SURFACES WITH ISOTHERMAL REPRESENTATION OF THEIR LINES OF CURVATURE AND THEIR TRANSFORMATIONS*

BY

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INTRODUCTION.

In 1897 THYBAUT† showed that minimal surfaces admit of transformations which are very similar to the BÄCKLUND transformations of pseudospherical surfaces. The given minimal surface $\bar{S}$ and the transform $\bar{S}_1$ are the focal sheets of a W-congruence. This transformation is such that the minimal surfaces S and S_1 adjoint to $\bar{S}$ and $\bar{S}_1$ respectively are the sheets of an envelope of spheres whose centers lie on a surface applicable to a paraboloid of revolution, by the theorem of GUICHARD. DARBOUX‡ and BIANCHI§ have considered the pairs of isothermic surfaces which are at the same time in conformal correspondence and form the envelope of a family of spheres, and have thus established a transformation from a given isothermic surface to another isothermic surface which together form such a pair. BIANCHI has shown that these transformations admit of a theorem of permutability very similar to the theorem of this kind which obtains for the BÄCKLUND transformations.

In the present paper we shall show that there is a transformation of surfaces with isothermal spherical representation of their lines of curvature, changing such a surface into one of the same kind. When in particular the given surface is minimal its transform is minimal and the surfaces are in the relation of S and S_1 mentioned above.

It is known that if tangents be drawn to a surface and perpendicular to the direction of an infinitesimal deformation of the surface, these tangents form a W-congruence for which the given surface is one of the focal sheets. If the given surface be minimal and it be required that the lines of curvature correspond on the two surfaces, the second focal surface also is a minimal surface.

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† *Sur la déformation du paraboloide*, etc., Annales de l' École Normale (3), vol. 14 (1897), pp. 45–96.

‡ *Sur les surfaces isothermiques*, Annales de l' École Normale (3), vol. 16 (1899), pp. 491–508.

§ *Ricerche sulle superficie isoterme e sulla deformazione delle quadriche*, Annali di Matematica (3), vol. 11 (1905), pp. 93–157.

In this way we get the THYBAUT transformations of minimal surfaces and obtain the equations of condition on two functions w and ϕ and a constant m, which determine the transformation. The relation between the two surfaces being perfectly reciprocal, the first is a transform of the second and the transformation functions are expressible simply in terms of m, ϕ, w.

We have remarked that the minimal surfaces S and S_1, adjoint to the minimal surfaces $\bar{S}$ and $\bar{S}_1$ which are THYBAUT transforms of one another, are the sheets of an envelope of spheres whose centers lie on a surface applicable to a paraboloid of revolution. By means of a theorem of MOUTARD we show in § 4 that with each surface whose lines of curvature have isothermal spherical representation there are associated ∞^4 surfaces of the same kind, each of which forms with the given one the envelope of a family of spheres depending upon two parameters. The determination of these new surfaces is the same problem as finding the functions m, ϕ, w of a THYBAUT transformation and quadratures. We call the surfaces under discussion the surfaces Σ.

The transformations which we have discovered possess the following *theorem of permutability: If a surface Σ be transformed into two surfaces Σ_1 and Σ_2 of the same kind by means of transformations involving the constants m_1 and m_2 respectively, there exists a surface Σ' which is the transform of Σ_1 and Σ_2 by means of transformations involving m_2 and m_1 respectively; and this surface can be found without quadrature.*

We shall say that these four surfaces Σ, Σ_1, Σ_2, Σ' form a *quatern.* These transformations of the surfaces Σ carry with them a transformation of the surfaces of centers of the spheres enveloped by the several pairs of surfaces Σ; and these latter transformations evidently possess a theorem of permutability similar to the above. In § 7 it is shown that if four surfaces Σ, Σ_1, Σ_2, Σ', form a quatern involving the constants m_1, m_2, each can be transformed by means of a transformation involving a third constant m_3 in such a way that the four new surfaces also form a quatern.

In § 8 we call attention to the fact that surfaces with plane lines of curvature in both systems are surfaces Σ and show that the determination of their transformations reduces to quadratures. When these transforms also have plane lines of curvature in both systems, the surfaces of centers of the spheres enveloped by pairs of them are surfaces of translation whose generators are in perpendicular planes. Moreover, the cyclides of DUPIN play an important rôle in the theory.

We close with a discussion of the surfaces Σ with spherical lines of curvature in one system. In particular, we find that when the curves $v =$ const. of a minimal surface are spherical there can be found by quadratures an infinity of surfaces Σ for which the curves $v =$ const. are spherical and another infinity for which the curves $u =$ const. are spherical.

§ 1. *Transformation of Thybaut.*

Let S be a minimal surface referred to its lines of curvature, and let the parameters be so chosen that the linear element of the surface and of its spherical representation can be written.*

$$ds^2 = e^{2\theta}(du^2 + dv^2), \tag{1}$$

and

$$ds'^2 = e^{-2\theta}(du^2 + dv^2) \tag{2}$$

respectively. Now θ is a solution of the equation

$$\frac{\partial^2\theta}{\partial u^2} + \frac{\partial^2\theta}{\partial v^2} = e^{-2\theta}. \tag{3}$$

Moreover, every solution of this equation gives a minimal surface.

From (1) and (2) it follows that the second fundamental coefficients of the surface are

$$D = -D'' = -1, \qquad D' = 0. \tag{4}$$

Denote by X_1, Y_1, Z_1; X_2, Y_2, Z_2; X, Y, Z; the direction cosines of the tangents to the curves $v =$ const., $u =$ const. on the surface and of its normal. Thus

$$X_1 = e^{-\theta}\frac{\partial x}{\partial u}, \qquad X_2 = e^{-\theta}\frac{\partial x}{\partial v}. \tag{5}$$

From these are found readily †

$$\begin{aligned} &\frac{\partial X_1}{\partial u} = -\frac{\partial\theta}{\partial v}X_2 - e^{-\theta}X, \qquad \frac{\partial X_2}{\partial u} = \frac{\partial\theta}{\partial v}X_1, \qquad \frac{\partial X}{\partial u} = e^{-\theta}X_1, \\ &\frac{\partial X_1}{\partial v} = \frac{\partial\theta}{\partial u}X_2, \qquad \frac{\partial X_2}{\partial v} = -\frac{\partial\theta}{\partial u}X_1 + e^{-\theta}X, \qquad \frac{\partial X}{\partial v} = -e^{-\theta}X_2, \end{aligned} \tag{6}$$

and similar expressions in the Y and Z.

The minimal surface $\bar{S}$ adjoint to S is given by quadratures of the form ‡

$$\frac{\partial\bar{x}}{\partial u} = \frac{\partial x}{\partial v}, \qquad \frac{\partial\bar{x}}{\partial v} = -\frac{\partial x}{\partial u}. \tag{7}$$

The linear element of $\bar{S}$ and of its spherical representation are the same as for S and the second fundamental coefficients have the values

$$\bar{D} = \bar{D}'' = 0, \qquad \bar{D}' = 1. \tag{8}$$

Hence the parametric curves on $\bar{S}$ are its asymptotic lines.

* BIANCHI, *Lezioni*, II, p. 335.

† Ibid., I, p. 123; German translation, p. 94.

‡ Ibid., II, p. 336.

Denote by w the characteristic function in an infinitesimal deformation of $\bar{S}$; then w is any solution of the equation

$$\frac{\partial^2 w}{\partial u \partial v} + \frac{\partial \theta}{\partial v}\frac{\partial w}{\partial u} + \frac{\partial \theta}{\partial u}\frac{\partial w}{\partial v} = 0.^* \tag{9}$$

The direction-cosines of the direction of deformation of $\bar{S}$ are proportional to the coördinates, x_0, y_0, z_0, of a surface S_0. When a solution w of equation (9) is known, these coördinates are given by the quadratures †

$$\frac{\partial x_0}{\partial u} = e^\theta\left(wX_1 - e^\theta \frac{\partial w}{\partial u} X\right), \qquad \frac{\partial x_0}{\partial v} = e^\theta\left(wX_2 + e^\theta \frac{\partial w}{\partial v} X\right). \tag{10}$$

If lines be drawn tangent to $\bar{S}$ and perpendicular to the direction of the infinitesimal deformation of $\bar{S}$, they form a W-congruence, for which $\bar{S}$ is one of the focal sheets. Moreover, this is a general construction for W-congruences. ‡

Denote by ω the angle which the line so drawn at a point of $\bar{S}$ makes with the tangent to the curve $v =$ const. at the point; then

$$\Sigma x_0(X_1 \cos\omega + X_2 \sin\omega) = 0,$$

or

$$\cos\omega : \sin\omega = \Sigma x_0 X_2 : -\Sigma x_0 X_1. \tag{11}$$

We shall put this result in a more suitable form.

We introduce the function T defined by

$$T = \Sigma X x_0, \tag{12}$$

and in accordance with equation (9) we put

$$\frac{\partial \psi}{\partial u} = e^{2\theta}\frac{\partial w}{\partial u}, \qquad \frac{\partial \psi}{\partial v} = -e^{2\theta}\frac{\partial w}{\partial v}. \tag{13}$$

From (12) we get by differentiation

$$\frac{\partial T}{\partial u} = e^{-\theta}\Sigma x_0 X_1 - \frac{\partial \psi}{\partial u}, \qquad \frac{\partial T}{\partial v} = -e^{-\theta}\Sigma x_0 X_2 - \frac{\partial \psi}{\partial v}. \tag{14}$$

It is readily found that

$$\frac{\partial}{\partial v}(e^\theta \Sigma x_0 X_1) = \frac{\partial}{\partial u}(e^\theta \Sigma x_0 X_2).$$

In consequence of this equation we define a function ϕ so that

$$\frac{\partial \phi}{\partial u} = me^\theta \Sigma x_0 X_1, \qquad \frac{\partial \phi}{\partial v} = me^\theta \Sigma x_0 X_2, \tag{15}$$

* Ibid., II, p. 7; German translation, p. 291.

† Ibid., p. 6; German translation, p. 290.

