THE EQUILONG TRANSFORMATIONS OF SPACE*

BY

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In August, 1904, SCHEFFERS presented to the International Mathematical Congress in Heidelberg a paper entitled *Ueber Isogonalkurven, Aequitangentialkurven, und komplexe Zahlen.* This is published in the official account of the Congress † and again in much greater detail, but under the same title, in the Mathematische Annalen of the following year.‡ In these papers he has discussed a type of transformation of the plane which he has called "equilong" and which carries straight lines into straight lines, keeping invariant the distance between the points of contact of a line with any two of its envelopes. He has exhibited § the beautiful duality existing between these transformations and those of the conformal group; even as the latter depend upon an arbitrary function of the usual complex variable, so are the equilong transformations of the plane expressed by an arbitrary function of a complex variable of a different type. There seems no *a priori* reason why this close analogy of conformal and equilong transformations should not subsist in three or more dimensions. Such is not, however, the case. In December, 1904, STUDY stated, in a short article, *Ueber mehrere Probleme der Geometrie, die dem Problem der konformen Abbildung analog sind:* ‖ "Alle diese Aufgaben, deren beide letzte Analoga zum Problem der konformen Abbildung sind, lassen sich durch explizite Formeln lösen, . . . Man findet in allen Fällen unendliche Gruppen. Die Mitteilung der zugehörigen Formeln, die nicht in der Art mit hyperkomplexen Grössen zusammenzuhängen scheinen . . . wollen wir bis zu einer ausführlicheren Darstellung aufschieben." Since these words were written, three years ago, nothing further has been published concerning equilong transformations of space. The object of the present paper, which, incidentally, was originally written in ignorance of Professor STUDY's work, is to give an analytic discussion of the problem and to demonstrate the remarkable theorem which he indicates, namely, that,

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† Verhandlungen des dritten internationalen Mathematiker-Kongresses, Leipzig, 1905.

‡ Vol. 60 (1905).

§ Mathematische Annalen, loc. cit., p. 528.

‖ Sitzungsberichte der Niederrheinischen Gesellschaft für Natur- und Heilkunde, December 5, 1904, p. 59.

whereas the conformal group in three dimensions depends on ten arbitrary parameters merely, the equilong group involves two arbitrary functions.

Let us define as "equilong" any analytic transformation of the assemblage of all oriented planes of space, which carries a plane into a plane, and preserves invariant the distance between the points of contact of a plane with any two envelopes which it may touch. We mean by "oriented plane," of course, a plane whose aspect, or the direction of whose normals, is well determined, and two oriented planes shall be considered as infinitely near when not only their cartesian coördinates, but their aspects differ infinitesimally. We may therefore restate our problem as follows: Consider an oriented plane, and three non-coaxial oriented planes infinitely near thereto. Then the triangle which these three determine in the original plane shall go over into an equal triangle determined in the same way by the corresponding planes. We shall take as coördinates of our oriented plane a, b, c, the direction cosines of its directed normal, and p the distance from the origin; all quantities involved being supposed, for simplicity, to be real. The corresponding plane shall be determined by A, B, C and P. It is clear from our definition that parallel planes must go into parallel planes; hence A, B, C will be functions of a, b, c only. Moreover, from the equality of the triangles just mentioned, we see that two planes infinitely near a given plane will cut in two lines whose angle is invariant. Therefore

$$\frac{da\,d'a + db\,d'b + dc\,d'c}{\sqrt{da^2 + db^2 + dc^2}\sqrt{d'a^2 + d'b^2 + d'c^2}} = \frac{dA\,d'A + dB\,d'B + dC\,d'C}{\sqrt{dA^2 + dB^2 + dC^2}\sqrt{d'A^2 + d'B^2 + d'C^2}}.$$

This equation is most easily fulfilled by looking upon our coördinates a, b, c, which satisfy the equation

$$a^2 + b^2 + c^2 = 1,$$

as representing points of a unit sphere, and seeking the most general conformal transformation thereof. We may therefore write

$$a = \frac{z + \bar{z}}{z\bar{z} + 1}, \qquad b = \frac{i(z - \bar{z})}{z\bar{z} + 1}, \qquad c = \frac{z\bar{z} - 1}{z\bar{z} + 1},$$

$$A = \frac{Z + \bar{Z}}{Z\bar{Z} + 1}, \qquad B = \frac{i(Z - \bar{Z})}{Z\bar{Z} + 1}, \qquad C = \frac{Z\bar{Z} - 1}{Z\bar{Z} + 1},$$

$$(1) \qquad Z = f(z), \qquad \bar{Z} = \bar{f}(\bar{z}),$$

$$(2) \qquad Z = f(\bar{z}), \qquad \bar{Z} = \bar{f}(z).$$

The distinction between cases (1) and (2) is that in the first the orienting coördinates are subjected to a direct, and in the second, to an inverse, conformal transformation. Let us also write, for the sake of later reductions, the

equations

$$da d'a + db d'b + dc d'c = \frac{2(dz d'\bar{z} + d'z d\bar{z})}{(z\bar{z}+1)^2},$$

$$dA d'A + dB d'B + dC d'C = \frac{2(dz d'\bar{z} + d'z d\bar{z})}{(f(z)\bar{f}(\bar{z})+1)^2} f'(z)\bar{f}'(\bar{z}).$$

Our problem is now solved so far as the direction cosines are concerned; it remains merely to find $P = P(a, b, c, p)$. We have already insured that the triangle cut in any oriented plane by the neighboring planes shall be carried into a similar triangle; all that remains is to arrange matters so that the area of this triangle shall be invariant in absolute value.

