 will coincide with that one which osculates the asymptotic curve of S which lies upon R_1. If $C \neq 0$, these two complexes determine a congruence whose directrices coincide if and only if the two branches of the flecnode curve of R_1 coincide.*

There is of course a corresponding theorem with respect to R_2.

Four linear complexes have, in general, two straight lines in common. We proceed to determine the straight lines which belong to the four complexes C_1, C_2, C', C''. We have already found the directrices of the congruence common to the complexes C' and C''. They were the lines d and d' of § 4 (equations (70a) and (70b)). The lines common to the four complexes must intersect d and d'.

The two points

$$p = -a'_u y + 2a'z, \qquad q = -b_v y + 2b_\rho, \tag{73}$$

are on d; the two points

$$y, \qquad r = -a'b_v z - ba'_u\rho + 2a'b\sigma, \tag{74}$$

are on d'. Therefore the coördinates of an arbitrary point on d will be

$$-(\lambda a'_u + \mu b_v)y + 2a'\lambda z + 2b\mu\rho$$

or, referred to the tetrahedron $P_y P_z P_\rho P_\sigma$,

$$x_1 = -(\lambda a'_u + \mu b_v), \qquad x_2 = 2a'\lambda, \qquad x_3 = 2b\mu, \qquad x_4 = 0.$$

Similarly the coördinates of an arbitrary point of d' will be

$$x'_1 = \lambda', \qquad x'_2 = -\mu' a' b_v, \qquad x'_3 = -\mu' b a'_u, \qquad x'_4 = 2\mu' a' b.$$

The Plückerian coördinates of the line joining these points will be

$$\begin{aligned}
\omega_{12} &= -2a'\lambda\lambda' + a' a'_u b_v \lambda\mu' + a' b_v^2 \mu\mu', \\
\omega_{13} &= -2b\mu\lambda' + b a_u'^2 \lambda\mu' + b b_v a'_u \mu\mu', \\
\omega_{14} &= \qquad\qquad -2a' b a'_u \lambda\mu' - 2a' b b_v \mu\mu', \\
\omega_{23} &= \qquad\qquad -2a' b a'_u \lambda\mu' + 2a' b b_v \mu\mu', \\
\omega_{24} &= \qquad\qquad 4a'^2 b \lambda\mu', \\
\omega_{34} &= \qquad\qquad\qquad\qquad 4a' b^2 \mu\mu'.
\end{aligned}$$

This line belongs to the other two complexes C_1 and C_2 if

$$\begin{aligned}
a'^2 b C\,[(2^6 b_v a'_u - 2^9 a'^2 b^2)\mu\mu' - 2^7\mu\lambda' + \theta\lambda\mu'] &= 0, \\
a' b^2 C'[-2^7\lambda\lambda' + 2^6(a'_u b_v - 2^3 a'^2 b^2)\lambda\mu' + \theta'\mu\mu'] &= 0.
\end{aligned}$$

Let us leave aside the case in which S is a ruled surface, in which case at least one of the four complexes is indeterminate, and also the case that one or both of the quantities C or C' are equal to zero, since then at least two of the complexes coincide. We shall then have a', b, C and C' all different from zero. The lines which are common to the four complexes must then be those for which

$$(75) \qquad \begin{aligned}
&-2^7\lambda\lambda' + 2^6(a'_u b_v - 2^3 a'^2 b^2)\lambda\mu' + \theta'\mu\mu' = 0, \\
&-2^7\mu\lambda' + \theta\lambda\mu' + 2^6(a'_u b_v - 2^3 a'^2 b^2)\mu\mu' = 0,
\end{aligned}$$

whence

$$(76) \qquad \begin{aligned}
&\lambda^2 : \mu^2 = \theta' : \theta, \\
&2^7 \frac{\lambda'}{\mu'} = 2^6(a'_u b_v - 8a'^2 b^2) + \theta\frac{\lambda}{\mu}.
\end{aligned}$$

The four complexes have, therefore, two lines in common unless both θ and θ' are equal to zero in which case they have an infinity of lines in common, which form a plane pencil. In fact equations (76) show that these lines join an arbitrary point of d to the point

$$x'_1 = a'_u b_v - 8a'^2 b^2, \qquad x'_2 = -2a' b_v, \qquad x'_3 = -2b a'_u, \qquad x'_4 = 4a' b$$

of d'. This latter point moreover is the point in which d' intersects the osculating hyperboloid H whose equation is

$$x_1 x_4 - x_2 x_3 + 2a' b x_4^2 = 0.$$

If θ and θ' do not both vanish, we have

$$
(77)\qquad
\begin{aligned}
&\lambda : \mu = \pm \sqrt{\theta'} : \sqrt{\theta}, \\
&\lambda' : \mu' = \tfrac{1}{2}(a'_u b_v - 8a'^2 b^2) \pm \frac{1}{2^7}\sqrt{\theta\theta'};
\end{aligned}
$$

i. e., the four complexes have two lines in common. They intersect d in two points which form a harmonic group with the points p and q; their intersections with d' are harmonic conjugates with respect to H. We may recapitulate as follows.

Consider a surface S which is not ruled, at a point P for which no two of the four fundamental linear complexes coincide. These four linear complexes have an infinity of lines in commom only in the case that the two branches of the flecnode curve coincide on both of the osculating ruled surfaces. The lines common to the four complexes then form a pencil whose vertex is the point where the directrix of the second kind intersects the osculating hyperboloid, and whose plane is determined by this point and the directrix of the first kind. Moreover the flecnodes of the osculating ruled surfaces are the points where their generators intersect the directrix of the first kind.

In every other case the four complexes have only two lines in common. Let there be marked upon the directrix of the first kind its intersections with the asymptotic tangents of P. The two lines common to the four complexes intersect the directrix of the first kind in two points which are harmonic conjugates of each other with respect to these intersections. They intersect the directrix of the second in two points which are harmonic conjugates of each other with respect to the pair of points in which this directrix intersects the osculating hyperboloid.

§ 6. *Canonical development of the non-homogeneous coördinates of a surface.*

We have the following expressions for the first and second derivatives of y;

$$
(78)\qquad
\begin{aligned}
&y_u = z, \qquad y_v = \rho, \\
&y_{uu} = -fy - 2b\rho, \qquad y_{uv} = \sigma, \qquad y_{vv} = -gy - 2a'z,
\end{aligned}
$$

whence

$$
(79)\qquad
\begin{aligned}
y_{uuu} &= f_u y - fz - 2b_u\rho - 2b\sigma, \\
y_{uuv} &= (2bg - f_v)y + 4a'bz - (f + 2b_v)\rho, \\
y_{uvv} &= (2a'f - g_u)y - (g + 2a'_u)z + 4a'b\rho, \\
y_{vvv} &= -g_v y - 2a'_v z - g\rho - 2a'\sigma.
\end{aligned}
$$

The expressions for the fourth derivatives of y are capable of several forms which become identical if one takes account of the integrability conditions.

We find

$$
(80)\qquad
\begin{aligned}
y_{uuuu} &= \alpha_1 y + \alpha_2 z + \alpha_3 \rho + \alpha_4 \sigma,\\
y_{uuuv} &= \beta_1 y + \beta_2 z + \beta_3 \rho + \beta_4 \sigma,\\
y_{uuvv} &= \gamma_1 y + \gamma_2 z + \gamma_3 \rho + \gamma_4 \sigma,\\
y_{uvvv} &= \delta_1 y + \delta_2 z + \delta_3 \rho + \delta_4 \sigma,\\
y_{vvvv} &= \epsilon_1 y + \epsilon_2 z + \epsilon_3 \rho + \epsilon_4 \sigma,
\end{aligned}
$$

where

$$
(81)\qquad
\begin{aligned}
\alpha_1 &= f^2 - 4b^2 g + 2bf_v - f_{uu}, & \alpha_2 &= -2(f_u + 4a'b^2),\\
\alpha_3 &= 4b(f + b_v) - 2b_{uu}, & \alpha_4 &= -4b_u,\\
\beta_1 &= 2bg_u + 2b_u g - 4a'bf - f_{uv}, & \beta_2 &= 4a'_u b + 4a'b_u + 2bg - f_v,\\
\beta_3 &= -(f_u + 2b_{uv} + 8a'b^2), & \beta_4 &= -(f + 2b_v),\\
\gamma_1 &= 4a'_u f + 2a'f_u + fg - g_{uu}, & \gamma_2 &= 2a'f - 2g_u - 2a'_{uu},\\
\gamma_3 &= 2bg + 8a'_u b + 4a'b_u, & \gamma_4 &= 4a'b,\\
\delta_1 &= 2a'f_v + 2a'_v f - 4a'bg - g_{uv}, & \delta_2 &= -(g_v + 2a'_{uv} + 8a'^2 b),\\
\delta_3 &= 4b_v a' + 4a'_v b + 2a'f - g_u, & \delta_4 &= -(g + 2a'_u),\\
\epsilon_1 &= g^2 - 4a'^2 f + 2a'g_u - g_{vv}, & \epsilon_2 &= 4a'(g + a'_u) - 2a'_{vv},\\
\epsilon_3 &= -2(g_v + 4a'^2 b), & \epsilon_4 &= -4a'_v.
\end{aligned}
$$

Assume that the point P of the surface S corresponds to the values (0, 0) of u and v, an assumption which involves no restriction of generality. Let the surface be analytical in the vicinity of this point. Then, for values of u and v sufficiently small, we shall have

$$Y = y + y_u u + y_v v + \tfrac{1}{2}(y_{uu} u^2 + 2y_{uv} uv + y_{vv} v^2) + \cdots,$$

which may be written in the form

$$Y = x_1 y + x_2 z + x_3 \rho + x_4 \sigma$$

by making use of equations (78), (79), (80). The expressions for $x_1, \cdots, x_4$ will be power-series proceeding according to powers of u and v and convergent in a certain domain. Moreover $x_1, \cdots, x_4$ will represent the coördinates of any point P' of the surface S within a certain vicinity of the point P, referred to the tetrahedron $P_y P_z P_\rho P_\sigma$.