‡ Ibid., II, pp. 52, 53; German translation, p. 316.

where m is an arbitrary constant. With the help of (6) and (10) the second differential coefficients of ϕ are reducible to

$$
(16)\qquad
\begin{aligned}
\frac{\partial^2\phi}{\partial u^2} &= \frac{\partial\theta}{\partial u}\frac{\partial\phi}{\partial u} - \frac{\partial\theta}{\partial v}\frac{\partial\phi}{\partial v} - mT + me^{2\theta}w,\\
\frac{\partial^2\phi}{\partial u\,\partial v} &= \frac{\partial\theta}{\partial v}\frac{\partial\phi}{\partial u} + \frac{\partial\theta}{\partial u}\frac{\partial\phi}{\partial v},\\
\frac{\partial^2\phi}{\partial v^2} &= -\frac{\partial\theta}{\partial u}\frac{\partial\phi}{\partial u} + \frac{\partial\theta}{\partial v}\frac{\partial\phi}{\partial v} + mT + me^{2\theta}w.
\end{aligned}
$$

We return to the consideration of the congruence of tangents to $\bar{S}$. The coördinates of a point on the tangent at the point $(\bar{x}, \bar{y}, \bar{z})$ are, in consequence of (11) and (15), of the form

$$
(17)\qquad \bar{x}_1 = \bar{x} + \frac{\lambda e^{-\theta}}{m}\left(\frac{\partial\phi}{\partial v}X_1 - \frac{\partial\phi}{\partial u}X_2\right),
$$

where λ is a function of u and v. Differentiating with respect to u and v, we get

$$
(18)\qquad
\begin{aligned}
\frac{\partial\bar{x}_1}{\partial u} &= [e^{-\theta}\lambda T + e^{\theta}(1-\lambda w)]X_2 - e^{-2\theta}\frac{\lambda}{m}\frac{\partial\phi}{\partial v}X + \frac{e^{-\theta}}{m}\left(\frac{\partial\phi}{\partial v}X_1 - \frac{\partial\phi}{\partial u}X_2\right)\frac{\partial\lambda}{\partial u},\\
\frac{\partial\bar{x}_1}{\partial v} &= [e^{-\theta}\lambda T - e^{\theta}(1-\lambda w)]X_1 - e^{-2\theta}\frac{\lambda}{m}\frac{\partial\phi}{\partial u}X + \frac{e^{-\theta}}{m}\left(\frac{\partial\phi}{\partial v}X_1 - \frac{\partial\phi}{\partial u}X_2\right)\frac{\partial\lambda}{\partial v}.
\end{aligned}
$$

We denote by $\bar{S}_1$ the second focal sheet of the above congruence and proceed to find the value of λ, in order that the coördinates of $\bar{S}_1$ be given by (17). Also we denote by X', Y', Z' the direction-cosines of the normal to $\bar{S}_1$. They are of the form

$$
(19)\qquad X' = aX + b\left(\frac{\partial\phi}{\partial u}X_1 + \frac{\partial\phi}{\partial v}X_2\right),
$$

where a and b are functions of u and v such that

$$
\Sigma X'\frac{\partial\bar{x}_1}{\partial u} = 0, \qquad \Sigma X'\frac{\partial x_1}{\partial v} = 0.
$$

These equations reduce to

$$
(20)\qquad
\begin{aligned}
e^{-2\theta}\frac{\lambda}{m}\frac{\partial\phi}{\partial v}a - [e^{-\theta}\lambda T + e^{\theta}(1-\lambda w)]\frac{\partial\phi}{\partial v}b &= 0,\\
e^{-2\theta}\frac{\lambda}{m}\frac{\partial\phi}{\partial u}a - [e^{-\theta}\lambda T - e^{\theta}(1-\lambda w)]\frac{\partial\phi}{\partial u}b &= 0.
\end{aligned}
$$

When S is not a surface of revolution, that is when θ is a function of both u and v, the function ϕ also is a function of both u and v.*

It will be found later that for the cases to be considered the function a is always different from zero.

Eliminating a and b from equations (20), we get $\lambda = 1/w$, so that the coördinates of the second focal sheet are of the form

$$\bar{x}_1 = \bar{x} + \frac{e^{-\theta}}{mw}\left(\frac{\partial \phi}{\partial v} X_1 - \frac{\partial \phi}{\partial u} X_2\right). \tag{21}$$

Now equations (18) become

$$\begin{aligned} \frac{\partial \bar{x}_1}{\partial u} &= -\frac{e^{-\theta}}{mw^2}\frac{\partial \phi}{\partial v}\frac{\partial w}{\partial u} X_1 + e^{-\theta}\left(\frac{T}{w} + \frac{1}{mw^2}\frac{\partial \phi}{\partial u}\frac{\partial w}{\partial u}\right) X_2 - e^{-2\theta}\frac{1}{mw}\frac{\partial \phi}{\partial v} X, \\ \frac{\partial \bar{x}_1}{\partial v} &= e^{-\theta}\left(\frac{T}{w} - \frac{1}{mw^2}\frac{\partial \phi}{\partial v}\frac{\partial w}{\partial v}\right) X_1 + \frac{e^{-\theta}}{mw^2}\frac{\partial \phi}{\partial u}\frac{\partial w}{\partial v} X_2 - e^{-2\theta}\frac{1}{mw}\frac{\partial \phi}{\partial u} X. \end{aligned} \tag{22}$$

Since the congruence is a W-congruence, the parametric lines on $\bar{S}_1$ are asymptotic. Hence the necessary and sufficient condition that $\bar{S}_1$ be a minimal surface is that these lines be orthogonal. We shall limit our discussion to W-congruences of this kind.†

From (22) it follows that the condition of orthogonality reduces to

$$\frac{\partial w}{\partial u}\frac{\partial w}{\partial v}\left[\left(\frac{\partial \phi}{\partial u}\right)^2 + \left(\frac{\partial \phi}{\partial v}\right)^2\right] + wmT\left[\frac{\partial \phi}{\partial u}\frac{\partial w}{\partial v} - \frac{\partial \phi}{\partial v}\frac{\partial w}{\partial u}\right] + e^{-2\theta} w^2 \frac{\partial \phi}{\partial u}\frac{\partial \phi}{\partial v} = 0.$$

Since w is a function of both u and v,‡ we can replace this equation by

$$e^{-2\theta}\left[\left(\frac{\partial \phi}{\partial u}\right)^2 + \left(\frac{\partial \phi}{\partial v}\right)^2\right] + mwT\left[\frac{\dfrac{\partial \phi}{\partial u}}{\dfrac{\partial \psi}{\partial u}} + \frac{\dfrac{\partial \phi}{\partial v}}{\dfrac{\partial \psi}{\partial v}}\right] - w^2 \frac{\dfrac{\partial \phi}{\partial u}}{\dfrac{\partial \psi}{\partial u}}\frac{\dfrac{\partial \phi}{\partial v}}{\dfrac{\partial \psi}{\partial v}} = 0,$$

where ψ is given by (13).

If we define a function $\bar{w}$ by

$$\phi = \psi + \bar{w}$$

* For, if $\partial\phi/\partial u = 0$, it follows from (15) that $\Sigma x_0 X_1 = 0$. Differentiate with respect to v; this gives $\Sigma x_0 X_2 = 0$, since $\partial\theta/\partial u \neq 0$. Hence $x_0/X = y_0/Y = z_0/Z = \rho$, where ρ is a factor of proportionality. If x_0, y_0 and z_0 be replaced by these values in equations (10) and similar equations in y_0 and z_0, we are brought to a set of inconsistent equations. Hence ϕ is a function of both u and v.

† From (19) it is seen that when $a = 0$ the normals to $\bar{S}$ and $\bar{S}_1$ are perpendicular, so that the congruence is normal. But the condition that $\bar{S}_1$ be minimal as well as $\bar{S}$ carries with it the correspondence of the lines of curvature on the two surfaces. However, the lines of curvature correspond on the two focal sheets of a normal W-congruence only when these sheets are pseudospherical (BIANCHI, I, p. 283; German translation, p. 244). Hence $a \neq 0$.

‡ As we have excluded the case where θ is a function of u or v alone, it is seen from (9) that w is a constant if it is not a function of both u and v.

and substitute this expression in the above equation, it becomes

$$e^{-2\theta}\left[\left(\frac{\partial\phi}{\partial u}\right)^2+\left(\frac{\partial\phi}{\partial v}\right)^2\right]+2mwT-w^2+mwT\left(\frac{\dfrac{\partial\bar{w}}{\partial u}}{\dfrac{\partial\psi}{\partial u}}+\frac{\dfrac{\partial\bar{w}}{\partial v}}{\dfrac{\partial\psi}{\partial v}}\right)$$

$$-w^2\left(\frac{\dfrac{\partial\bar{w}}{\partial u}}{\dfrac{\partial\psi}{\partial u}}+\frac{\dfrac{\partial\bar{w}}{\partial v}}{\dfrac{\partial\psi}{\partial v}}+\frac{\dfrac{\partial\bar{w}}{\partial u}\dfrac{\partial\bar{w}}{\partial v}}{\dfrac{\partial\psi}{\partial u}\dfrac{\partial\psi}{\partial v}}\right)=0.$$

If this equation be differentiated with respect to u and v and we make use of equations (13) and (16), we find that in each case the resulting equation can be made to vanish identically by taking $\bar{w}$ constant. Now the above equation of condition reduces to

$$(23)\qquad e^{-2\theta}\left[\left(\frac{\partial\phi}{\partial u}\right)^2+\left(\frac{\partial\phi}{\partial v}\right)^2\right]+2mwT-w^2=0,$$

and from (13) it follows that

$$(A)\qquad \frac{\partial\phi}{\partial u}=e^{2\theta}\frac{\partial w}{\partial u},\qquad \frac{\partial\phi}{\partial v}=-e^{2\theta}\frac{\partial w}{\partial v}.$$

In consequence of these equations and (15), equations (14) can be written in the form

$$m\frac{\partial T}{\partial u}=\frac{\partial w}{\partial u}-m\frac{\partial\phi}{\partial u},\qquad m\frac{\partial T}{\partial v}=\frac{\partial w}{\partial v}-m\frac{\partial\phi}{\partial v}.$$

Thus far the function ϕ has been defined only to within an additive constant, hence in all generality we can write the integral of the above equations in the form

$$(24)\qquad mT=w-m\phi.$$

Now equations (16) and (23) assume the fundamental forms

$$(B)\qquad \begin{aligned}\frac{\partial^2\phi}{\partial u^2}&=\frac{\partial\theta}{\partial u}\frac{\partial\phi}{\partial u}-\frac{\partial\theta}{\partial v}\frac{\partial\phi}{\partial v}+me^{2\theta}w+m\phi-w,\\ \frac{\partial^2\phi}{\partial u\partial v}&=\frac{\partial\theta}{\partial v}\frac{\partial\phi}{\partial u}+\frac{\partial\theta}{\partial u}\frac{\partial\phi}{\partial v},\\ \frac{\partial^2\phi}{\partial v^2}&=-\frac{\partial\theta}{\partial u}\frac{\partial\phi}{\partial u}+\frac{\partial\theta}{\partial v}\frac{\partial\phi}{\partial v}+me^{2\theta}w-m\phi+w,\end{aligned}$$

$$(C)\qquad e^{-2\theta}\left[\left(\frac{\partial\phi}{\partial u}\right)^2+\left(\frac{\partial\phi}{\partial v}\right)^2\right]=2m\phi w-w^2.$$

Moreover equations (20) reduce to

$$a=e^{\theta}mTb.$$

Substituting in (19) and expressing the condition

$$X'^2 + Y'^2 + Z'^2 = 1,$$

we get

$$b^2 e^{2\theta} m^2 \phi^2 = 1.$$

From (18) it is found that we must take

$$b = -e^{-\theta}/m\phi,$$

so that (19) becomes

$$X' = \left(1 - \frac{w}{m\phi}\right) X - \frac{e^{-\theta}}{m\phi}\left(\frac{\partial \phi}{\partial u} X_1 + \frac{\partial \phi}{\partial v} X_2\right). \tag{25}$$

From (22) we find for the linear element of $\bar{S}_1$

$$d\bar{s}_1^2 = e^{-2\theta} \frac{\phi^2}{w^2}(du^2 + dv^2); \tag{26}$$

and consequently the linear element of its spherical representation is

$$ds_1'^2 = e^{2\theta} \frac{w^2}{\phi^2}(du^2 + dv^2). \tag{27}$$

The transformation (21) from the surface $\bar{S}$ to the second minimal surface $\bar{S}_1$ was discovered by THYBAUT.* By retracing the steps in the foregoing development it is readily shown that *every pair of functions ϕ, w satisfying the fundamental equations (A), (B), (C), determine a transformation of Thybaut.*