Let us start with the plane

$$ax + by + cz = p,$$

or (a, b, c, p), and find the area of the triangle cut therein by the planes

$$(a + da, b + db, c + dc, p + dp),\ (a + d'a, b + d'b, c + d'c, p + d'p),$$
$$(a + d''a, b + d''b, c + d''c, p + d''p).$$

The most symmetrical, if not the most immediately obvious, way to proceed, is to find the coördinates of the point where the last three planes meet, to find its distance from the first plane, and to divide into the volume of the tetraedron formed by the four planes. Proceeding thus we find that the square root of the area of our triangle differs by a constant factor from

$$\frac{\begin{vmatrix} a & b & c & p \\ da & db & dc & dp \\ d'a & d'b & d'c & d'p \\ d''a & d''b & d''c & d''p \end{vmatrix}}{\sqrt{\begin{vmatrix} a & b & c \\ da & db & dc \\ d'a & d'b & d'c \end{vmatrix}}\ \sqrt{\begin{vmatrix} a & b & c \\ d'a & d'b & d'c \\ d''a & d''b & d''c \end{vmatrix}}\ \sqrt{\begin{vmatrix} a & b & c \\ d''a & d''b & d''c \\ da & db & dc \end{vmatrix}}}.$$

This is our second invariant. It should be noticed that the cofactor of p in the numerator is zero, in view of the equations

$$\Sigma a da = \Sigma a d'a = \Sigma a d''a = 0;$$

we have, therefore, but one type of expression in either numerator or denominator,

$$\begin{vmatrix} a & b & c \\ da & db & dc \\ d'a & d'b & d'c \end{vmatrix} = \sqrt{\begin{vmatrix} 1 & 0 & 0 \\ 0 & \Sigma da^2 & \Sigma da d'a \\ 0 & \Sigma da d'a & \Sigma d'a^2 \end{vmatrix}} = \frac{2i(dz d'\bar{z} - d'z d\bar{z})}{(z\bar{z}+1)^2}.$$

Our invariant will therefore differ by a numerical factor from

$$\frac{\begin{vmatrix} dp & d'p & d''p \\ dz & d'z & d''z \\ d\bar{z} & d'\bar{z} & d''\bar{z} \end{vmatrix} (z\bar{z}+1)}{\sqrt{(dzd'\bar{z}-d'zd\bar{z})(d'zd''\bar{z}-d''zd\bar{z})(d''zd\bar{z}-dzd''\bar{z})}}$$

$$= \frac{\begin{vmatrix} dP & d'P & d''P \\ dz & d'z & d''z \\ d\bar{z} & d'\bar{z} & d''\bar{z} \end{vmatrix} [f(z)\bar{f}(\bar{z})+1]}{\sqrt{f'(z)\bar{f}'(\bar{z})}\sqrt{(dzd'\bar{z}-d'zd\bar{z})(d'zd''\bar{z}-d''zd'\bar{z})(d''zd\bar{z}-dzd''\bar{z}}}$$

$$= \frac{\dfrac{\partial P}{\partial p}\begin{vmatrix} dp & d'p & d''p \\ dz & d'z & d''z \\ d\bar{z} & d'\bar{z} & d''\bar{z} \end{vmatrix} [f(z)\bar{f}(\bar{z})+1]}{\sqrt{f'(z)\bar{f}'(\bar{z})}\sqrt{\begin{vmatrix} dz & d\bar{z} \\ d'z & d'\bar{z} \end{vmatrix}}\sqrt{\begin{vmatrix} d'z & d'\bar{z} \\ d''z & d''\bar{z} \end{vmatrix}}\sqrt{\begin{vmatrix} d''z & d''\bar{z} \\ dz & d\bar{z} \end{vmatrix}}},$$

the last reduction being effected by means of the formula

$$dP = \frac{\partial P}{\partial p}dp + \frac{\partial P}{\partial z}dz + \frac{\partial P}{\partial \bar{z}}d\bar{z}.$$

We thus reach the simple differential equation

$$\frac{\partial P}{\partial p} = \frac{(z\bar{z}+1)}{[f(z)\bar{f}(\bar{z})+1]}\sqrt{f'(z)\bar{f}'(\bar{z})},$$

and this yields the final forms for our transformation:

$$P = \frac{(z\bar{z}+1)p}{[f(z)\bar{f}(\bar{z})+1]}\sqrt{f'(z)\bar{f}'(\bar{z})} + Q(z, \bar{z}),$$

(I) (1) $Z = f(z), \qquad \bar{Z} = \bar{f}(\bar{z}),$

(2) $Z = f(\bar{z}), \qquad \bar{Z} = \bar{f}(z).$

These equations may also be written in terms of real coördinates as follows:

$$z = x + iy, \qquad Z = X + iY,$$

$$X = \phi(x, y), \qquad Y = \psi(x, y).$$

$$P = \frac{(x^2+y^2+1)p}{(\phi^2+\psi^2+1)}\sqrt{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial x}\right)^2} + \chi(x, y),$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad \frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y}, \tag{1}$$

$$\frac{\partial \phi}{\partial x} = -\frac{\partial \psi}{\partial y}, \qquad \frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y}. \tag{2}$$

It is easy to verify analytically the geometrically evident fact that our equilong transformations form a group.

It is clear that our method of reasoning may be carried at once into n dimensions. If we call the parameters corresponding to a, b, c and to p respectively the direction and the distance parameters, we have

The most general equilong transformation of a euclidean space of n dimensions depends on the most general conformal transformation of a space of $n-1$ dimensions and an arbitrary function of the direction parameters. The distance parameter enters linearly.

We see from this that in a space of more than three dimensions there is but one arbitrary function involved. It is well to insist on the word "euclidean" for in Riemannian or Lobatschewskian space, conformal and equilong transformations are identical. For this reason, the differences here noted for the euclidean case seem the more striking.

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