We find

$$
\begin{aligned}
x_1 &= 1 - \tfrac{1}{2}fu^2 - \tfrac{1}{2}gv^2 - \tfrac{1}{6}f_u u^3 + \tfrac{1}{2}(2bg - f_v)u^2v + \tfrac{1}{2}(2a'f - g_u)uv^2 \\
&\qquad - \tfrac{1}{6}g_v v^3 + \tfrac{1}{24}(\alpha_1 u^4 + 4\beta_1 u^3 v + 6\gamma_1 u^2 v^2 + 4\delta_1 uv^3 + \epsilon_1 v^4) + \cdots, \\
x_2 &= u - a'v^2 - \tfrac{1}{6}fu^3 + 2a'bu^2v - \tfrac{1}{2}(g + 2a'_u)uv^2 - \tfrac{1}{3}a'_v v^3 \\
&\qquad + \tfrac{1}{24}(\alpha_2 u^4 + 4\beta_2 u^3 v + 6\gamma_2 u^2 v^2 + 4\delta_2 uv^3 + \epsilon_2 v^4) + \cdots, \\
x_3 &= v - bu^2 - \tfrac{1}{3}b_u u^3 - \tfrac{1}{2}(f + 2b_v)u^2v + 2a'buv^2 - \tfrac{1}{6}gv^3 \\
&\qquad + \tfrac{1}{24}(\alpha_3 u^4 + 4\beta_3 u^3 v + 6\gamma_3 u^2 v^2 + 4\delta_3 uv^3 + \epsilon_3 v^4) + \cdot \\
x_4 &= uv - \tfrac{1}{3}bu^3 - \tfrac{1}{3}a'v^3 \\
&\qquad + \tfrac{1}{24}(\alpha_4 u^4 + 4\beta_4 u^3 v + 6\gamma_4 u^2 v^2 + 4\delta_4 uv^3 + \epsilon_4 v^4) + \cdots
\end{aligned}
\tag{82}
$$

Introduce non-homogeneous coördinates by putting

$$
\xi = \frac{x_2}{x_1}, \qquad \eta = \frac{x_3}{x_1}, \qquad \zeta = \frac{x_4}{x_1}. \tag{83}
$$

Then

$$
\begin{aligned}
\xi &= u - a'v^2 + \tfrac{1}{3}fu^3 + 2a'bu^2v - a'_u uv^2 - \tfrac{1}{3}a'_v v^3 \\
&\quad + \tfrac{1}{24}[(\alpha_2 + 4f_u)u^4 + (4\beta_2 - 24bg + 12f_v)u^3v + (6\gamma_2 - 36a'f + 12g_u)u^2v^2 \\
&\qquad + (4\delta_2 + 4g_v)uv^3 + (\epsilon_2 - 12a'g)v^4] + \cdots, \\
\eta &= v - bu^2 - \tfrac{1}{3}b_u u^3 - b_v u^2 v + 2a'buv^2 + \tfrac{1}{3}gv^3 \\
&\quad + \tfrac{1}{24}[(\alpha_3 - 12bf)u^4 + (4\beta_3 + 4f_u)u^3v + (6\gamma_3 - 36bg + 12f_v)u^2v^2 \\
&\qquad + (4\delta_3 - 24a'f + 12g_u)uv^3 + (\epsilon_3 + 4g_v)v^4] + \cdots, \\
\zeta &= uv - \tfrac{1}{3}bu^3 - \tfrac{1}{3}a'v^3 + \tfrac{1}{24}[\alpha_4 u^4 + (4\beta_4 + 12f)u^3v + 6\gamma_4 u^2v^2 \\
&\qquad + (4\delta_4 + 12g)uv^3 + \epsilon_4 v^4] + \cdots,
\end{aligned}
$$

whence

$$
\begin{aligned}
\xi\eta &= uv - bu^3 - a'v^3 - \tfrac{1}{3}b_u u^4 + \tfrac{1}{3}(f - 3b_v)u^3v + 5a'bu^2v^2 \\
&\qquad + \tfrac{1}{3}(g - 3a'_u)uv^3 - \tfrac{1}{3}a'_v v^4 + \cdots,
\end{aligned}
$$

and consequently, substituting the values of α_4, β_4, γ_4, δ_4, ϵ_4, from (81),

$$
\begin{aligned}
\zeta - \xi\eta &= \tfrac{2}{3}bu^3 + \tfrac{2}{3}a'v^3 + \tfrac{1}{6}b_u u^4 + \tfrac{2}{3}b_v u^3 v - 4a'bu^2v^2 \\
&\qquad + \tfrac{2}{3}a'_u uv^3 + \tfrac{1}{6}a'_v v^4 + \cdots.
\end{aligned}
\tag{84}
$$

We find further, up to terms of the fourth order,

$$
\xi^3 = u^3 - 3a'u^2v^2 + \cdots, \qquad \eta^3 = v^3 - 3bu^2v^2 + \cdots, \qquad \xi^4 = u^4 + \cdots,
$$

$$
\xi^3\eta = u^3v + \cdots, \qquad \xi^2\eta^2 = u^2v^2 + \cdots, \qquad \xi\eta^3 = uv^3 + \cdots, \qquad \eta^4 = v^4 + \cdots,
$$

so that we may write in (84)

$$u^3 = \xi^3 + 3a'\xi^2\eta^2, \qquad v^3 = \eta^3 + 3b\xi^2\eta^2, \qquad u^4 = \xi^4, \text{ etc.}$$

The result is the development, exact up to terms of the fourth order,

$$(85) \quad \zeta = \xi\eta + \tfrac{2}{3}b\xi^3 + \tfrac{2}{3}a'\eta^3 + \tfrac{1}{6}(b_u\xi^4 + 4b_v\xi^3\eta + 4a'_u\xi\eta^3 + a'_v\eta^4) + \cdots.$$

The tetrahedron of reference which gives rise to the development (85) has for two of its edges the asymptotic tangents of the point P, the equations of which are: $\eta = \zeta = 0$ and $\xi = \zeta = 0$. We proceed to make the most general transformation of coördinates which will retain these two lines as edges of the tetrahedron of reference, viz.;

$$(86) \quad x = \frac{\lambda\xi + \lambda'\zeta}{1+\alpha\xi+\beta\eta+\gamma\zeta}, \qquad y = \frac{\mu\eta + \mu'\zeta}{1+\alpha\xi+\beta\eta+\gamma\zeta}, \qquad z = \frac{\nu\zeta}{1+\alpha\xi+\beta\eta+\gamma\zeta},$$

where $\lambda, \lambda', \mu, \mu', \nu, \alpha, \beta, \gamma$ are arbitrary constants, which we shall choose in such a way as to simplify the development of the equation of the surface into series.