Denote by S_1 the adjoint minimal surface of $\bar{S}_1$; its coördinates are given by

$$\frac{\partial x_1}{\partial u} = -\frac{\partial \bar{x}_1}{\partial v} = e^{-\theta}\frac{\phi}{w} X_1', \qquad \frac{\partial x_1}{\partial v} = \frac{\partial \bar{x}_1}{\partial u} = e^{-\theta}\frac{\phi}{w} X_2', \tag{28}$$

where

$$\begin{aligned} X_1' &= \left[\frac{e^{-2\theta}}{m\phi w}\left(\frac{\partial \phi}{\partial u}\right)^2 - 1\right] X_1 + \frac{e^{-2\theta}}{m\phi w}\frac{\partial \phi}{\partial u}\frac{\partial \phi}{\partial v} X_2 + \frac{e^{-\theta}}{m\phi}\frac{\partial \phi}{\partial u} X, \\ X_2' &= -\frac{e^{-2\theta}}{m\phi w}\frac{\partial \phi}{\partial u}\frac{\partial \phi}{\partial v} X_1 + \left[1 - \frac{e^{-2\theta}}{m\phi w}\left(\frac{\partial \phi}{\partial v}\right)^2\right] X_2 - \frac{e^{-\theta}}{m\phi}\frac{\partial \phi}{\partial v} X. \end{aligned} \tag{29}$$

As thus defined the functions X_1', Y_1', Z_1'; X_2', Y_2', Z_2', are the direction-cosines of the tangents to the curves $v =$ const., $u =$ const., respectively, on S_1.

In consequence of equations (7) we can write the integral of equations (28) in the form

$$x_1 = x - \frac{1}{m}\left[\frac{e^{-\theta}}{w}\frac{\partial \phi}{\partial u} X_1 + \frac{e^{-\theta}}{w}\frac{\partial \phi}{\partial v} X_2 + X\right]. \tag{30}$$

This equation and similar ones for y_1 and z_1 define the transformation from

* *Sur la déformation du paraboloide et sur quelques problèmes qui s'y rattachent.* Annales de l'École Normale, ser. 3, vol. 14 (1897), p. 45.

S to S_1 which is entirely complementary to the THYBAUT transformation of their adjoint surfaces $\bar{S}$ and $\bar{S}_1$.

Referring to equations (25), we see that equation (30) can be written in the form

$$x_1 - \frac{\phi}{w} X' = x - \frac{\phi}{w} X. \tag{30'}$$

Hence the normals to S and S_1 at corresponding points M, M_1, meet in a point M_0 which is distant ϕ/w from M and M_1. Therefore the surfaces S and S_1 are the sheets of the envelope of the spheres of radius ϕ/w whose centers are on the surface which is the locus of M_0. It can be shown that the latter surface is applicable to a paraboloid of revolution.*

§ 2. *Inverse transformation of Thybaut.*

Since $\bar{S}$ and $\bar{S}_1$ are the focal sheets of a congruence the relation between them is entirely reciprocal so that there is a THYBAUT transformation from the latter into the former. We denote the transformation functions by ϕ_1, w_1, m_1 and seek their form.

The equations of transformation are of the form

$$\bar{x} = \bar{x}_1 + \frac{e^{-\theta_1}}{m_1 w_1}\left(\frac{\partial \phi_1}{\partial v} X_1' - \frac{\partial \phi_1}{\partial u} X_2'\right),$$

where, in consequence of (27),

$$e^{-\theta_1} = e^{\theta} \frac{w}{\phi}.$$

Replacing $\bar{x}_1$, X_1', X_2' by their values from (21) and (29), we get an equation of the form

$$AX_1 + BX_2 + CX = 0,$$

where A, B, C are determinate functions of the above quantities. Since the same relation holds for Y and Z, these functions must vanish identically. This gives the three equations

$$\frac{1}{mw}\frac{\partial \phi}{\partial v} + \frac{1}{m_1 w_1 \phi}\left[\frac{1}{m\phi}\left(\frac{\partial \phi}{\partial u}\right)^2 - e^{2\theta} w\right]\frac{\partial \phi_1}{\partial v} + \frac{1}{m m_1 w_1 \phi^2}\frac{\partial \phi}{\partial u}\frac{\partial \phi}{\partial v}\frac{\partial \phi_1}{\partial u} = 0,$$

$$-\frac{1}{mw}\frac{\partial \phi}{\partial u} + \frac{1}{m m_1 w_1 \phi^2}\frac{\partial \phi}{\partial u}\frac{\partial \phi}{\partial v}\frac{\partial \phi_1}{\partial v} + \frac{1}{m_1 w_1 \phi}\left[\frac{1}{m\phi}\left(\frac{\partial \phi}{\partial v}\right)^2 - e^{2\theta} w\right]\frac{\partial \phi_1}{\partial u} = 0,$$

$$\frac{\partial \phi}{\partial u}\frac{\partial \phi_1}{\partial v} + \frac{\partial \phi}{\partial v}\frac{\partial \phi_1}{\partial u} = 0.$$

We replace the last equation by

$$\frac{\partial \phi_1}{\partial u} = -\rho \frac{\partial \phi}{\partial u}, \qquad \frac{\partial \phi_1}{\partial v} = \rho \frac{\partial \phi}{\partial v},$$

* BIANCHI, II, p. 333.

and find by substitution that the first two equations are satisfied if

$$\rho = \frac{\phi m_1 w_1}{m w^2} e^{-2\theta}.$$

Hence

$$(31) \qquad \frac{\partial \phi_1}{\partial u} = -\frac{m_1 w_1 \phi}{m w^2} e^{-2\theta} \frac{\partial \phi}{\partial u}, \qquad \frac{\partial \phi_1}{\partial v} = \frac{m_1 w_1 \phi}{m w^2} e^{-2\theta} \frac{\partial \phi}{\partial v}.$$

But ϕ_1 and w_1 must satisfy equations of the form (A), which are for the present case

$$(32) \qquad \frac{\partial \phi_1}{\partial u} = e^{-2\theta} \frac{\phi^2}{w^2} \frac{\partial w_1}{\partial u}, \qquad \frac{\partial \phi_1}{\partial v} = -e^{-2\theta} \frac{\phi^2}{w^2} \frac{\partial w_1}{\partial v}.$$

Subtracting these equations from (31), we get

$$\frac{\partial w_1}{\partial u} = -\frac{m_1 w_1}{m\phi} \frac{\partial \phi}{\partial u}, \qquad \frac{\partial w_1}{\partial v} = -\frac{m_1 w_1}{m\phi} \frac{\partial \phi}{\partial v},$$

of which the integral is

$$w_1 = \left(\frac{1}{\phi}\right)^{\frac{m_1}{m}},$$

with a particular choice of the constant of integration which has no effect upon the generality of the solution.

Now equations (31) reduce to

$$(33) \qquad \frac{\partial \phi_1}{\partial u} = -\frac{m_1}{m w^2} \left(\frac{1}{\phi}\right)^{\frac{m_1}{m}-1} e^{-2\theta} \frac{\partial \phi}{\partial u}, \qquad \frac{\partial \phi_1}{\partial v} = \frac{m_1}{m w^2} \left(\frac{1}{\phi}\right)^{\frac{m_1}{m}-1} e^{-2\theta} \frac{\partial \phi}{\partial v}.$$

The functions ϕ_1 and w_1 must satisfy equations of the form (B). The second of these is

$$\frac{\partial^2 \phi_1}{\partial u \partial v} = \frac{\partial}{\partial v} \log\left(e^{-\theta} \frac{\phi}{w}\right) \frac{\partial \phi_1}{\partial u} + \frac{\partial}{\partial u} \log\left(e^{-\theta} \frac{\phi}{w}\right) \frac{\partial \phi_1}{\partial v}.$$

When the values from (33) are substituted in this equation, it reduces to $m_1 = m$. Consequently by means of (A) equations (33) can be reduced to

$$\frac{\partial \phi_1}{\partial u} = -\frac{1}{w^2} \frac{\partial w}{\partial u}, \qquad \frac{\partial \phi_1}{\partial v} = -\frac{1}{w^2} \frac{\partial w}{\partial v}.$$

Hence the functions determining the transformation from $\bar{S}_1$ into $\bar{S}$ are given by

$$(34) \qquad \phi_1 = \frac{1}{w}, \qquad w_1 = \frac{1}{\phi}, \qquad m_1 = m.$$

§ 3. *The associate surfaces* Σ_0.

Equation (9) is the tangential equation of the surfaces with the same spherical representation of their lines of curvature as S. Each solution W of this

equation determines such a surface. We call them *surfaces* Σ. The function W gives the distance from the origin to the tangent plane and the rectangular coördinates (ξ, η, ζ) are of the form*

$$\xi = WX + e^{2\theta}\left(\frac{\partial W}{\partial u}\frac{\partial X}{\partial u} + \frac{\partial W}{\partial v}\frac{\partial X}{\partial v}\right),$$

or in consequence of (6),

$$\xi = WX + e^{\theta}\left(\frac{\partial W}{\partial u}X_1 - \frac{\partial W}{\partial v}X_2\right). \tag{35}$$

These surfaces Σ are evidently associate to $\bar{S}$, for the lines of curvature of the former have the same representation as the asymptotic lines of the latter.

Since the function w of a THYBAUT transformation of $\bar{S}$ is a solution of equation (9), every transformation of this kind carries with it the determination of a surface associate to $\bar{S}$. We denote this surface by Σ_0 and by $2q$ the square of the distance from a point on Σ_0 to the origin. From (35), (A) and (C) we get

$$2q = 2m\phi w.$$

Denoting by w_{rs} the second derived covariants of the function w with respect to the linear element of the spherical representation of S we get from (A) and (B)

$$w_{11} = \frac{\partial^2 w}{\partial u^2} + \frac{\partial \theta}{\partial u}\frac{\partial w}{\partial u} - \frac{\partial \theta}{\partial v}\frac{\partial w}{\partial v} = mw + e^{-2\theta}(m\phi - w),$$

$$w_{22} = \frac{\partial^2 w}{\partial v^2} - \frac{\partial \theta}{\partial u}\frac{\partial w}{\partial u} + \frac{\partial \theta}{\partial v}\frac{\partial w}{\partial v} = -mw + e^{-2\theta}(m\phi - w).$$

The sum of the radii of principal curvature of Σ_0 has the value†

$$\rho_1 + \rho_2 = e^{2\theta}(w_{11} + w_{22}) + 2w = 2m\phi,$$

so that the above equation can be written

$$2q = (\rho_1 + \rho_2)w. \tag{36}$$

§ 4. *General transformations of surfaces* Σ.

MOUTARD has established the following theorem:‡ Given two solutions ψ_1, ψ_2, of an equation of the form

$$\frac{\partial^2 \psi}{\partial u \partial v} = M\psi; \tag{37}$$

the equations

$$\frac{\partial}{\partial u}(\psi_1\psi_1') = \psi_1^2\frac{\partial}{\partial u}\left(\frac{\psi_2}{\psi_1}\right), \qquad \frac{\partial}{\partial v}(\psi_1\psi_1') = -\psi_1^2\frac{\partial}{\partial v}\left(\frac{\psi_2}{\psi_1}\right), \tag{38}$$

* BIANCHI, *Lezioni*, I, p. 173; German translation, p. 141.

† BIANCHI, *Lezioni*, I, p. 173; German translation, p. 141.

‡ Ibid., II, p. 69.

are consistent, and the function ψ_1' thus defined is such that

$$\frac{\partial^2 \psi_1'}{\partial u \partial v} = \psi_1 \frac{\partial^2}{\partial u \partial v}\left(\frac{1}{\psi_1}\right)\psi_1'. \tag{39}$$

By means of this theorem we shall establish a transformation of a surface Σ into a surface Σ_1 with the same spherical representation of its lines of curvature as a transform S_1 of S.

If we put

$$w = e^{-\theta}\psi, \tag{40}$$

equation (9) becomes

$$\frac{\partial^2 \psi}{\partial u \partial v} = e^{-\theta}\frac{\partial^2 e^{\theta}}{\partial u \partial v}\psi, \tag{41}$$

which evidently is of the form (37).

Suppose now that we have given a surface Σ determined by a solution W of equation (9) and a pair of functions ϕ, w, giving a THYBAUT transformation of S. Hence two solutions of (41) are

$$\psi_1 = e^{\theta} w, \qquad \psi_2 = e^{\theta} W.$$

Equations (38) reduce to

$$\frac{\partial}{\partial u}(e^{\theta} w \psi_1') = e^{2\theta} w^2 \frac{\partial}{\partial u}\left(\frac{W}{w}\right), \qquad \frac{\partial}{\partial v}(e^{\theta} w \psi_1') = -e^{2\theta} w^2 \frac{\partial}{\partial v}\left(\frac{W}{w}\right) \tag{42}$$

and the corresponding equation (39) is

$$\frac{\partial^2 \psi_1'}{\partial u \partial v} = e^{\theta} w \frac{\partial^2}{\partial u \partial v}\left(\frac{e^{-\theta}}{w}\right)\psi_1'. \tag{43}$$

The tangential equation of the surfaces Σ_1 is evidently

$$\frac{\partial^2 W_1}{\partial u \partial v} - \frac{\partial}{\partial v}\log\left(e^{\theta}\frac{w}{\phi}\right)\frac{\partial W_1}{\partial u} - \frac{\partial}{\partial u}\log\left(e^{\theta}\frac{w}{\phi}\right)\frac{\partial W_1}{\partial v} = 0. \tag{44}$$

If we put

$$W_1 = -\frac{e^{\theta} w}{\phi}\psi_1',$$

the equation for ψ_1' reduces to (43), in consequence of equations (A) and (C). Hence the function W_1, determined by the quadratures

$$\frac{\partial}{\partial u}(\phi W_1) = -e^{2\theta} w^2 \frac{\partial}{\partial u}\left(\frac{W}{w}\right), \qquad \frac{\partial}{\partial v}(\phi W_1) = e^{2\theta} w^2 \frac{\partial}{\partial v}\left(\frac{W}{w}\right), \tag{45}$$

is a solution of equation (44).

The surface whose coördinates are of the form

$$\xi_1 = W_1 X' + e^{-\theta}\frac{\phi}{w}\left(\frac{\partial W_1}{\partial u}X_1' - \frac{\partial W_1}{\partial v}X_2'\right) \tag{46}$$

is therefore associate to the transform $\bar{S}_1$ of $\bar{S}$ by means of the functions ϕ, w. If the expressions for X', X'_1, X'_2, given by (25) and (29) and of $\partial W_1/\partial u$, $\partial W_1/\partial v$, be substituted in (46), it can be reduced to the form

$$\xi_1 = \xi + (X' - X)R, \tag{47}$$

where we have put

$$R = \frac{1}{w}\left[m\phi W_1 + \left(\frac{\partial W}{\partial u}\frac{\partial \phi}{\partial u} - \frac{\partial W}{\partial v}\frac{\partial \phi}{\partial v}\right) - W(m\phi - w)\right]. \tag{48}$$

From (47) it is evident that the surfaces Σ and Σ_1 are the sheets of the envelopes of spheres of radius R whose centers lie on the surface $\bar{\Sigma}$ defined by

$$\bar{\xi} = \xi - RX, \qquad \bar{\eta} = \eta - RY, \qquad \bar{\zeta} = \zeta - RZ. \tag{49}$$

From these expressions we get by differentiation

$$\frac{\partial \bar{\xi}}{\partial u} = Ae^{\theta}X_1 - \frac{\partial R}{\partial u}X, \qquad \frac{\partial \bar{\xi}}{\partial v} = -Be^{\theta}X_2 - \frac{\partial R}{\partial v}X,$$

where we have put for the sake of brevity

$$\begin{aligned} A &= \frac{\partial^2 W}{\partial u^2} + \frac{\partial \theta}{\partial u}\frac{\partial W}{\partial u} - \frac{\partial \theta}{\partial v}\frac{\partial W}{\partial v} + (W - R)e^{-2\theta}, \\ B &= \frac{\partial^2 W}{\partial v^2} - \frac{\partial \theta}{\partial u}\frac{\partial W}{\partial u} + \frac{\partial \theta}{\partial v}\frac{\partial W}{\partial v} + (W - R)e^{-2\theta}. \end{aligned} \tag{49'}$$

But it is readily found from (48) that

$$\frac{\partial R}{\partial u} = \frac{A}{w}\frac{\partial \phi}{\partial u}, \qquad \frac{\partial R}{\partial v} = -\frac{B}{w}\frac{\partial \phi}{\partial v},$$

so that the above equations can be written

$$\frac{\partial \bar{\xi}}{\partial u} = A\left(e^{\theta}X_1 - \frac{1}{w}\frac{\partial \phi}{\partial u}X\right), \qquad \frac{\partial \bar{\xi}}{\partial v} = B\left(-e^{\theta}X_2 + \frac{1}{w}\frac{\partial \phi}{\partial v}X\right). \tag{50}$$

From these expressions and similar ones for $\bar{\eta}$ and $\bar{\zeta}$ we find that the direction-cosines of the normal to the surface $\bar{\Sigma}$ are of the form

$$\bar{X} = \sqrt{\frac{w}{2m\phi}}\left[X + \frac{e^{-\theta}}{w}\left(\frac{\partial \phi}{\partial u}X_1 + \frac{\partial \phi}{\partial v}X_2\right)\right]. \tag{51}$$

As defined, a surface $\bar{\Sigma}$ is the locus of the centers of spheres whose envelope consists of a surface Σ and its transform by means of a set of functions m, w, ϕ. Since the expression (51) for $\bar{X}$ involves only the latter functions, all the transforms, by means of these functions, of the surfaces Σ correspond with parallelism of tangent planes. And furthermore it follows from (50) that the tangents to the parametric curves on these surfaces $\bar{\Sigma}$ are parallel; hence the parametric curves form a conjugate system.

§ 5. *Special cases.*

We have seen that the minimal surfaces S and S_1 are the sheets of the envelope of a family of spheres. Now we shall show that this transformation is the same as that just given for any surface Σ.

From (30) and (25) we get

$$(52)\qquad W_1 = \Sigma x_1 X' = \left(1 - \frac{w}{m\phi}\right) W - \frac{e^{-\theta}}{m\phi}\left[\Sigma x X_1 \frac{\partial \phi}{\partial u} + \Sigma x X_2 \frac{\partial \phi}{\partial v}\right] + \frac{1}{m}.$$

If we differentiate the equation

$$W = \Sigma x X,$$

we get

$$\frac{\partial W}{\partial u} = e^{-\theta}\Sigma x X_1, \qquad \frac{\partial W}{\partial v} = -e^{-\theta}\Sigma x X_2.$$

When these values are substituted in (52) and (48), we have

$$R = \frac{\phi}{w},$$

in which case equations (47) reduce to (30′). Hence S_1 is the transform of S in the sense of the generalized transformation of any surface Σ.

Let us now apply this transformation to the surface Σ_0, that is, we have $W = w$. From (45) it is seen that $W_1\phi$ is constant. Taking $W_1 = 1/\phi$ we get, as we have seen before, the surface Σ_{10} which determines the inverse THYBAUT transformation from $\bar{S}_1$ into $\bar{S}$. Now from (48) we have

$$R = m\left(\frac{1}{w} + \phi\right).$$

From (34) it is seen that this expression may be written

$$R = m\left(\phi_1 + \frac{1}{w_1}\right),$$

which puts in evidence the reciprocal character of the transformation.

In consequence of equations (A) equations (45) can be written in the form

$$(45')\qquad \begin{aligned} \phi\frac{\partial W_1}{\partial u} &= -e^{2\theta} w \frac{\partial W}{\partial u} + (W - W_1)\frac{\partial \phi}{\partial u}, \\ \phi\frac{\partial W_1}{\partial v} &= e^{2\theta} w \frac{\partial W}{\partial v} + (W - W_1)\frac{\partial \phi}{\partial v}. \end{aligned}$$

From this it is seen that if W_1 and W are any functions satisfying this equation, so also are $W_1 + a$ and $W + a$ for all values of the constant a. Hence *surfaces parallel to a given surface Σ are transformed into surfaces parallel to Σ_1 and at the same distance.*

Suppose we have two solutions W and W' of equation (9) and that the corresponding solutions of (45′) are denoted by W_1 and W'_1. It is evident that $aW + bW'$ is a solution of equation (9) for all values of the constants a, b, and the corresponding solution of (45′) is $aW_1 + bW'_1$. Hence *the locus of a point dividing in constant ratio the join of corresponding points on two surfaces Σ, Σ' is a surface of the same kind. And the transform of this locus by means of the functions m, ϕ, w, divides in the same ratio the join of corresponding points of the transforms of Σ and Σ' by means of the same functions.*

When in particular W' is w, then W_1 is $1/\phi$ as we have seen. From this we remark that by varying the constant of integration obtained from the quadratures (45), we get the surfaces which divide in constant ratio the joins of corresponding points on the surface for which the constant is zero, and the surface Σ_{10} previously defined.

§ 6. *Theorem of permutability.*

There is a theorem of permutability for the transformations of THYBAUT very similar to the theorem of permutability of the BÄCKLUND transformations, as discovered by BIANCHI. It is as follows: Given two transforms $\bar{S}_1$ and $\bar{S}_2$ of a minimal surface $\bar{S}$ by means of transformations involving respective constants m_1, m_2; there exists a minimal surface $\bar{S}'$ which is the THYBAUT transform of $\bar{S}_1$ and $\bar{S}_2$ by means of transformations involving the same constants in inverse order.

This theorem is a direct result of a more general theorem due to BIANCHI.* Instead of considering the surfaces $\bar{S}$, $\bar{S}_1$, $\bar{S}_2$, we take their adjoints S, S_1, S_2. We have seen that S and S_1 are sheets of an envelope of spheres whose centers lie on a surface; and the same is true of S and S_2. DARBOUX has considered pairs of surfaces which are the sheets of an envelope of spheres in such a way that the two sheets are represented conformally upon one another; he has found that both the surfaces are isothermic and their lines of curvature correspond. BIANCHI * has made a profound study of these transformations of one isothermic surface into another and has denoted such a transformation by D_m, thus putting in evidence the constant m which appears in the equations. He has established a theorem of permutability of these transformations which leads to the theorem stated above. We shall write down the results of his investigation for the case where the given isothermal surface is minimal and refer the reader to his memoir for their derivation.