We now perform the calculations by substituting in (86) for ζ the series (85). We find

$$\frac{1}{1+\alpha\xi+\beta\eta+\gamma\zeta} = 1 - \alpha\xi - \beta\eta + \alpha^2\xi^2 + (2\alpha\beta - \gamma)\xi\eta + \beta^2\eta^2$$
$$- \tfrac{2}{3}b\gamma\xi^3 + 2\alpha\gamma\xi^2\eta + 2\beta\gamma\xi\eta^2 - \tfrac{2}{3}a'\gamma\eta^3 + (\ )_4,$$

where the symbol $(\ )_n$ denotes terms of the nth order. Consequently

$$(87) \quad \begin{aligned} x &= \lambda\xi - \lambda\alpha\xi^2 + (\lambda' - \lambda\beta)\xi\eta + (\lambda\alpha^2 + \tfrac{2}{3}\lambda' b)\xi^3 \\ &\quad + [\lambda(2\alpha\beta - \gamma) - \lambda'\alpha]\xi^2\eta + \beta(\lambda\beta - \lambda')\xi\eta^2 + \tfrac{2}{3}\lambda' a'\eta^3 + (\ )_4, \end{aligned}$$

whence

$$(88) \quad \begin{aligned} x^2 &= \lambda^2\xi^2 - 2\lambda^2\alpha\xi^3 + 2\lambda(\lambda' - \lambda\beta)\xi^2\eta + (\ )_4, \\ x^3 &= \lambda^3\xi^3 - 3\lambda^3\alpha\xi^4 + 3\lambda^2(\lambda' - \lambda\beta)\xi^3\eta + (\ )_5, \\ x^4 &= \lambda^4\xi^4 + (\ )_5. \end{aligned}$$

Similarly

$$\begin{aligned} y &= \mu\eta + (\mu' - \mu\alpha)\xi\eta - \mu\beta\eta^2 + \tfrac{2}{3}\mu' b\xi^3 + \alpha(\mu\alpha - \mu')\xi^2\eta \\ &\quad + [\mu(2\alpha\beta - \gamma) - \mu'\beta]\xi\eta^2 + (\mu\beta^2 + \tfrac{2}{3}\mu'a')\eta^3 + (\ )_4, \end{aligned}$$

$$(89) \quad \begin{aligned} y^2 &= \mu^2\eta^2 + 2\mu(\mu' - \mu\alpha)\xi\eta^2 - 2\mu^2\beta\eta^3 + (\ )_4, \\ y^3 &= \mu^3\eta^3 + 3\mu^2(\mu' - \mu\alpha)\xi\eta^3 - 3\mu^3\beta\eta^4 + (\ )_5, \\ y^4 &= \mu^4\eta^4 + (\ )_5, \end{aligned}$$

and

$$(90) \quad x^3y = \mu\lambda^3\xi^3\eta + (\ )_5, \quad x^2y^2 = \lambda^2\mu^2\xi^2\eta^2 + (\ )_5, \quad xy^3 = \lambda\mu^3\xi\eta^3 + (\ )_5.$$

We have further

$$(91)\qquad \frac{1}{\nu}z=\xi\eta+\tfrac{2}{3}b\xi^3-\alpha\xi^2\eta-\beta\xi\eta^2+\tfrac{2}{3}a'\eta^3+(\tfrac{1}{6}b_u-\tfrac{2}{3}b\alpha)\xi^4+(\alpha^2-\tfrac{2}{3}b\beta+\tfrac{2}{3}b_v)\xi^3\eta$$
$$+(2\alpha\beta-\gamma)\xi^2\eta^2+(\beta^2-\tfrac{2}{3}a'\alpha+\tfrac{2}{3}a'_u)\xi\eta^3+(\tfrac{1}{6}a'_v-\tfrac{2}{3}a'\beta)\eta^4+(\ )_5,$$

$$\frac{xy}{\lambda\mu}=\xi\eta+\left(\frac{\mu'}{\mu}-2\alpha\right)\xi^2\eta+\left(\frac{\lambda'}{\lambda}-2\beta\right)\xi\eta^2+\tfrac{2}{3}\frac{\mu'}{\mu}b\xi^4$$
$$+\left(3\alpha^2+\tfrac{2}{3}\frac{\lambda'}{\lambda}b-2\frac{\mu'}{\mu}\alpha\right)\xi^3\eta$$
$$+\left(6\alpha\beta-2\gamma-2\frac{\lambda'}{\lambda}\alpha-2\frac{\mu'}{\mu}\beta+\frac{\lambda'}{\lambda}\frac{\mu'}{\mu}\right)\xi^2\eta^2$$
$$+\left(3\beta^2+\tfrac{2}{3}\frac{\mu'}{\mu}a'-2\frac{\lambda'}{\lambda}\beta\right)\xi\eta^3+\tfrac{2}{3}\frac{\lambda'}{\lambda}a'\eta^4+(\ )_5;$$

and we find from (88) and (89)

$$\frac{x^3}{\lambda^3}=\xi^3-3\alpha\xi^4+3\left(\frac{\lambda'}{\lambda}-\beta\right)\xi^3\eta+(\ )_5,$$
$$\frac{x^2y}{\lambda^2\mu}=\xi^2\eta+\left(\frac{\mu'}{\mu}-3\alpha\right)\xi^3\eta+\left(2\frac{\lambda'}{\lambda}-3\beta\right)\xi^2\eta^2+(\ )_5,$$
$$\frac{xy^2}{\lambda\mu^2}=\xi\eta^2+\left(2\frac{\mu'}{\mu}-3\alpha\right)\xi^2\eta^2+\left(\frac{\lambda'}{\lambda}-3\beta\right)\xi\eta^3+(\ )_5,$$
$$\frac{y^3}{\mu^3}=\eta^3+3\left(\frac{\mu'}{\mu}-\alpha\right)\xi\eta^3-3\beta\eta^4+(\ )_5.$$

The development (85) contains of the third degree only the terms ξ^3 and η^3. If we now compute the development for z in terms of x and y we find that there will be introduced, in general, terms of the form x^2y and xy^2. In fact we find that

$$\frac{z}{\nu}-\frac{xy}{\lambda\mu}-\tfrac{2}{3}b\frac{x^3}{\lambda^3}-\left(\alpha-\frac{\mu'}{\mu}\right)\frac{x^2y}{\lambda^2\mu}-\left(\beta-\frac{\lambda'}{\lambda}\right)\frac{xy^2}{\lambda\mu^2}-\tfrac{2}{3}a'\frac{y^3}{\mu^3}$$

involves only terms of the fourth order. The form of the development (85) will be preserved if and only if

$$(92)\qquad \frac{\mu'}{\mu}=\alpha,\qquad \frac{\lambda'}{\lambda}=\beta.$$

Assuming these relations to be satisfied we find

$$\frac{z}{\nu}-\frac{xy}{\lambda\mu}-\tfrac{2}{3}b\frac{x^3}{\lambda^3}-\tfrac{2}{3}a'\frac{y^3}{\mu^3}=(\tfrac{1}{6}b_u+\tfrac{2}{3}b\alpha)\frac{x^4}{\lambda^4}+(\tfrac{2}{3}b_v-\tfrac{4}{3}b\beta)\frac{x^3y}{\lambda^3\mu}$$
$$+(\gamma-\alpha\beta)\frac{x^2y^2}{\lambda^2\mu^2}+(\tfrac{2}{3}a'_u-\tfrac{4}{3}a'\alpha)\frac{xy^3}{\lambda\mu^3}+(\tfrac{1}{6}a'_v+\tfrac{2}{3}a'\beta)\frac{y^4}{\mu^4}.$$

Let neither a' nor b be zero at the point considered. In fact a' or b can vanish only at special points of the surface unless it is a ruled surface. We may then determine α, β and γ so as to make the coefficients of x^3y, x^2y^2, and xy^3 disappear. This gives

$$(93) \qquad \alpha = \frac{a'_u}{2a'}, \qquad \beta = \frac{b_v}{2b}, \qquad \gamma = \alpha\beta = \frac{a'_u b_v}{4a'b}.$$

The development now becomes

$$z = \frac{\nu}{\lambda\mu}xy + \tfrac{2}{3}b\frac{\nu}{\lambda^3}x^3 + \tfrac{2}{3}a'\frac{\nu}{\mu^3}y^3 + \tfrac{1}{6}\left(b_u + 2b\frac{a'_u}{a'}\right)\frac{\nu}{\lambda^4}x^4$$
$$+ \tfrac{1}{6}\left(a'_v + 2a'\frac{b_v}{b}\right)\frac{\nu}{\mu^4}y^4 + \cdots.$$

Let λ, μ, ν be chosen as non-vanishing quantities so that

$$(94) \qquad \lambda^3 = 4b\nu, \qquad \mu^3 = 4a'\nu, \qquad \nu = \lambda\mu.$$

Then

$$\lambda^2 = 4b\mu, \qquad \mu^2 = 4a'\lambda.$$

whence

$$(95) \qquad \lambda = 4\sqrt[3]{A}, \qquad \mu = 4\sqrt[3]{B}, \qquad \nu = 16a'b,$$

where

$$(95a) \qquad A = a'b^2, \qquad B = a'^2b,$$

and where the symbol $\sqrt[3]{A}$ may be taken to have any one of its three values, the corresponding value of $\sqrt[3]{B}$ being then determined from the relation

$$4b\mu = \lambda^2.$$

We thus find the canonical development

$$(96) \qquad z = xy + \tfrac{1}{6}(x^3 + y^3) + \tfrac{1}{24}(Ix^4 + Jy^4) + \cdots,$$

where I and J,

$$(96a) \qquad I = \frac{B_u}{4B\sqrt[3]{A}}, \qquad J = \frac{A_v}{4A\sqrt[3]{B}},$$

are absolute invariants. All further coefficients will be absolute invariants.

This development was given for the first time in 1893 by TRESSE in his Paris thesis. His proof, however, is entirely different from the above, and the geometrical significance of this development, which we shall now proceed to study, is not found in TRESSE's thesis nor, so far as I am aware, elsewhere.