The surfaces S_1 and S_2 are defined by equations of the form (30), when the functions m, ϕ, w are replaced by m_1, ϕ_1, w_1 and m_2, ϕ_2, w_2 successively. In like manner transforms S' and S'' of S_1 and S_2 respectively are defined by

* *Ricerche sulle superficie isoterme*, etc., loc. cit.

equations of the form

$$x' = x_1 - \frac{1}{m_2}\left[\frac{e^{-\theta_1}}{w_1'}\frac{\partial \phi_1'}{\partial u}X_1^{(1)} + \frac{e^{-\theta_1}}{w_1'}\frac{\partial \phi_1'}{\partial v}X_2^{(1)} + X^{(1)}\right],$$

$$x'' = x_2 - \frac{1}{m_1}\left[\frac{e^{-\theta_2}}{w_2'}\frac{\partial \phi_2'}{\partial u}X_1^{(2)} + \frac{e^{-\theta_2}}{w_2'}\frac{\partial \phi_2'}{\partial v}X_2^{(2)} + X^{(2)}\right].$$

From (27) it follows that

$$e^{-\theta_1} = e^{\theta}\frac{w_1}{\phi_1}, \qquad e^{-\theta_2} = e^{\theta}\frac{w_2}{\phi_2}; \tag{53}$$

and $X_1^{(1)}$, $X_2^{(1)}$, $X^{(1)}$; $X_1^{(2)}$, $X_2^{(2)}$, $X^{(2)}$; are given by (25) and (29) when ϕ, w, m are replaced by ϕ_1, w_1, m_1 and ϕ_2, w_2, m_2 respectively.

BIANCHI shows* that S' and S'' coincide when

$$\begin{aligned} w_1' &= \frac{\Phi_1}{\phi_1} + (m_2 - m_1)w_2, & \phi_1' &= \frac{\Phi_1}{w_1} + (m_2 - m_1)\phi_2, \\ w_2' &= \frac{\Phi_2}{\phi_2} + (m_1 - m_2)w_1, & \phi_2' &= \frac{\Phi_2}{w_2} + (m_1 - m_2)\phi_1, \end{aligned} \tag{54}$$

where

$$\begin{aligned} \Phi_1 &= e^{-2\theta}\left(\frac{\partial \phi_1}{\partial u}\frac{\partial \phi_2}{\partial u} + \frac{\partial \phi_1}{\partial v}\frac{\partial \phi_2}{\partial v}\right) + w_1 w_2 - m_2(w_1\phi_2 + w_2\phi_1), \\ \Phi_2 &= e^{-2\theta}\left(\frac{\partial \phi_1}{\partial u}\frac{\partial \phi_2}{\partial u} + \frac{\partial \phi_1}{\partial v}\frac{\partial \phi_2}{\partial v}\right) + w_1 w_2 - m_1(w_1\phi_2 + w_2\phi_1). \end{aligned} \tag{55}$$

Moreover, the above functions satisfy fundamental equations for S_1 and S_2 similar to (A), (B), (C) for S.

From (21) it follows that the THYBAUT transform of $\bar{S}_1$ by means of the above functions is defined by equations of the form

$$\bar{x}' = \bar{x}_1 + \frac{e^{-\theta_1}}{m_2 w_1'}\left(\frac{\partial \phi_1'}{\partial v}X_1^{(1)} - \frac{\partial \phi_1'}{\partial u}X_2^{(1)}\right).$$

By means of (53) we get

$$\begin{aligned} e^{-\theta_1}\frac{\partial \phi_1'}{\partial u} &= -\frac{\Phi_1}{\phi_1 w_1}e^{-\theta}\frac{\partial \phi_1}{\partial u} - (m_2 - m_1)e^{-\theta}\frac{\partial \phi_2}{\partial u}, \\ e^{-\theta_1}\frac{\partial \phi_1'}{\partial v} &= \frac{\Phi_1}{\phi_1 w_1}e^{-\theta}\frac{\partial \phi_1}{\partial v} + (m_2 - m_1)e^{-\theta}\frac{\partial \phi_2}{\partial v}. \end{aligned} \tag{56}$$

* Loc. cit., p. 119. It must be remarked that BIANCHI has put $\lambda = e^{-\theta}(\partial\phi/\partial u)$, $\mu = e^{-\theta}(\partial\phi/\partial v)$ to reduce his formulæ to the form, p. 119, and that his function σ reduces to w when S is minimal.

Hence the above equation can be reduced to

$$
\begin{aligned}
\bar{x}' = \bar{x} + \frac{m_2 - m_1}{m_1 m_2 w_1' \phi_1} e^{-\theta} \Bigg\{ & \left[(m_1\phi_1 - w_1)\frac{\partial \phi_2}{\partial v} - (m_2\phi_2 - w_2)\frac{\partial \phi_1}{\partial v} \right] X_1 \\
(57) \qquad & - \left[(m_1\phi_1 - w_1)\frac{\partial \phi_2}{\partial u} - (m_2\phi_2 - w_2)\frac{\partial \phi_1}{\partial u} \right] X_2 \\
& + e^{-\theta}\left(\frac{\partial \phi_2}{\partial v}\frac{\partial \phi_1}{\partial u} - \frac{\partial \phi_2}{\partial u}\frac{\partial \phi_1}{\partial v} \right) X \Bigg\}.
\end{aligned}
$$

From (54) and (55) it follows that

$$w_1' \phi_1 = w_2' \phi_2.$$

Hence the right-hand member of equation (57) is symmetrical with respect to the sets of functions m_1, ϕ_1, w_1; m_2, ϕ_2, w_2. Consequently $\bar{S}'$ is also the transform of $\bar{S}_2$ by means of the functions m_1, ϕ_2', w_2'.

It follows from (57) that $\bar{S}$ and $\bar{S}'$ coincide when $m_1 = m_2$ and only in this case.

We shall now show that the theorem of permutability of THYBAUT transformations, as just derived, can be extended to the transformations, which we have found, of surfaces with the same spherical representation of their lines of curvature as S.

Given a surface Σ and as before denote by W the distance from the origin of its tangent plane. By means of the functions (m_1, ϕ_1, w_1) and (m_2, ϕ_2, w_2) we transform Σ into Σ_1 and Σ_2 respectively. Denote by W_1 and W_2 the distances of the tangent planes of these respective surfaces from the origin; these functions are given by the quadratures (45)

$$(58) \qquad \frac{\partial}{\partial u}(\phi_1 W_1) = -e^{2\theta} w_1^2 \frac{\partial}{\partial u}\left(\frac{W}{w_1}\right), \qquad \frac{\partial}{\partial v}(\phi_1 W_1) = e^{2\theta} w_1^2 \frac{\partial}{\partial v}\left(\frac{W}{w_1}\right),$$

$$(59) \qquad \frac{\partial}{\partial u}(\phi_2 W_2) = -e^{2\theta} w_2^2 \frac{\partial}{\partial u}\left(\frac{W}{w_2}\right), \qquad \frac{\partial}{\partial v}(\phi_2 W_2) = e^{2\theta} w_2^2 \frac{\partial}{\partial v}\left(\frac{W}{w_2}\right).$$

By means of the functions (m_2, ϕ_1', w_1') and (m_1, ϕ_2', w_2'), given by (54) the surfaces Σ_1 and Σ_2 can be transformed into surfaces with the same representation of their lines of curvature as S'. We shall show that the two new surfaces coincide.

In order that this may be true the function W', representing the distance from the origin to the tangent plane to Σ', must satisfy the four equations

$$(60) \qquad \frac{\partial}{\partial u}(\phi_1' W') = -e^{-2\theta}\frac{\phi_1^2}{w_1^2} w_1'^2 \frac{\partial}{\partial u}\left(\frac{W_1}{w_1'}\right), \qquad \frac{\partial}{\partial v}(\phi_1' W') = e^{-2\theta}\frac{\phi_1^2}{w_1^2} w_1'^2 \frac{\partial}{\partial v}\left(\frac{W_1}{w_1'}\right).$$

$$(61) \qquad \frac{\partial}{\partial u}(\phi_2' W') = -e^{-2\theta}\frac{\phi_2^2}{w_2^2} w_2'^2 \frac{\partial}{\partial u}\left(\frac{W_2}{w_2'}\right), \qquad \frac{\partial}{\partial v}(\phi_2' W') = e^{-2\theta}\frac{\phi_2^2}{w_2^2} w_2'^2 \frac{\partial}{\partial v}\left(\frac{W_2}{w_2'}\right).$$

Similar to equations (A) we have the set

$$\frac{\partial \phi_i'}{\partial u} = e^{-2\theta}\frac{\phi_i^2}{w_i^2}\frac{\partial w_i'}{\partial u}, \qquad \frac{\partial \phi_i'}{\partial v} = -e^{-2\theta}\frac{\phi_i^2}{w_i^2}\frac{\partial w_i'}{\partial v} \qquad (i=1,2).$$

If we make use of these equations and (56) in the result obtained by eliminating $\partial W'/\partial u$ from the first of equations (60) and (61), we can reduce it to the form

$$(62)\quad \Omega'\left(\frac{1}{w_2}\frac{\partial \phi_2}{\partial u} - \frac{1}{w_1}\frac{\partial \phi_1}{\partial u}\right)[\Omega' W' - \Omega W + (m_2 - m_1)(\phi_1 W_1 w_2 - \phi_2 W_2 w_1)] = 0,$$

where for the sake of brevity we have put

$$(63)\quad \begin{aligned} \Omega' &= \Phi_1 + (m_2 - m_1)\phi_2 w_1 = \Phi_2 + (m_1 - m_2)\phi_1 w_2, \\ \Omega &= \Phi_1 + (m_2 - m_1)\phi_1 w_2 = \Phi_2 + (m_1 - m_2)\phi_2 w_1. \end{aligned}$$

In like manner the elimination of $\partial W'/\partial v$ from the second of equations (60) and (61) leads to

$$(64)\quad \Omega'\left(\frac{1}{w_2}\frac{\partial \phi_2}{\partial v} - \frac{1}{w_1}\frac{\partial \phi_1}{\partial v}\right)[\Omega' W' - \Omega W + (m_2 - m_1)(\phi_1 W_1 w_2 - \phi_2 W_2 w_1)] = 0.$$

From (63) we get by differentiation

$$\frac{\partial \Omega'}{\partial u} = (m_2 - m_1)e^{-2\theta}\left(\phi_2\frac{\partial \phi_1}{\partial u} - \phi_1\frac{\partial \phi_2}{\partial u}\right),$$

$$\frac{\partial \Omega'}{\partial v} = (m_1 - m_2)e^{-2\theta}\left(\phi_2\frac{\partial \phi_1}{\partial v} - \phi_1\frac{\partial \phi_2}{\partial v}\right).$$

Hence Ω' cannot vanish unless ϕ_2 is a function of ϕ_1. Moreover, it is only in this case that the quantities in parentheses in (62) and (64) can vanish simultaneously. Hence the expression in brackets in these two equations must vanish, so that

$$(65)\qquad \Omega' W' = (m_1 - m_2)(\phi_1 W_1 w_2 - \phi_2 W_2 w_1) + \Omega W.$$

It is readily shown that this value of W' satisfies equations (60) and (61). Hence we have the following theorem of permutability:

If a surface Σ with isothermal representation of its lines of curvature be transformed into two surfaces Σ_1 and Σ_2 of the same kind by means of transformations T_{m_1}, T_{m_2} respectively, there exists a surface Σ' which is the transform of Σ_1 and Σ_2 by transformations T'_{m_2}, T'_{m_1} respectively; and this surface can be obtained without quadratures.

We have seen that the surfaces Σ and Σ_1 form the envelope of a doubly infinite family of spheres whose centers lie on a surface $\bar{\Sigma}_1$ with a conjugate system of lines in correspondence with the lines of curvature on Σ. In like manner there are surfaces $\bar{\Sigma}_2$, $\bar{\Sigma}_1'$, $\bar{\Sigma}_2'$, which are the loci of the centers of the spheres envel-

oped by the pairs of surfaces Σ, Σ_2; Σ_1, Σ'; Σ_2, Σ' respectively. The transformations which we have been considering can be looked upon as transforming the surfaces $\bar{\Sigma}_1$, $\bar{\Sigma}_2$, $\bar{\Sigma}_1'$, $\bar{\Sigma}_2'$ into one another; we have consequently for these transformations a theorem of permutability.

We have seen that all the surfaces $\bar{\Sigma}_1$ corresponding to transformations of surfaces Σ by means of the same values of the functions m, ϕ, w have a conjugate system in correspondence with the lines of curvature of the surfaces Σ; and the tangents to the curves of this system at corresponding points of two surfaces $\bar{\Sigma}_1$ are parallel. Hence the congruences formed by joining corresponding points of two of these surfaces $\bar{\Sigma}_1$ have their developable surfaces in correspondence with the lines of curvature of the surfaces Σ. We can, therefore, look upon the above transformations as transformations of such a congruence into another whose developables correspond to the developables of the former. Furthermore, there is a theorem of permutability for these congruence transformations.

The function w_1' whose expression is given by (54) is a solution of the tangential LAPLACE equation (44) for S_1. Hence there is a surface Σ_1 determined by this value of W_1. In order to determine the surface Σ of which it is the transform by means of m_1, ϕ_1, w_1, we substitute its value (54) in equation (58) and find that

$$W = (m_2 - m_1) w_2.$$

In like manner it is found that the surface Σ determined by

$$W = (m_1 - m_2) w_1$$

is transformed by means of the functions m_2, ϕ_2, w_2 into the surface Σ_2 whose tangent plane is at the distance w_2' from the origin.

It is readily found from (48) that the radius of the spheres whose envelope is formed of the surfaces Σ and Σ_1 is given by $R_1 = m_2 \Phi_2 / w_1$, and for Σ and Σ_2 $R_2 = m_1 \Phi_1 / w_2$.

§ 7. *Groups of eight surfaces* Σ.

Given a surface Σ and three surfaces Σ_1, Σ_2, Σ_3 transforms of it by transformations $T_1(w_1, \phi_1, m_1)$, $T_2(w_2, \phi_2, m_2)$, $T_3(w_3, \phi_3, m_3)$. In consequence of the theorem of permutability we can find without quadrature surfaces of the same kind Σ_{12}, Σ_{23}, Σ_{31}, which are transforms of the respective pairs of surfaces (Σ_1, Σ_2), (Σ_2, Σ_3), (Σ_3, Σ_1). Thus a surface Σ_{ik} is the transform of Σ by the successive application of transformations T_i, T_k' or T_k, T_i'; consequently Σ_{ik} is the same surface as Σ_{ki}. Consider now three surfaces Σ_i, Σ_{ik}, Σ_{il}, where $k \neq l \neq i$; the surfaces Σ_{ik} and Σ_{il} are supposed to be obtained from Σ_i by transformations T_k', T_l' respectively. By the theorem of permutability there is a fourth surface Σ_{ikl} which is the transform of Σ_{ik} by a transformation T_l'' and

of Σ_{il} by a transformation T''_k. We are now going to show that all the surfaces obtained as Σ_{ikl} and corresponding to a permutation of the subscripts i, k, l are one and the same surface.

For the sake of brevity we put

$$
(66)\qquad
\begin{aligned}
L_{12} &= e^{-2\theta}\left(\frac{\partial\phi_1}{\partial u}\frac{\partial\phi_2}{\partial u}+\frac{\partial\phi_1}{\partial v}\frac{\partial\phi_2}{\partial v}\right)+w_1w_2,\\
L_{23} &= e^{-2\theta}\left(\frac{\partial\phi_2}{\partial u}\frac{\partial\phi_3}{\partial u}+\frac{\partial\phi_2}{\partial v}\frac{\partial\phi_3}{\partial v}\right)+w_2w_3,\\
L_{31} &= e^{-2\theta}\left(\frac{\partial\phi_3}{\partial u}\frac{\partial\phi_1}{\partial u}+\frac{\partial\phi_3}{\partial v}\frac{\partial\phi_1}{\partial v}\right)+w_3w_1.
\end{aligned}
$$

Evidently $L_{ik}=L_{ki}$, for $k \neq i$.

Denote by (m_1, w_{21}, ϕ_{21}) and (m_3, w_{23}, ϕ_{23}) the functions determining the transformations which change Σ_2 into two surfaces Σ_{21} and Σ_{23} respectively. From (54) and (55) it follows that

$$
(67)\qquad
\begin{aligned}
w_{21} &= \frac{L_{12}-m_1w_2\phi_1-m_2w_1\phi_2}{\phi_2}, & \phi_{21} &= \frac{L_{12}-m_1w_1\phi_2-m_2w_2\phi_1}{w_2},\\
w_{23} &= \frac{L_{23}-m_3w_2\phi_3-m_2w_3\phi_2}{\phi_2}, & \phi_{23} &= \frac{L_{23}-m_3w_3\phi_2-m_2w_2\phi_3}{w_2}.
\end{aligned}
$$

If we write the linear element of the spherical representation of Σ_{21} in the form

$$ds'^2_{12}=e^{-2\theta_{12}}(du^2+dv^2),$$

it follows from (53) that

$$
(68)\qquad e^{-\theta_{12}}=e^{\theta_2}\frac{w_{21}}{\phi_{21}}=e^{-\theta}\frac{L_{12}-m_1w_2\phi_1-m_2w_1\phi_2}{L_{12}-m_1w_1\phi_2-m_2w_2\phi_1}.
$$

We denote by Σ_{213}, or Σ_{231}, the surface which forms with Σ_2, Σ_{21}, Σ_{23} a quatern and write the linear element of its spherical representation in the form

$$ds'^2_{213}=e^{-2\theta_{213}}(du^2+dv^2).$$

From (68) and (53) it follows that

$$
(69)\qquad e^{-\theta_{213}}=e^{-\theta_2}\frac{K}{J}=e^{\theta}\frac{w_2K}{\phi_2J},
$$

where we have put for the sake of brevity

$$
(70)\qquad
\begin{aligned}
J &= e^{-2\theta_2}\left(\frac{\partial\phi_{21}}{\partial u}\frac{\partial\phi_{23}}{\partial u}+\frac{\partial\phi_{21}}{\partial v}\frac{\partial\phi_{23}}{\partial v}\right)+w_{21}w_{23}-m_1w_{21}\phi_{23}-m_3w_{23}\phi_{21},\\
K &= e^{-2\theta_2}\left(\frac{\partial\phi_{21}}{\partial u}\frac{\partial\phi_{23}}{\partial u}+\frac{\partial\phi_{21}}{\partial v}\frac{\partial\phi_{23}}{\partial v}\right)+w_{21}w_{23}-m_1w_{23}\phi_{21}-m_3w_{21}\phi_{23}.
\end{aligned}
$$

From (67) we get by differentiation

$$
(71)\quad
\begin{aligned}
-e^{-\theta_2}\frac{\partial\phi_{21}}{\partial u} &= \frac{L_{12}-m_1w_1\phi_2-m_1w_2\phi_1}{\phi_2w_2}e^{-\theta}\frac{\partial\phi_2}{\partial u}+(m_1-m_2)e^{-\theta}\frac{\partial\phi_1}{\partial u},\\
e^{-\theta_2}\frac{\partial\phi_{21}}{\partial v} &= \frac{L_{12}-m_1w_1\phi_2-m_1w_2\phi_1}{\phi_2w_2}e^{-\theta}\frac{\partial\phi_2}{\partial v}+(m_1-m_2)e^{-\theta}\frac{\partial\phi_1}{\partial v},\\
-e^{-\theta_2}\frac{\partial\phi_{23}}{\partial u} &= \frac{L_{23}-m_3w_3\phi_2-m_3w_2\phi_3}{\phi_2w_2}e^{-\theta}\frac{\partial\phi_2}{\partial u}+(m_3-m_2)e^{-\theta}\frac{\partial\phi_3}{\partial u},\\
e^{-\theta_2}\frac{\partial\phi_{23}}{\partial v} &= \frac{L_{23}-m_3w_3\phi_2-m_3w_2\phi_3}{\phi_2w_2}e^{-\theta}\frac{\partial\phi_2}{\partial v}+(m_3-m_2)e^{-\theta}\frac{\partial\phi_3}{\partial v}.
\end{aligned}
$$

Substituting these values and those from (67) in (70), we get

$$
(72)\quad
\begin{aligned}
\phi_2J &= (m_1-m_2)(m_1-m_3)\phi_1L_{23}+(m_2-m_3)(m_2-m_1)\phi_2L_{31}\\
&\quad+(m_3-m_1)(m_3-m_2)\phi_3L_{12}-m_1(m_2-m_3)^2w_1\phi_2\phi_3\\
&\quad-m_2(m_3-m_1)^2w_2\phi_3\phi_1-m_3(m_1-m_2)^2w_3\phi_1\phi_2,\\
w_2K &= (m_1-m_2)(m_1-m_3)w_1L_{23}+(m_2-m_3)(m_2-m_1)w_2L_{31}\\
&\quad+(m_3-m_1)(m_3-m_2)w_3L_{12}-m_1(m_2-m_3)^2\phi_1w_2w_3\\
&\quad-m_2(m_3-m_1)^2\phi_2w_3w_1-m_3(m_1-m_2)^2\phi_3w_1w_2.
\end{aligned}
$$

Since these expressions are symmetric in the functions, (m_1, ϕ_1, w_1), (m_2, ϕ_2, w_2), (m_3, ϕ_3, w_3), it follows from (69) that all the surfaces Σ_{ikl}, for any permutation of the subscripts i, k, l, have the same representation of their lines of curvature. It remains for us to show that the function W_{ik}, which determines the distance from the origin to the tangent plane to Σ_{ikl}, is likewise symmetric in the above functions, in order to prove that all of these surfaces Σ_{ikl} coincide.