The transformation which connects the tedrahedron $P_yP_zP_\rho P_\sigma$ with the tetrahedron, for which the development assumes the canonical form, is

$$(97) \qquad x = \frac{\lambda(\xi+\beta\zeta)}{1+\alpha\xi+\beta\eta+\alpha\beta\zeta}, \qquad y = \frac{\mu(\eta+\alpha\zeta)}{1+\alpha\xi+\beta\eta+\alpha\beta\zeta}, \qquad z = \frac{\nu\zeta}{1+\alpha\xi+\beta\eta+\alpha\beta\zeta},$$

where

$$(97a)\qquad \alpha = \frac{a'_u}{2a'}, \qquad \beta = \frac{b_v}{2b}, \qquad \lambda = 4\sqrt{A}, \qquad \mu = 4\sqrt[2]{B},$$

$$\nu = \lambda\mu = 16a'b, \qquad A = a'b^2, \qquad B = a'^2b.$$

If we write

$$(98)\qquad x = \frac{X}{\omega}, \qquad y = \frac{Y}{\omega}, \qquad z = \frac{Z}{\omega},$$

so as to introduce homogeneous coördinates, and make use of (83), if moreover we denote by κ a factor of proportionality, we shall have;

$$(99)\qquad \kappa X = \lambda(x_2 + \beta x_4), \qquad \kappa Y = \mu(x_3 + \alpha x_4), \qquad \kappa Z = \lambda\mu x_4,$$

$$\kappa\omega = x_1 + \alpha x_2 + \beta x_3 + \alpha\beta x_4,$$

whence

$$(100)\qquad \frac{\lambda\mu}{\kappa}x_1 = -\alpha\mu X - \lambda\beta Y + \alpha\beta Z + \lambda\mu\omega,$$

$$\frac{\lambda\mu}{\kappa}x_2 = \mu X - \beta Z, \qquad \frac{\lambda\mu}{\kappa}x_3 = \lambda Y - \alpha Z, \qquad \frac{\lambda\mu}{\kappa}x_4 = Z.$$

The plane $Z = 0$ is the plane tangent to the surface at the given point; the lines $X = Z = 0$ and $Y = Z = 0$ are the two asymptotic tangents at this point. The line $\omega = Z = 0$ according to (70a) is the directrix of the first kind. The line $X = Y = 0$ is the directrix of the second kind. We have found, therefore, the geometrical significance of three of the faces, of three of the vertices, and of four of the edges of our canonical tetrahedron. We shall be able to describe it completely as a result of the investigation contained in the following paragraph, which is based upon the consideration of a certain class of surfaces of the third order.

§ 7. *The canonical cubic of a point of the surface.*

Let the surface S be referred to the canonical tetrahedron which belongs to one of its points P, and consider the cubic surface

$$(101)\qquad z = xy + \tfrac{1}{6}(x^3 + y^3),$$

which shall be called the canonical cubic of the point. Its equation in the homogeneous coördinates (98) is

$$(102)\qquad F = 6Z\omega^2 - 6XY\omega - X^3 - Y^3 = 0.$$

We find

$$F_X = -6Y\omega - 3X^2, \quad F_Y = -6X\omega - 3Y^2, \quad F_Z = 6\omega^2, \quad F_\omega = 12Z\omega - 6XY,$$

where

$$F_X = \frac{\partial F}{\partial X},$$

etc. Therefore, the only singular point of this surface is

$$X = Y = \omega = 0.$$

Moreover, the examination of the second derivatives shows that this point is a *unode;* i. e., the quadric cone which, in general, replaces the tangent plane at a double point, degenerates into a pair of coincident planes. These planes, moreover, in this case coincide with the plane $\omega = 0$, which is, therefore, the *uniplane* of the cubic surface.

What is the exact and characteristic relation of the cubic surface (102) to the surface S? We verify without difficulty that the most general cubic surface which has the point $X = Y = \omega = 0$, $Z = 1$, as a unode and the plane $\omega = 0$ as uniplane is

$$F \equiv Z\omega^2 + \phi(X, Y, \omega) = 0, \tag{103}$$

where $\phi(X, Y, \omega)$ is a general ternary cubic in X, Y, ω, so that the equation contains ten arbitrary constants. Let the uniplane, instead of being $\omega = 0$, be the arbitrary plane

$$\omega + pX + qY + rZ = 0,$$

and let the unode be an arbitrary point

$$X = -\lambda, \qquad Y = -\mu, \qquad Z = 1, \qquad \omega = p\lambda + q\mu - r,$$

of this plane; the equation of the most general unodal cubic surface will then be obtained by putting

$$\omega + pX + qY + rZ, \qquad X + \lambda Z, \qquad Y + \mu Z, \qquad Z,$$

in place of

$$\omega, \qquad X, \qquad Y, \qquad Z,$$

respectively in (103). We thus find

$$F \equiv Z(\omega + pX + qY + rZ)^2 + \phi(X + \lambda Z,\ Y + \mu Z,\ \omega + pX + qY + rZ) = 0, \tag{104}$$

an equation involving fifteen arbitrary constants as that of the most general unodal cubic or, in non-homogeneous coördinates,

$$z(1 + px + qy + rz)^2 + \phi(x + \lambda z,\ y + \mu z,\ 1 + px + qy + rz) = 0. \tag{105}$$

We shall assume in the first place that this cubic surface has contact of the third order with S at the point P. Then, in particular, its asymptotic tangents at P will coincide with those of S. The plane $z = 0$ must therefore intersect the cubic in a cubic curve which has P as a double point, the asymptotic tangents of P being the tangents of the double point.

The plane $Z=0$ intersects the surface (103) in the cubic curve

$$\phi(X, Y, \omega)=0,$$

or in the notation of SALMON*

$$aX^3+bY^3+c\omega^3+3a_2X^2Y+3a_3X^2\omega+3b_1Y^2X+3b_3Y^2\omega$$
$$+3c_1\omega^2X+3c_2\omega^2Y+6mXY\omega=0.$$

In order that the point $X=Y=0$ may be a double point with the lines $XY=0$ as tangents, we must have

$$c=c_1=c_2=a_3=b_3=0,$$

so that, with a slight change of notation, we may write

$$\phi(X, Y, \omega)=aX^3+bY^3+3a'X^2Y+3b'XY^2+6mXY\omega.$$

Introducing non-homogeneous coördinates, we find as the equation of the most general unodal cubic which touches the surface S at the point P and whose asymptotic tangents at that point coincide with those of S, the following:

$$z(1+px+qy+rz)^2+\phi(x+\lambda z, y+\mu z, 1+px+qy+rz)=0, \tag{106a}$$

where

$$\phi=a(x+\lambda z)^3+b(y+\mu z)^3+3a'(x+\lambda z)^2(y+\mu z)+3b'(x+\lambda z)(y+\mu z)^2$$
$$+6m(x+\lambda z)(y+\mu z)(1+px+qy+rz). \tag{106b}$$

If this surface has contact of the third order with the surface S at the point P, the substitution of the development

$$z=xy+\tfrac{1}{6}(x^3+y^3)+\tfrac{1}{24}(Ix^4+Jy^4)+\cdots \tag{96}$$

into (106) must satisfy these equations up to terms of the third order inclusive. Since we shall need them for a later investigation, we shall compute the terms of the fourth order which result from the substitution of (96) into (106), as well as those of the third.

We find

$$1+px+qy+rz=1+px+qy+rxy+\frac{r}{6}(x^3+y^3)+\frac{r}{24}(Ix^4+Jy^4)+\cdots,$$

$$z(1+px+qy+rz)^2=xy+\tfrac{1}{6}(x^3+12px^2y+12qxy^2+y^3)+(\tfrac{1}{24}I+\tfrac{1}{3}p)x^4$$
$$+(\tfrac{1}{3}q+p^2)x^3y+2(pq+r)x^2y^2+(\tfrac{1}{3}p+q^2)xy^3+(\tfrac{1}{24}J+\tfrac{1}{3}q)y^4+\cdots,$$
$$(x+\lambda z)^2(y+\mu z)=x^2y+2\lambda x^2y^2+\mu x^3y+\cdots, \tag{107}$$
$$(x+\lambda z)(y+\mu z)^2=xy^2+2\mu x^2y^2+\lambda xy^3+\cdots,$$
$$(x+\lambda z)(y+\mu z)(1+px+qy+rz)=xy+(p+\mu)x^2y+(q+\lambda)xy^2+\tfrac{1}{6}\mu x^4$$
$$+(\tfrac{1}{6}\lambda+p\mu)x^3y+(\lambda\mu+p\lambda+q\mu+r)x^2y^2+(\tfrac{1}{6}\mu+q\lambda)xy^3+\tfrac{1}{6}\lambda y^4+\cdots.$$

* Higher plane curves, 3d edition, p. 189.