As in (63) we put

$$
(73)\quad
\begin{aligned}
\Omega'_{21} &= L_{12}-m_1w_1\phi_2-m_2w_2\phi_1, & \Omega_{21} &= L_{12}-m_1\phi_1w_2-m_2\phi_2w_1,\\
\Omega'_{23} &= L_{23}-m_3w_3\phi_2-m_2w_2\phi_3, & \Omega_{23} &= L_{23}-m_3\phi_3w_2-m_2\phi_2w_3.
\end{aligned}
$$

Hence by (65) the functions W_{21} and W_{23} determining the surfaces Σ_{21} and Σ_{23} are given by

$$
(74)\quad
\begin{aligned}
W_{21}\Omega'_{21} &= (m_1-m_2)(\phi_1W_1w_2-\phi_2W_2w_1)+\Omega_{21}W,\\
W_{23}\Omega'_{23} &= (m_3-m_2)(\phi_3W_3w_2-\phi_2W_2w_3)+\Omega_{23}W.
\end{aligned}
$$

By analogy we have that W_{213} is such that

$$W_{213}J=(m_3-m_1)(\phi_{23}W_{23}w_{21}-\phi_{21}W_{21}w_{23})+W_2K,$$

which reduces to

$$
\begin{aligned}
W_{213}\phi_2J &= (m_1-m_2)(m_1-m_3)W_1[L_{23}-m_2\phi_2w_3-m_3\phi_3w_2]\\
&\quad+(m_2-m_3)(m_2-m_1)W_2[L_{31}-m_3\phi_3w_1-m_1\phi_1w_3]\\
&\quad+(m_3-m_1)(m_3-m_2)W_3[L_{12}-m_1\phi_1w_2-m_2\phi_2w_1].
\end{aligned}
$$

Since by (72) $\phi_2 J$ is symmetric in the functions (m_1, ϕ_1, w_1), (m_2, ϕ_2, w_2), (m_3, ϕ_3, w_3) and the right-hand number is evidently of this kind, it is true also of W_{213}. Hence Σ_{213} is a transform of Σ_{21} in two ways by means of transformations involving m_3; and in like manner it is a transform of Σ_{23} and Σ_{31}, each in two ways, by means of transformations involving the constants m_1 and m_2 respectively.

Thus we have groups of eight surfaces, with isothermal representation of their lines of curvature, each of which is related to three others by transformations involving three different constants, the constants being the same three for each member of the group.

An immediate consequence of this result is the theorem.*

Given four surfaces Σ, Σ_1, Σ_2, Σ_{12} forming a quatern; each of these surfaces can be transformed by a transformation involving a constant m_3 different from m_1 and m_2 such that the new surfaces form a quatern also.

§ 8. *Surfaces with plane lines of curvature in both systems.*

Surfaces whose lines of curvature in both systems are plane have isothermal representation of these lines. For the curves upon the sphere must be circles and consequently have constant geodesic curvature. Hence they form an isothermal system.† Moreover, if the curves in one family of an isothermal system have constant geodesic curvature, the same is true of the curves in the other family. Hence the surfaces which we are about to consider are the only surfaces with isothermal representation with plane lines of curvature.

An orthogonal system of circles on the sphere is obtained, in the most general manner, by intersecting the sphere with two pencils of planes whoses axes are polar reciprocal with respect to the sphere.‡ There are two cases; 1°, when the axes are tangent to the sphere; 2°, when the axes are not tangent.

1°. By taking for axes the two tangents to the sphere at the point $(0, 0, 1)$, which are parallel to the axes Ox, Oy, the coördinates of the sphere can be given in the form

$$(75) \qquad X = \frac{2u}{u^2+v^2+1}, \qquad Y = \frac{2v}{u^2+v^2+1}, \qquad Z = \frac{u^2+v^2-1}{u^2+v^2+1},$$

from which we get for the linear element

$$(76) \qquad ds'^2 = \frac{4(du^2+dv^2)}{(u^2+v^2+1)^2}.$$

2°. If the axes are parallel to the axes Ox, Oy, and meet the axis Oz in the

* BIANCHI (loc. cit., p. 127) has established a similar theorem for the transformations D_m.

† BIANCHI, *Lezioni*, I, p. 210; German translation, p. 177.

‡ Ibid. I, p. 108; German translation, p. 81.

points $(0, 0, 1/a)$, $(0, 0, a)$, the coördinates of the sphere are expressible thus

$$(77)\quad X=\frac{\sqrt{1-a^2}\sin u}{\cosh v+a\cos u},\qquad Y=\frac{\sqrt{1-a^2}\sinh v}{\cosh v+a\cos u},\qquad Z=\frac{\cos u+a\cosh v}{\cosh v+a\cos u},$$

and the linear element is given by

$$(78)\qquad ds'^2=\frac{(1-a^2)(du^2+dv^2)}{(\cosh v+a\cos u)^2}.$$

From (76) and (78) it is seen that in either case e^θ is the sum of functions of u and v alone, so that equation (41) reduces to

$$\frac{\partial^2\psi}{\partial u\,\partial v}=0.$$

Consequently the most general solution of equation (9) is

$$(79)\qquad W=(U+V)e^{-\theta},$$

where U and V are arbitrary functions of u and v alone.

We denote by U_1 and V_1 the values of these functions which give a function w determining a THYBAUT transformation; thus

$$(80)\qquad w=(U_1+V_1)e^{-\theta}.$$

We consider the first case (75). Now

$$(81)\qquad e^\theta=\frac{u^2+v^2+1}{2},$$

and equations (A) reduce to

$$\frac{\partial\phi}{\partial u}=\frac{u^2+v^2+1}{2}U_1'-u(U_1+V_1),$$

$$\frac{\partial\phi}{\partial v}=-\frac{u^2+v^2+1}{2}V_1'+v(U_1+V_1).$$

If these values be substituted in equations (B) and (C), it is found that

$$m\phi=\frac{u^2+v^2+1}{2}m(U_1-V_1)-(uU_1'+vV_1')+(U_1+V_1),$$

and

$$(82)\qquad U_1=a\cosh\sqrt{2m}\,u,\qquad V_1=\pm a\cos\sqrt{2m}\,v,$$

where a is an arbitrary constant.

When, in the second case,

$$(83)\qquad e^\theta=\frac{(\cosh v+a\cos u)}{\sqrt{1-a^2}},$$

equations (A) reduce to

$$\frac{\partial \phi}{\partial u} = e^{\theta} U_1' + (U_1 + V_1) \frac{a \sin u}{\sqrt{1-a^2}},$$

$$\frac{\partial \phi}{\partial v} = -e^{\theta} V_1' + (U_1 + V_1) \frac{\sinh v}{\sqrt{1-a^2}}.$$

Proceeding as in the former case we find that we must consider separately the case where $m = \frac{1}{2}$.

When $m \neq \frac{1}{2}$, we get

$$2m\phi = \frac{1}{\sqrt{1-a^2}} [(\cosh v + a \cos u)(U_1 - V_1) + 2(a \sin u U_1' - \sinh v V_1') + (U_1 + V_1)(\cosh v - a \cos u)],$$

and

$$U_1 = a \cosh(\sqrt{2m-1}\, u), \qquad V_1 = \pm a \cos(\sqrt{2m-1}\, v). \tag{84}$$

For the particular case $m = \frac{1}{2}$ we find

$$\phi = 4\alpha e^{\theta} + \frac{2}{\sqrt{1-a^2}} (a \sin u U_1' - \sinh v V_1') + \frac{(U_1 + V_1)}{\sqrt{1-a^2}} (\cosh v - \cos u)$$

and

$$U_1 = \alpha u^2 + a_1 u + a_2, \qquad V_1 = \alpha v^2 + b_1 v + b_2, \tag{85}$$

where the constants of integration must satisfy the relation

$$a_1^2 + b_1^2 = 4\alpha(a_2 + b_2) \tag{86}$$

in order that equation (C) be satisfied.

When these values of ϕ and w are substituted in (45) and W is replaced by its expression (79), the function W_1 determining the transform Σ_1 of the given surface Σ is given by quadratures. We can thus find all the transforms of surfaces with plane lines of curvature in both systems and from the theorem of permutability it follows that all the transformations of these new surfaces are given without quadrature.

We inquire now whether any of these transforms Σ_1 have plane lines of curvature in both systems. From (27) and (80) it follows that the linear element of the spherical representation of the surfaces Σ_1 is

$$ds_1'^2 = \frac{(U_1 + V_1)^2}{\phi^2} (du^2 + dv^2). \tag{87}$$

Hence the necessary and sufficient condition that the lines of curvature be plane in both systems is that

$$\frac{\partial^2}{\partial u \partial v} \left(\frac{\phi}{U_1 + V_1} \right) = 0.$$

One finds that for both cases (81) and (83) this equation reduces to $\phi U_1' V_1' = 0$. Hence either U_1 or V_1 is a constant.

Let U_1 be a constant. This is not possible for the cases (82) and (84). However, it is possible for (85) by taking $a = a_1 = 0$; it follows from (86) that $b_1 = 0$ also. Hence V_1 is a constant. Without loss of generality we can take $U_1 + V_1 = 1$. Now

$$\phi = \frac{\cosh v - a \cos u}{\sqrt{1 - a^2}}. \tag{88}$$

Then the linear element (87) becomes

$$ds_1'^2 = \frac{(1 - a^2)(du^2 + dv^2)}{(\cosh v - a \cos u)^2}, \tag{89}$$

and by means of (25) it is found that

$$X' = -\frac{\sqrt{1 - a^2} \sin u}{\cosh v - a \cos u}, \quad Y' = \frac{\sqrt{1 - a^2} \sinh v}{\cosh v - a \cos u}, \quad Z' = -\frac{\cos u - a \cosh v}{\cosh v - a \cos u}. \tag{90}$$

Equations (45) reduce in the present case to

$$\frac{\partial}{\partial u}(\phi W_1) = -U', \qquad \frac{\partial}{\partial v}(\phi W_1) = V',$$

of which the integral is

$$\phi W_1 = -(U - V) + 2c_1,$$

where c_1 is a constant. The expression (48) for the radius R of the sphere tangent to Σ and Σ_1 becomes

$$R = \frac{1}{\sqrt{1 - a^2}}[a(\sin u \cdot U' - \cos u \cdot U) - \sinh v \cdot V' + \cosh v \cdot V + c_1(\cosh v + a \cos u]. $$

The functions A and B defined by (49′) have now the forms

$$A = (U'' + U - c_1)e^{-\theta}, \qquad B = (V'' - V - c_1)e^{-\theta}.$$

From (49) it follows that the coördinates of the center of the sphere are

$$\bar{\xi} = \int (U'' + U - c_1) \cos u \, du, \qquad \bar{\eta} = \int (V'' - V - c_1) \cosh v \, dv,$$

$$\bar{\zeta} = \frac{1}{\sqrt{1 - a^2}} \int -(U'' + U - c_1) \sin u \, du + a(V'' - V - c_1) \sinh v \, dv.$$

Hence the locus of the centers of the spheres is a surface of translation whose generating curves lie in planes parallel to the coördinate planes $\bar{\xi} = 0$, $\bar{\eta} = 0$. Moreover, the most general surfaces of translation of this kind can be defined by equations of the above form. By repeating the steps in inverse order we can establish the theorem: *Given a surface of translation generated by the*

motion of a plane curve when a point of the latter describes a second plane curve whose plane is perpendicular to the plane of the former; it is the locus of the centers of a family of spheres which envelope two surfaces with plane lines of curvature in both families.

When $a = 0$ in (77), the curves $u =$ const. on the sphere are great circles with a common diameter and the curves $v =$ const. are their parallels. From (89) and (90) it is seen that the same curves serve for the representation of Σ and Σ_1, the correspondence being given by

$$X' = -X, \qquad Y' = Y, \qquad Z' = -Z.$$

Now the surface $\bar{\Sigma}$ is a cylinder whose generators $u =$ const. are parallel to the y-axis. Moreover, the spheres whose centers lie on a generator have the same radius.

Returning to the general case, $a \neq 0$, we get from equations (A) and (88)

$$w = \frac{\sqrt{1-a^2}}{\cosh v + a \cos u}.$$

The coördinates of the surface Σ_0 whose tangent plane is at the distance w from the origin are

$$(91) \quad \xi = \frac{\sin u \cosh v}{\cosh v + a \cos u}, \qquad \eta = \frac{-a \cos u \sinh v}{\cosh v + a \cos u}, \qquad \zeta = \frac{\sqrt{1-a^2} \cos u \cosh v}{\cosh v + a \cos u}.$$

We find that the fundamental functions for this surface have the values

$$E, F, G = \frac{\cosh^2 v, \; 0, \; a^2 \cos^2 u}{(\cosh v + a \cos u)^2},$$

$$D, D', D'' = \frac{-\sqrt{1-a^2} \cosh v, \; 0, \; \sqrt{1-a^2} \cdot a \cos u}{(\cosh v + a \cos u)^2}.$$

From these expressions it is seen that the centers of curvatures of the surface are defined by

$$x_1 = \xi - \frac{\cosh v}{\sqrt{1-a^2}} X, \qquad x_2 = \xi + \frac{a \cos u}{\sqrt{1-a^2}} X,$$

and similar expressions for the y and z.

Replacing the various terms in the right-hand members by their values from (77) and (91), we get

$$x_1 = 0, \qquad y_1 = -\sinh v, \qquad z_1 = -\frac{a}{\sqrt{1-a^2}} \cosh v,$$

$$x_2 = \sin u, \qquad y_2 = 0, \qquad z_2 = \frac{\cos u}{\sqrt{1-a^2}}.$$

Hence when $a \neq 0$ the surface of centers consists of the two focal conics

$$x_1 = 0, \qquad (1-a^2)z_1^2 - a^2 y_1^2 = 1,$$
$$y_2 = 0, \qquad x_2^2 + (1-a^2)z_2^2 = 1.$$

And when $a = 0$ the surface of centers reduces to the y-axis and the circle

$$y_2 = 0, \qquad x_2^2 + z_2^2 = 1.$$

From this it follows that the surfaces Σ_0 are the cyclides of Dupin.

§ 9. *Surfaces Σ with spherical lines of curvature in one system.*

ENNEPER* has shown that the necessary and sufficient condition that the lines of curvature $v =$ const. be spherical is that the following relation obtain

$$\sqrt{E} = R\cos\sigma \cdot \frac{\sqrt{E}}{\rho_1} + R\sin\sigma \frac{\rho_2}{\sqrt{G}} \frac{\partial}{\partial v} \frac{\sqrt{E}}{\rho_1},$$

where ρ_1 and ρ_2 are the principal radii of curvature, R the radius of the sphere and σ the angle under which the latter cuts the surface; both R and σ are functions of v alone. For the surfaces Σ

$$\sqrt{E} = -\rho_1 e^{-\theta}, \quad \sqrt{G} = \rho_2 \cdot e^{-\theta},$$

so that the above equation becomes

$$\sqrt{E} = \alpha e^{-\theta} + \beta \frac{\partial \theta}{\partial v}, \tag{92}$$

where α and β are functions of v alone.

We consider first the case where Σ is a minimal surface; now (92) becomes

$$e^{2\theta} = \alpha + \beta \frac{\partial}{\partial v} e^{\theta}.$$

If this be differented with respect to u, we get

$$\frac{\partial}{\partial u} e^{2\theta} = \beta \frac{\partial^2}{\partial u \partial v} e^{\theta}.$$

Eliminating β from this equation and its derivative with respect to u, we get

$$\frac{\partial \theta}{\partial u} \frac{\partial^3 \theta}{\partial u^2 \partial v} - \frac{\partial^2 \theta}{\partial u^2} \frac{\partial^2 \theta}{\partial u \partial v} - \left(\frac{\partial \theta}{\partial u}\right)^2 \frac{\partial \theta}{\partial v} = 0. \tag{93}$$

This equation is satisfied by solutions of the equation

$$\frac{\partial^2}{\partial u \partial v} e^{\theta} = 0,$$

*Göttinger Nachrichten, 1868, p. 421; BIANCHI, *Lezioni*, II, p. 304.

that is, when the lines of curvature in both systems are plane. Excluding this case, every common solution of equations (3) and (93) determines a minimal surface whose lines of curvature $v = \text{const.}$ are spherical. DOBRINER * showed that there are minimal surfaces of this kind and found the expressions for their coördinates in the WEIERSTRASS form; they involve ϑ-functions.

In a similar manner it can be shown that when θ is a solution of equation (3) and of

$$(94) \qquad \frac{\partial\theta}{\partial v}\frac{\partial^3\theta}{\partial u\partial v^2} - \frac{\partial^2\theta}{\partial v^2}\frac{\partial^2\theta}{\partial u\partial v} - \frac{\partial\theta}{\partial u}\left(\frac{\partial\theta}{\partial v}\right)^3 = 0,$$

the lines of curvature $u = \text{const.}$ on the corresponding minimal surface are spherical.

In order to consider the case where Σ is not a minimal surface we remark that the functions E and G are determined by the equations

$$\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v} = \frac{\partial\theta}{\partial v}, \qquad \frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u} = \frac{\partial\theta}{\partial u}.$$

If the expression (92) for E be substituted in the first of these equations, we get

$$(95) \qquad \sqrt{G} = \left(\alpha' e^{-\theta} + \beta\frac{\partial^2\theta}{\partial v^2}\right)\left(\frac{\partial\theta}{\partial v}\right)^{-1} + \beta' - \alpha e^{-\theta},$$

where the accents denote differentiation with respect to v. Substituting this value in the second of the above equations, we get

$$(96) \quad \alpha' e^{-\theta}\left(\frac{\partial^2\theta}{\partial u\partial v} + \frac{\partial\theta}{\partial u}\frac{\partial\theta}{\partial v}\right) - \beta\left[\frac{\partial\theta}{\partial v}\frac{\partial^3\theta}{\partial u\partial v^2} - \frac{\partial^2\theta}{\partial u\partial v}\frac{\partial^2\theta}{\partial v^2} - \frac{\partial\theta}{\partial u}\left(\frac{\partial\theta}{\partial u}\right)^3\right] = 0.$$

Comparing this equation with (94) we see that if the lines of curvature $u = \text{const.}$ on the minimal surface S are spherical, the lines of curvature $v = \text{const.}$ are spherical on the surfaces with the same representation as S and for which E and G are given by

$$(97) \qquad \sqrt{E} = \alpha e^{-\theta} + \beta\frac{\partial\theta}{\partial v}, \qquad \sqrt{G} = \beta' - \alpha e^{-\theta} + \beta\frac{\partial^2\theta}{\partial v^2}\left(\frac{\partial\theta}{\partial v}\right)^{-1}$$

where now α is any constant and β any function whatever of v. When in particular $\alpha = 0$, it follows from (92) that the spheres upon which the curves $v = \text{const.}$ lie cut the surface orthogonally; and for any other value of α the projection of the radius of the sphere upon the tangent plane is constant.

If equation (96) be differentiated with respect to u and α' and β be eliminated

Die Minimalflächen mit einem System sphärischer Krümmungslinien, Acta Mathematica, vol. 10 (1887), pp. 145–152.

we get the equation of condition

$$(98)\quad \begin{vmatrix} e^{-\theta}\left(\dfrac{\partial^2\theta}{\partial u\,\partial v}+\dfrac{\partial\theta}{\partial u}\dfrac{\partial\theta}{\partial v}\right), & \dfrac{\partial\theta}{\partial v}\dfrac{\partial^3\theta}{\partial u\,\partial v^2}-\dfrac{\partial^2\theta}{\partial u\,\partial v}\dfrac{\partial^2\theta}{\partial v^2}-\dfrac{\partial\theta}{\partial u}\left(\dfrac{\partial\theta}{\partial v}\right)^3 \\ \dfrac{\partial}{\partial u}\left[e^{-\theta}\left(\dfrac{\partial^2\theta}{\partial u\,\partial v}+\dfrac{\partial\theta}{\partial u}\dfrac{\partial\theta}{\partial v}\right)\right], & \dfrac{\partial}{\partial u}\left[\dfrac{\partial\theta}{\partial v}\dfrac{\partial^3\theta}{\partial u\,\partial v^2}-\dfrac{\partial^2\theta}{\partial u\,\partial v}\dfrac{\partial^2\theta}{\partial v^2}-\dfrac{\partial\theta}{\partial u}\left(\dfrac{\partial\theta}{\partial v}\right)^3\right] \end{vmatrix}=0.$$

Given a solution of this equation which is also a solution of equation (3), there is an infinity of surfaces Σ with the given spherical representation of their lines of curvature and for which the curves $v=$ const. are spherical. The fundamental functions are given by (97) in which α is an arbitrary function of v and β is determined by (96).

In particular, if θ be a solution of (93) it satisfies (98); that is, when the curves $v=$ const. are spherical on a minimal surface there are an infinity of surfaces Σ with the same spherical representation for which the curves $v=$ const. are spherical.

From (92) it follows that the coördinates ξ, η, ζ of a point on one of the surfaces Σ with spherical lines of curvature $v=$ const. are given by quadratures of the form

$$\frac{\partial\xi}{\partial u}=\left(\alpha e^{-\theta}+\beta\frac{\partial\theta}{\partial v}\right)X_1=\frac{\partial}{\partial u}(\alpha X+\beta X_2);$$

hence

$$(99)\quad \xi=\alpha X+\beta X_2+V_1,\qquad \eta=\alpha Y+\beta Y_2+V_2,\qquad \zeta=\alpha Z+\beta Z_2+V_3,$$

where V_1, V_2, V_3 are functions of v alone. But from (95) we have

$$\frac{\partial\xi}{\partial v}=\left(\frac{\alpha' e^{-\theta}+\beta\dfrac{\partial^2\theta}{\partial v^2}}{\dfrac{\partial\theta}{\partial v}}+\beta'-\alpha e^{-\theta}\right)X_2.$$

Substituting the above value of ξ, we get

$$(100)\qquad V_1'=\beta\frac{\partial\theta}{\partial u}X_1+\frac{\alpha' e^{-\theta}+\beta\dfrac{\partial^2\theta}{\partial v^2}}{\dfrac{\partial\theta}{\partial u}}X_2-(\alpha'+\beta e^{-\theta})X.$$

It is readily verified that the expression on the right is a function of v alone. Hence the functions V_1, V_2, V_3, and consequently the surfaces Σ, can be found by quadratures.

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