This gives, on substitution in the left member of (106),

$$
\begin{aligned}
&(1+6m)xy+(a+\tfrac{1}{6})x^3+[2p+3a'+6m(\mu+p)]\,x^2y \\
&\qquad\qquad +[2q+3b'+6m(\lambda+q)]\,xy^2+(b+\tfrac{1}{6})y^3, \\
(108)\quad &+(\tfrac{1}{24}I+\tfrac{1}{3}p+m\mu)x^4+(p^2+\tfrac{1}{3}q+3a\lambda+3a'\mu+m\lambda+6mp\mu)x^3y \\
&\qquad +[2(pq+r)+6b'\mu+6a'\lambda+6m(\lambda\mu+p\lambda+q\mu+r)]\,x^2y^2, \\
&+(q^2+\tfrac{1}{3}p+3b\mu+3b'\lambda+m\mu+6mq\lambda)xy^3+(\tfrac{1}{24}J+\tfrac{1}{3}q+m\lambda)y^4+\cdots.
\end{aligned}
$$

If the surface (106) has contact of the third order with S at P, the terms of the third order in (108) must vanish, i. e.,

$$(109)\quad a=-\tfrac{1}{6},\quad b=-\tfrac{1}{6},\quad m=-\tfrac{1}{6},\quad p+3a'-\mu=0,\quad q+3b'-\lambda=0.$$

The uniplane of the unodal cubic surface will pass through the directrix d of the first kind if and only if

$$p=q=0,$$

since the equations of d are $Z=\omega=0$ in homogeneous form. Assuming this to be the case, the unode will be the intersection of the three planes

$$\omega+rZ=0,\qquad X+\lambda Z=0,\qquad Y+\mu Z=0.$$

This can be a point of the directrix of the second kind d' if and only if $\lambda=\mu=0$, since the equations of d' are

$$X=Y=0.$$

If we combine these conditions with (129), we notice that to every ordinary point P of a surface S there belongs a family of ∞^1 unodal cubic surfaces, each of which has contact of the third order with S at P, whose uniplane contains the directrix of the first kind and whose unode is on the directrix of the second kind of the point P. The equation of the most general unodal cubic of this kind is

$$(110)\qquad z(1+rz)^2-\tfrac{1}{6}x^3-\tfrac{1}{6}y^3-xy(1+rz)=0.$$

Consider a surface U of this family; it has contact of the third order with S at P, i. e., *every* plane which contains P will intersect the surfaces U and S in plane curves which have contact of the third order with each other at P. We can determine four tangents through P which may be called tangents of fourth order contact with respect to U. For, every plane which passes through such a tangent will intersect S and U in two plane curves which have at least fourth order contact at P. In order to find these tangents we introduce the values of the coefficients, a, b, etc., for the surface U into (108) and equate to zero the terms of the fourth order which do not vanish identically. We thus find that the equation

$$Ix^4 + 24rx^2y^2 + Jy^4 = 0$$

determines the four tangents of fourth order contact between U and S.

The condition that the quartic

$$ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4 = 0$$

may represent a harmonic pencil is

$$j = ace + 2bcd - ad^2 - b^2e - c^3 = 0.$$

The four tangents of fourth order contact between U and S will therefore form a harmonic group if and only if

$$4IJr - 64r^3 = 0,$$

i. e., if

$$r = 0 \qquad \text{or} \qquad r = \pm \tfrac{1}{4}\sqrt{IJ}.$$

Only in the case $r = 0$ will it be possible moreover to arrange these tangents into pairs which divide each other harmonically and which are at the same time harmonic conjugates with respect to the asymptotic tangents of the point P. But for $r = 0$, equation (110) becomes identical with (101), the canonical cubic.

The canonical cubic of a point P is, therefore, completely characterized by the following properties.

1. *It has a unode situated upon the directrix of the second kind of the point P.*
2. *Its uniplane passes through the directrix of the first kind of the point P.*
3. *It has contact of the third order with the given surface S at the point P.*
4. *The four tangents of fourth order contact, between it and the surface S at the point P, form a harmonic pencil.*
5. *If these four tangents be arranged in pairs which divide each other harmonically, the members of each pair are harmonic conjugates also with respect to the asymptotic tangents of P.*

The plane $z = 0$ intersects the canonical cubic in the plane cubic curve

$$x^3 + y^3 + 6xy = 0,$$

or in homogeneous coördinates

$$f \equiv X^3 + Y^3 + 6XY\omega = 0. \tag{111}$$

This plane cubic curve has, as we already know, the point $X = Y = 0$ as double point, the two tangents to the curve at this point being the asymptotic tangents $X = 0$ and $Y = 0$ of the surface S. The Hessian of f, omitting a numerical factor, is

$$H \equiv X^3 + Y^3 - 2XY\omega = 0. \tag{112}$$

The two curves intersect in the double point $X = Y = 0$, which counts for six intersections. The remaining three points of intersection, viz.

$$(113)\quad X=-1,\ Y=1,\ \omega=0;\ X=-\epsilon,\ Y=1,\ \omega=0;\ X=-\epsilon^2\ Y=1,\ \omega=0,$$

where ϵ is an imaginary third root of unity, are all on the line $\omega = 0$. They are the points of inflection of $f = 0$. Therefore we have a further property of the canonical cubic.

The tangent plane intersects the canonical cubic surface in a plane cubic curve which has the point of contact as double point and the asymptotic tangents of this point as tangents. Its three points of inflection are situated upon the directrix of the first kind.

The plane $\omega = 0$, and therefore the canonical tetrahedron, has now been determined. In order to complete the geometrical determination of the system of coördinates which leads to the canonical development, it remains to determine the arbitrary numerical factors. This may be done as follows.

Consider the point of inflection:

$$X = 1, \qquad Y = 1, \qquad \omega = 0,$$

of the plane cubic $f = 0$. The tangent to the curve at that point will be

$$Z = 0, \qquad X + Y - 2\omega = 0.$$

The hyperboloid H, which osculates R_1 and R_2, has the equation

$$8(Z\omega - XY) + Z^2 = 0,$$

when referred to the canonical system of coördinates. The line $X = Y = 0$ intersects this quadric for $Z = 0$ and for $Z + 8\omega = 0$, i. e., in the point of contact and in the further point

$$X = 0, \qquad Y = 0, \qquad Z = -8, \qquad \omega = 1.$$

The plane

$$4(X + Y - 2\omega) - Z = 0$$

contains this point and passes through the inflectional tangent of the plane cubic which was considered above.

We may recapitulate our results as follows:

Two of the edges of the canonical tetrahedron of a regular point P of a non-ruled surface are the asymptotic tangents of that point. Two other edges are the directrices of the first and second kind which belong to the point P. The face of the tetrahedron opposite to P is the uniplane of its canonical cubic surface. This completes the determination of the tetrahedron. The system of coördinates which gives rise to the canonical development refers to this tetrahedron. The numerical factors which still remain arbitrary in a

projective system of coördinates, after the tetrahedron of reference has been chosen, are determined in such a way that the equation

$$4(x+y-2)-z=0 \tag{114}$$

represents a plane, passing through the second point of intersection of the osculating hyperboloid with the directrix of the second kind, as well as through one of the inflectional tangents of the plane curve in which the tangent plane intersects the canonical cubic surface. Since this plane cubic curve has three inflectional tangents, the canonical development may be made in three ways. It is for this reason that there enters into its expression the cube root of a rational invariant.

The quartic

$$Ix^4+Jy^4=0 \tag{115}$$

determines the tangents of fourth-order contact between the surface S and its canonical cubic. The expressions for I and J (96a) show, therefore, that if A is a function of u alone, or if B is a function of v alone, these four tangents coincide with each other and with one of the two asymptotic tangents of the point considered.

§ 8. *Osculating unodal cubic surfaces.*

Since the most general unodal cubic surface depends upon fifteen arbitrary constants, and since the requirement that one surface shall have contact of the fourth order with another is equivalent to fifteen conditions, there will be a finite number of unodal cubics which have contact of the fourth order with the surface S at a given point P. We shall speak of them as its *osculating unodal cubic surfaces.*

If equations (106) represent such an osculating unodal cubic, the coefficients in the expression (108), as far as computed, must all be equal to zero. This gives

$$\begin{gathered}
a=b=m=-\tfrac{1}{6}, \\
p-\mu+3a'=0, \qquad 8p-4\mu+I=0, \\
q-\lambda+3b'=0, \qquad 8q-4\lambda+J=0, \\
p^2+\tfrac{1}{3}q-\tfrac{2}{3}\lambda+3a'\mu-p\mu=0, \\
q^2+\tfrac{1}{3}p-\tfrac{2}{3}\mu+3b'\lambda-q\lambda=0, \\
2pq+r+6b'\mu+6a'\lambda-\lambda\mu-p\lambda-q\mu=0,
\end{gathered} \tag{116}$$

whence

$$\begin{gathered}
\lambda=2q+\tfrac{1}{4}J, \qquad \mu=2p+\tfrac{1}{4}I, \qquad 3a'=p+\tfrac{1}{4}I, \\
3b'=q+\tfrac{1}{4}J, \qquad r=-2pq-\tfrac{3}{4}(Iq+Jp+\tfrac{1}{4}IJ),
\end{gathered} \tag{117}$$

where

$$p^2 - q - \tfrac{1}{6}J + \tfrac{1}{2}Ip + \tfrac{1}{16}I^2 = 0,$$

$$q^2 - p - \tfrac{1}{6}I + \tfrac{1}{2}Jq + \tfrac{1}{16}J^2 = 0,$$

or

$$(118)\qquad \begin{aligned} (p + \tfrac{1}{4}I)^2 - q - \tfrac{1}{6}J &= 0, \\ (q + \tfrac{1}{4}J)^2 - p - \tfrac{1}{6}I &= 0. \end{aligned}$$

Equations (118) will determine, in general, four pairs of values p, q to which will correspond therefore *four osculating unodal cubics.*

Put

$$(119)\qquad p + \tfrac{1}{4}I = p', \qquad q + \tfrac{1}{4}J = q',$$

so that (118) becomes

$$(120)\qquad p'^2 - q' + \tfrac{1}{12}J = 0, \qquad q'^2 - p' + \tfrac{1}{12}I = 0,$$

whence the biquadratic

$$(121)\qquad p'^4 + \tfrac{1}{6}Jp'^2 - p' + \tfrac{1}{12}I + \tfrac{1}{144}J^2 = 0$$

for p'. Let $p'_1, \cdots p'_4$ be its four roots. The uniplanes of the four osculating unodal cubics will be

$$(122)\qquad 1 + p_k x + q_k y + r_k z = 0 \qquad (k = 1, 2, 3, 4)$$

where

$$(123)\qquad \begin{aligned} p_k &= p'_k - \tfrac{1}{4}I, \qquad q_k = q'_k - \tfrac{1}{4}J = p'^2_k - \tfrac{1}{6}J, \\ r_k &= -2p'^3_k - \tfrac{1}{4}Ip'^2_k - \tfrac{5}{12}Jp'_k + \tfrac{1}{24}IJ, \qquad (k = 1, 2, 3, 4). \end{aligned}$$

The determinant of the four planes (122), writing only one term of each column, is

$$(124)\qquad \begin{aligned} \Delta = |1, p_k, q_k, r_k| &= |1, p'_k - \tfrac{1}{4}I, p'^2_k - \tfrac{1}{6}J, \\ &\quad -2p'^3_k - \tfrac{1}{4}Ip'^2_k - \tfrac{5}{12}Jp'_k + \tfrac{1}{24}IJ|, \\ &= (-2)|1, p'_k, p'^2_k, p'^3_k|. \end{aligned}$$

The determinant in the right member is the square root of the discriminant of the biquadratic (121). It is different from zero unless (121) has a pair of coincident roots. Under the same condition Δ will, therefore, be different from zero.

Hence, *unless two of the four osculating unodal cubic surfaces coincide, their four uniplanes form a non-degenerate tetrahedron.*

Moreover, the vertices of this tetrahedron are not in general identical with the four unodes, which form a second non-degenerate tetrahedron.

§ 9. *Geometric interpretation of the semi-covariants.*

We are now in a position to see just what is the geometrical significance of the fundamental semi-covariants z, ρ and σ.

The equation of the osculating hyperboloid referred to the tetrahedron $P_y P_s P_\rho P_\sigma$ was found to be

$$x_1 x_4 - x_2 x_3 + 2a'bx_4^2 = 0. \tag{11}$$

Therefore, the quadrics of the one-parameter family:

$$x_1 x_4 - x_2 x_3 + \lambda x_4^2 = 0 \tag{125}$$

all have contact of the second order with the surface S at the point P and are tangent to each other along the asymptotic tangents of P. There is just one surface of this family which is tangent to the uniplane of the canonical cubic, the equation of which is

$$x_1 + \alpha x_2 + \beta x_3 + \alpha\beta x_4 = 0, \qquad \alpha = \frac{a'_u}{2a'}, \qquad \beta = \frac{b_v}{2b}.$$

In fact, if $(x'_1, \cdots x'_4)$ is a point of (125) its tangent plane is

$$x'_4 x_1 - x'_3 x_2 - x'_2 x_3 + (x'_1 + 2\lambda x'_4) x_4 = 0.$$

If this is identical with the above plane, we must have

$$-\frac{x'_3}{x'_4} = \alpha, \qquad -\frac{x'_2}{x'_4} = \beta, \qquad \frac{x'_1}{x'_4} + 2\lambda = \alpha\beta,$$

whence

$$x'_1 x'_4 - x'_2 x'_3 + 2\lambda x'^2_4 = 0.$$

But, since the point $(x'_1, \cdots x'_4)$ is on the quadric (125), we have also

$$x'_1 x'_4 - x'_2 x'_3 + \lambda x'^2_4 = 0,$$

so that λ must be equal to zero.

Therefore, *there exists a quadric which is uniquely determined by the following properties:*

1. *It has contact of the second order with the surface S at the point P.*
2. *It touches the osculating hyperboloid H, at all of the points of the asymptotic tangents of P.*
3. *It is tangent to the uniplane of the canonical cubic of the point P.*

The equation of this quadric is

$$x_1 x_4 - x_2 x_3 = 0, \tag{126}$$

or in canonical coördinates

$$\zeta - \xi\eta = 0.$$

It is, therefore, the quadric obtained by omitting all but the quadratic terms in the canonical development of S in the vicinity of the point P. We shall, therefore, speak of it as the *canonical quadric* of P.

The points P_z and P_ρ are obviously situated on the asymptotic tangents of P_y, one on each of them. We now notice that the lines P_zP_σ, $P_\rho P_\sigma$ are the generators of the canonical quadric which pass through P_z and P_ρ respectively, so that P_σ is defined as their point of intersection.

Consider the ruled surface R_1 which is generated by P_yP_z if v alone is variable. The point P_z describes a curve C_z on this ruled surface, whose tangent is obtained by joining P_z to P_σ, since $z_v = \sigma$. For every value of u we obtain such a ruled surface, and upon each of them there is a one-parameter family of such curves C_z. Altogether we obtain in this way a double infinity of curves associated with S, situated upon the osculating ruled surfaces of the first kind. We shall call them the *characteristics of* S of the first kind. The characteristics of the second kind are defined in a similar manner.

We have shown in the first memoir * that the transformation

$$\bar{u} = \alpha(u), \qquad \bar{v} = \beta(v),$$

changes y, z, ρ, σ, into

$$(127)\qquad \begin{aligned} &\bar{y} = y, \qquad \bar{z} = \frac{1}{\alpha_u}(z + \tfrac{1}{2}\eta y), \qquad \bar{\rho} = \frac{1}{\beta_v}(\rho + \tfrac{1}{2}\zeta y), \\ &\bar{\sigma} = \frac{1}{\alpha_u\beta_v}(\sigma + \tfrac{1}{2}\eta\rho + \tfrac{1}{2}\zeta z + \tfrac{1}{4}\eta\zeta y), \end{aligned}$$

where

$$(128)\qquad \eta = \frac{\alpha_{uu}}{\alpha}, \qquad \zeta = \frac{\beta_{vv}}{\beta},$$

$\alpha(u)$ and $\beta(v)$ being arbitrary functions of their respective arguments. It will be noticed that, by means of these equations, any characteristic may be transformed into any other of the same kind.

Moreover, if u and v are allowed to vary simultaneously, P_z and P_ρ will describe two surfaces S_z and S_ρ. We shall call S_z the derivative of S_y with respect to u; S_ρ the derivative of S_y with respect to v. The surface S_z is obviously the locus of a single infinity of characteristics of the first kind. The function $\alpha(u)$ may be chosen in such a way that the transformed surface $S_{\bar{z}}$ shall coincide with an arbitrary surface made up of characteristics of the first kind. In fact, the intersection of this surface with the developable of the asymptotic curve $v =$ const., which passes through P_y, is an arbitrary curve on this developable, since η is an arbitrary function of u.

Thus a system of form (1) not only defines its integral surfaces, but two families of ∞^2 curves, its characteristics. Two other surfaces, each being the locus of a single infinity of characteristics, are associated with every integral surface of (1) as its derivatives with respect to u and v respectively. The independent variables of the system may be chosen in such a way that these two

* These Transactions, vol. 8 (1907), p. 256, formulæ (68).

derivatives become arbitrarily assigned surfaces which are loci of characteristics of the first and of the second kind respectively. If a surface S is given, together with its derivatives with respect to u and v, the independent variables are thereby determined except for a linear transformation. The derivatives serve therefore as geometrical images for the independent variables of the system.

Equations (127) show that the asymptotic curves of S_y are contained as limiting case among the characteristics of the surface. In the case of a ruled surface one of the above families of ∞^2 characteristics consists of straight lines; it gives rise to the flecnode congruence of the ruled surface.* The other degenerates into the one-parameter family of curved asymptotic lines.

The surface S_σ, the locus of P_σ, as u and v assume all of their values, may be called the second derivative of S_y with respect to u and v. We have already noticed that the point P_σ is the intersection of the generators of the canonical quadric which pass through P_z and P_ρ respectively. Clearly, P_σ by itself suffices to determine the two points P_z and P_ρ, so that S_σ *is a sufficient image of the two independent variables.* In other words, let there be associated with every point of a surface S a point of its canonical quadric. If the surface which is the locus of these points be regarded as the second derivative of S with respect to two independent variables u and v, the latter are completely determined thereby, except for a linear transformation.

§ 10. *The directrix congruences and the directrix curves of a surface.*

We have seen in § 4 that the osculating linear complexes of the two asymptotic curves, which meet in a point P of the surface S, have a congruence in common whose directrices we denoted by d and d'. One of them, the directrix of the first kind, denoted by d, is situated in the plane tangent to S at P. The other, that of the second kind, denoted by d', passes through the point P.

We shall first consider the directrix of the second kind d'. Referred to the tetrahedron $P_y P_z P_\rho P_\sigma$ its equations are

$$(70b) \qquad 2bx_2 + b_v x_4 = 0, \qquad 2a'x_3 + a'_u x_4 = 0.$$

It passes through P_y and the point P_τ where

$$(129) \qquad \tau = -a'b_v z - ba'_u \rho + 2a'b\sigma.$$

There exists such a line d' for every point P of the surface S, for which the osculating linear complexes of the two asymptotic curves are determinate and distinct. The totality of these lines forms a congruence, one line of which passes through every (general) point of the surface. We shall call it the *directrix congruence of the second kind.*

* Proj. Diff. Geom., p. 175.

If the point P describes an arbitrary curve on S, its directrix of the second kind d' generates a ruled surface of this congruence. We ask ourselves this question: Can we determine a curve on S such that the ruled surface made up of the directrices of the second kind of its points, shall be a developable? According to the general theory of congruences we know that there will be, in general, two one-parameter families of curves of this kind, because the lines of a congruence can be assembled into a single infinity of developables in two ways. The curves on S obtained in this way shall be called the *directrix curves of the second kind.* Through every point P which is not special, i. e., for which the lines d and d' are not indeterminate, there will pass two such curves, one of each family.

We proceed to determine the directrix curves on S, or what amounts to the same thing, the developables of the directrix congruence of the second kind.

Let u and v increase by δu and δv respectively, where δu and δv are infinitesimals. The point P_y will then move to $P_{y+y_u\delta u+y_v\delta v}$, and the directrix of the second kind of this latter point will join it to the point given by

$$\tau + \tau_u \delta u + \tau_v \delta v,$$

where τ is defined by (129).

We find

$$\tau_u = Py + Qz + R\rho + S\sigma, \qquad \tau_v = P'y + Q'z + R'\rho + S'\sigma, \tag{130}$$

where we have put

$$\begin{aligned} P &= a'b_v f + 4a'b^2 g - 2a'bf_v, & Q &= 8a'^2b^2 - a'b_{uv} - a'_u b_v, \\ R &= -(2a'bf + 2a'bb_v + ba'_{uu} + b_u a'_u), & S &= a'_u b + 2a'b_u, \end{aligned} \tag{131}$$

and

$$\begin{aligned} P' &= ba'_u g + 4a'^2bf - 2a'bg_u, & Q' &= -(2a'bg + 2a'ba'_u + a'b_{vv} + a'_v b_v), \\ R' &= 8a'^2b - ba'_{uv} - a'_u b_v, & S' &= a'b_v + 2a'_v b. \end{aligned} \tag{132}$$

The directrix of the second kind which belongs to the point $P_{y+y_u\delta u+y_v\delta v}$ joins it to the point $P_{\tau+\tau_u\delta u+\tau_v\delta v}$. An arbitrary point on this directrix will be given by

$$\lambda(y + y_u\delta u + y_v\delta v) + \mu(\tau + \tau_u\delta u + \tau_v\delta v). \tag{133}$$

The coördinates of such a point will, therefore, be

$$\begin{aligned} x_1 &= \lambda + \mu(P\delta u + P'\delta v), & x_2 &= \lambda\delta u + \mu(-a'b_v + Q\delta u + Q'\delta v), \\ x_3 &= \lambda\delta v + \mu(-ba'_u + R\delta u + R'\delta v), & x_4 &= \mu(2a'b + S\delta u + S'\delta v). \end{aligned}$$

In order that this point may also be a point of d', its coördinates must satisfy equations (70b), which give the conditions

$$\begin{aligned} &2b\delta u\lambda + [2b(Q\delta u + Q'\delta v) + b_v(S\delta u + S'\delta v)]\mu = 0, \\ &2a'\delta v\lambda + [2a'(R\delta u + R'\delta v) + a'_u(S\delta u + S'\delta v)]\mu = 0. \end{aligned} \tag{134}$$

whence

$$(135)\qquad \begin{vmatrix} b\delta u, & (2bQ + b_v S)\delta u + (2bQ' + b_v S')\delta v \\ a'\delta v, & (2a'R + a'_u S)\delta u + (2a'R' + a'_u S')\delta v \end{vmatrix} = 0.$$

Let us put

$$(136)\qquad \begin{aligned} L &= 2a'R + a'_u S = -2a'(2a'bf + 2a'bb_v + ba'_{uu}) + ba'^2_u, \\ 2M &= b(2a'R' + a'_u S') - a'(2bQ + b_v S) \\ &\qquad = 2a'b(a'b_{uv} - ba'_{uv}) + 2b^2 a'_u a'_v - 2a'^2 b_u b_v, \\ N &= 2bQ' + b_v S' = -2b(2a'bg + 2a'ba'_u + a'b_{vv}) + a'b^2_v. \end{aligned}$$

Then (135) becomes

$$(137)\qquad bL\delta u^2 + 2M\delta u\,\delta v - a'N\delta v^2 = 0.$$

This is the quadratic equation which determines the tangents of the two directrix curves which pass through the point P.

From their presence in equation (137) it is clear that L, M, N must be invariants. In fact, if $M = 0$, the directrix curves will be conjugate curves of the surface; if either L or N vanishes, one of the directrix curves is also an asymptotic line of the surface.

We proceed to express L, M and N in terms of the standard invariants of the first memoir. We showed there that all invariants are functions of

$$A = a'b^2, \qquad B = a'^2 b, \qquad H = a'h, \qquad K = bk,$$

where

$$(138)\qquad \begin{aligned} h &= b^2(f + b_v) - \tfrac{1}{4}bb_{uu} + \tfrac{5}{16}b^2_u, \\ k &= a'^2(g + a'_u) - \tfrac{1}{4}a'a'_{vv} + \tfrac{5}{16}a'^2_v, \end{aligned}$$

and of those that could be obtained from these by the operators

$$(139)\qquad U = a'\frac{\partial}{\partial u}, \qquad V = b\frac{\partial}{\partial v},$$

or by the Wronskian process. A rather lengthy calculation gives the following expressions for L, M, and N:

$$(140)\qquad \begin{aligned} L &= \frac{5U(B)^2 - 4a'^2 bU^2(B) - 16a'^6 h}{4a'^4 b}, \\ M &= \frac{a'^3(A, V(A)_u) - b^3(B, U(B)_v)}{3a'^3 b^3}, \\ N &= \frac{5V(A)^2 - 4a'b^2 V^2(A) - 16b^6 k}{4a'b^4}. \end{aligned}$$

Equation (137) determines the tangents of the directrix curves which pass through P. If the point P moves along either of these curves, the directrix of the second kind d' describes a developable. The cuspidal edges of these two developables intersect d' in two points P_i whose coördinates are given by

$$\lambda_i y + \mu_i \tau \qquad (i=1, 2), \tag{141}$$

where, according to (134)

$$\frac{\lambda_i}{\mu_i} = -\frac{2bQ + b_v S + N\dfrac{\delta v_i}{\delta u_i}}{2b} = -\frac{L + (2a'R + a'_u S')\dfrac{\delta v_i}{\delta u_i}}{2a'\dfrac{\delta v_i}{\delta u_i}}, \tag{142}$$

so that λ_1/μ_1 and λ_2/μ_2 are the two roots of the quadratic:

$$\begin{aligned} -4a'b^2\left(\frac{\lambda}{\mu}\right)^2 - 4b\left[a'(2bQ + b_v S) + M\right]\frac{\lambda}{\mu} \\ + bLN - a'(2bQ + b_v S)^2 - 2M(2bQ + b_v S) = 0. \end{aligned} \tag{143}$$

Let us introduce the canonical tetrahedron of reference. The coördinates of the two points P_1 and P_2 are given by (141), so that their coördinates referred to the tetrahedron $P_y P_z P_\rho P_\sigma$ will be

$$x_1 = \lambda_i, \qquad x_2 = -a' b_v \mu_i, \qquad x_3 = -ba'_u \mu_i, \qquad x_4 = 2a' b\mu_i \qquad (i=1, 2).$$

According to (97) the non-homogeneous canonical coördinates of these two points will be

$$x = 0, \qquad y = 0, \qquad z = \frac{64a'^2 b^2 \mu_i}{2\lambda_i - a'_u b_v \mu_i} = \frac{1}{K_i} \qquad (i=1, 2), \tag{144}$$

where

$$K = \frac{2\lambda - a'_u b_v \mu}{64a'^2 b^2 \mu} = -\frac{\alpha\beta}{16a'b} + \frac{1}{32a'^2 b^2}\frac{\lambda}{\mu}, \tag{145}$$

whence

$$\frac{\lambda}{\mu} = 2a' b\alpha\beta + 32a'^2 b^2 K, \qquad \alpha = \frac{a'_u}{2a'}, \qquad \beta = \frac{b_v}{2b}. \tag{146}$$

The two values of K are, consequently, the roots of the quadratic equation:

$$\begin{aligned} 2^{10} a'^4 b^4 K^2 + 2^5 a'^2 b^2 \left[\frac{M}{a'b} + 2Q + 2\beta S + 4a' b\alpha\beta\right] K \\ + (Q + \beta S + 2a' b\alpha\beta)\left(\frac{M}{a'b} + Q + \beta S + 2a' b\alpha\beta\right) - \frac{LN}{4a'b} = 0. \end{aligned} \tag{147}$$

The coefficients of this equation are invariants. We proceed to express them

in terms of some of those introduced previously. We have (cf. equations (25) and (46)),

$$C = 8a'_{uv} - 8\frac{a'_u a'_v}{a'} - 32a'^2 b,$$

$$C' = 8b_{uv} - 8\frac{b_u b_v}{b} - 32a' b^2.$$

If we compute the explicit expression for M, we shall find

$$M = \frac{a'b}{4}(a'C' - bC), \tag{148}$$

and similarly

$$Q + \beta S + 2a'b\alpha\beta = -\tfrac{1}{8}a'C' + 4a'^2b^2. \tag{149}$$

Consequently the quadratic for K becomes

$$\begin{aligned} &2^{10}a'^4b^4K^2 + 2^5a'^2b^2(2a'^2b^2 - \tfrac{1}{4}bC)K \\ &\qquad + (4a'^2b^2 - \tfrac{1}{8}a'C')(4a'^2b^2 + \tfrac{1}{8}a'C' - \tfrac{1}{4}bC) - \frac{LN}{4a'b} = 0, \end{aligned} \tag{150}$$

the invariance of the coefficients being now apparent. Further we find, upon eliminating λ/μ between (142) and (145),

$$K = \frac{C' - 2^5a'b^2}{2^8a'b^2} - \frac{N}{2^6a'^2b^3}\frac{\delta v}{\delta u}. \tag{151}$$

We may recapitulate as follows. *As the point P of the surface moves along one of the two directrix curves, the directrix of the second kind d' describes one of the two developables of the directrix congruence of which d' is a generator and the ratio $\delta u : \delta v$ will satisfy equation* (137). *The non-homogeneous canonical coördinates of the points P_1 and P_2 of d' in which d' meets the respective edges of regression are*

$$x = y = 0, \qquad z = \frac{1}{K},$$

where K is connected with $\delta v/\delta u$ by (151), *and consequently satisfies the quadratic equation* (150).

Let us consider the congruence made up of the directrices of the first kind d. The line d which belongs to a point P_y of the surface joins the points

$$r = -a'_u y + 2a'z, \qquad s = -b_v y + 2b\rho,$$

and its equations are

$$x_4 = 0, \qquad 2a'bx_1 + ba'_u x_2 + a'b_v x_3 = 0.$$

Let infinitesimal increments δu and δv be given to u and v so that the point P_y moves into $P_{y+y_u\delta u+y_v\delta v}$. The directrix of the first kind which belongs to this new point will be the line joining the point

$$r + r_u\delta u + r_v\delta v = [-a'_u - (a'_{uu} + 2a'f)\delta u - a'_{uv}\delta v]y$$
$$+ (2a' + a'_u\delta u + 2a'_v\delta v)z + (-2a'b\delta u - a_u\delta v)\rho + 2a'\delta v\cdot\sigma.$$

to the point

$$s + s_u\delta u + s_v\delta v = [-b_v - b_{uv}\delta u - (b_{vv} + 2bg)\delta v]y$$
$$+ (-b_v\delta u - 2a'b\delta v)z + (2b + 2b_u\delta u + b_v\delta v)\rho + 2b\delta u\sigma,$$

The coördinates of any point

$$\lambda'(r + r_u\delta u + r_v\delta v) + \mu'(s + s_u\delta u + s_v\delta v)$$

of this line, will be

$$(152)\quad \begin{aligned} x_1 &= -[a'_u + (a'_{uu} + 2a'f)\delta u + a'_{uv}\delta v]\lambda' - [b_v + b_{uv}\delta u + (b_{vv} + 2bg)\delta v]\mu', \\ x_2 &= (2a' + a'_u\delta u + 2a'_v\delta v)\lambda' - (b_v\delta u + 4a'b\delta v)\mu', \\ x_3 &= -(4a'b\delta u + a'_u\delta v)\lambda' + (2b + 2b_u\delta u + b_v\delta v)\mu', \\ x_4 &= 2a'\delta v\lambda' + 2b\delta u\mu'. \end{aligned}$$

This point will be upon the line d if its coördinates satisfy the equations

$$x_4 = 0, \qquad 2a'bx_1 + ba'_u x_2 + a'b_v x_3 = 0,$$

of d, i. e., if

$$a'\lambda'\delta v + b\mu'\delta u = 0,$$

$$(153)\quad \begin{aligned} &[2a'b^2(2a'f + a'_{uu}) - b^2a'^2_u + 4a'^2b^2b_v]\,\delta u^2 \\ &+ [2a'b(a'_{uv}b - a'b_{uv}) - ba'_u(2a'_v b + b_v a') + a'b_v(a'_u b + 2b_u a')]\,\delta u\delta v \\ &- [2a'^2b(2bg + b_{vv}) + 4a'^2b^2a'_u - a'^2b_v^2]\,\delta v^2 = 0. \end{aligned}$$

It is found, upon computing the coefficients of this quadratic, that it coincides with (137), so that *the curves determined by* (137) *have the same significance for the directrices of the first as for those of the second kind.* In other words, *if a point P moves along a directrix curve of the surface S, both of the directrices of the point P will describe developables at the same time.*

Moreover we have found

$$(154)\quad \frac{\delta u}{\delta v} = -\frac{a'\lambda'}{b\mu'},$$

so that $\lambda' : \mu'$ will be the two roots of the equation

$$(155)\quad La'\lambda'^2 - 2M\lambda'\mu' - Nb\mu'^2 = 0.$$

Therefore, every directrix of the first kind belongs to two developables of the congruence formed by their totality. The points P'_i where the directrix d, belonging to the point P, intersects the cuspidal edges of the two developables of which it is a generator, are given by

$$\lambda_i' r + \mu_i' s = \lambda_i'(-a_u' y + 2a'z) + \mu_i'(-b_v y + 2b\rho) \qquad (i=1,2),$$

where $\lambda_i' : \mu_i'$ are the two roots of the quadratic (155).

The tangent t_k of one of the directrix curves which passes through P joins this point to the point

$$y + y_u \delta u_k + y_v \delta v_k = y + z\delta u_k + \rho \delta v_k,$$

where $\delta u_k : \delta v_k$ is one of the roots of (137). If l and m are arbitrary quantities, the expression

$$ly + m\delta u_k \cdot z + m\delta v_k \cdot \rho$$

will consequently represent an arbitrary point of this tangent t_k. The point in which t_k intersects d will be obtained if

$$2a'bl + ba_u' m\delta u_k + a'b_v \cdot m\delta v_k = 0,$$

i. e., if

$$\frac{l}{m} = -\frac{ba_u' \delta u_k + a'b_v \delta v_k}{2a'b}.$$

Consequently the point of intersection of t_k and d will be given by

$$(156) \qquad -(ba_u'\delta u_k + a'b_v \delta v_k)y + 2a'b(z\delta u_k + \rho\delta v_k) = b\delta u_k \cdot r + a'\delta v_k \cdot s.$$

But the edge of regression of the developable which d describes as P moves along the directrix curve considered, is the locus of the point

$$\lambda_k' r + \mu_k' s,$$

which, according to (154) is the harmonic conjugate of the point (156). We have the following theorem.

As a point P describes a directrix curve of a surface, its directrices of both kinds describe developables. Let the tangent of one of the directrix curves be constructed at the point P, as well as the asymptotic tangents of this point. The harmonic conjugate of the first line with respect to the other two intersects the directrix of the first kind in a point of the cuspidal edge of the developable which it describes.

If the invariants θ and θ' (cf. equations (8) and (14), be expressed in terms of the fundamental invariants, it will be found that

$$(157) \qquad \theta = 64L, \qquad \theta' = 64N.$$

We find, therefore, the following result. *If the osculating ruled surfaces of the first kind have flecnode curves with coincident branches, the asymptotic curves of the first kind are also directrix curves.* It is obvious from this remark what will be the state of affairs if θ is zero for particular values of u and v only, instead of vanishing identically.

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