THE HYPERGEOMETRIC FUNCTIONS OF \(N\) VARIABLES

BY

J. I. HUTCHINSON

The generalization of the hypergeometric series in one variable and three parameters to that in two variables and four parameters was first made by Appell in 1880.\footnote{Presented to the Society April 25, 1908.} He showed that this series satisfies a system of partial differential equations of the second order, and that any four solutions of these equations are connected by a linear relation. Immediately after, Picard\footnote{His results were published in a completed form in a memoir entitled *Sur les fonctions hypergéométriques de deux variables*, Journal de Mathématiques, ser. 3, vol. 8 (1882), pp. 173–216.} showed that on the assumption of three linearly independent integrals which behave in a prescribed way in the vicinity of certain singularities, the differential equations of Appell were completely deducible. Appell also expressed the hypergeometric series in terms of a double definite integral in which the two variables occur as parameters, while Picard expressed it in a more useful form as a simple definite integral. These results were generalized by Lauricella.\footnote{* Sulle funzioni ipergeometriche a piu variabili, Rendiconti del Circolo Matematico di Palermo, vol. 7 (1893), pp. 111–158.}

The next question to be considered was the behavior of the three linearly independent solutions of the hypergeometric differential equations when the two independent variables described closed paths about the singular points. It was found by Picard\footnote{Sur une extension aux fonctions de deux variables, etc., Journal de l'École Normale, ser. 2, vol. 10 (1881), pp. 305–322.} that they undergo linear transformations, any of which can be expressed by a combination of five particular ones. Picard also showed that a certain Hermitian form is invariant for the substitutions of the group, but he went no further than to establish this fact for the cases in which the four parameters in the hypergeometric functions are rational numbers.

In the following paper I consider the hypergeometric functions of \(n\) variables \(x_1, x_2, \ldots, x_n\) and \(n + 2\) parameters. I deduce the generating substitutions of the group of linear homogeneous transformations on the \(n + 1\) linearly independent hypergeometric integrals produced by a variation of the \(x_i\) along closed paths.
paths surrounding the critical points. I then show that a certain Hermitian form is invariant for this group.

§ 1. The group of linear transformations.

Denote by $U$ the function

$$(u - u_0)^{a-1}(u - u_1)^{b-1}(u - x_1)^{\lambda_1-1}(u - x_2)^{\lambda_2-1}\ldots(u - x_n)^{\lambda_n-1}$$

and regard the $n + 1$ hypergeometric integrals

$$\omega_0 = \int_{u_0}^{x_1} Udu, \quad \omega_i = \int_{u_i}^{x_1} Udu \quad (i = 1, 2, \ldots, n)$$

as functions of the variables $x_i$. In order that these integrals may have a meaning it is necessary that the real parts of $\alpha$, $\beta$, $\lambda_i$, and $(n + 1) - (\alpha + \beta + \lambda_1 + \lambda_2 + \cdots + \lambda_n)$ be positive. For convenience we will assume $u_0 = 0$, $u_1 = 1$ as long as there is no restriction of generality. The following abbreviations will also be used:

$$a = e^{2\pi i \alpha}, \quad b = e^{2\pi i \beta}, \quad l_k = e^{2\pi i \lambda_k}.$$ 

Suppose that $x_1$ describes a closed path positively about the zero point. By expanding $\omega_0$ in ascending powers of $x_1$ it is easily seen to be reproduced multiplied by $a l_1$ when $x_1$ undergoes this change. The path of integration for $\omega_1$ is altered into the one indicated by the dotted lines in Fig. 1. The integral taken along the portion of the path from 1 to 0 is equal to $\omega_1 - \omega_0$. In going around the point 0 the factor $x_1^{a-1}$ in the integrand is reproduced multiplied by $a$. This factor occurs in every element of the integral (regarded as the limit of a sum) along the path $0x_1$ and we obtain $a \omega_0$ for the result of integration along the second part of the path. Hence, denoting the transformed integral by $\omega'_1$, we have

$$\omega'_1 = (a - 1) \omega_0 + \omega_1.$$ 

The remaining integrals $\omega_i (i > 1)$ are not affected by this monodromy of the branch-point $x_1$. Hence we obtain the substitution

$$(\Sigma_0) \quad \omega'_0 = a l_1 \omega_0, \quad \omega'_1 = (a - 1) \omega_0 + \omega_1, \quad \omega'_i = \omega_i \quad (i = 2, 3, \ldots, n).$$

We next consider the effect produced by moving $x_1$ positively around 1. By expanding $\omega_1$ in powers of $x_1 - 1$ we readily find that it is reproduced multiplied by $bl_1$. The path of integration for $\omega_0$ is changed into a path consisting of a line from 0 to 1, then an infinitesimal circle about 1 in the positive direction, and finally a line from 1 to $x_1$. This gives for the new integral $\omega_0 + l_1(b - 1) \omega_1$.  

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
The path of integration for each of the other integrals we imagine to be gradually deformed by being pushed away from the moving point $x_i$. The new path is indicated by the dotted line in Fig. 2, and the result of integrating along this is $\omega_i + b \cdot (l_i - 1) \cdot \omega_i$. Thus we obtain the substitution

$$\begin{align*}
\omega'_0 &= \omega_0 + l_i \cdot (b - 1) \cdot \omega_i, \\
\omega'_i &= \omega_i + b \cdot (l_i - 1) \cdot \omega_i \\
&\quad \text{for } i = 2, \ldots, n.
\end{align*}$$

(1)

A similar mode of procedure will show that if $x_i \,(i > 1)$ move positively about the point 1 and then return to its original position, the integrals undergo the transformation

$$\begin{align*}
\omega'_0 &= \omega_0, \\
\omega'_h &= \omega_h + (l_i - 1) \cdot \omega_i \\
\omega'_k &= \omega_k + b \cdot (l_i - 1) \cdot \omega_i \\
&\quad \text{for } k = i + 1, \ldots, n.
\end{align*}$$

(2)

We now consider the effect of moving $x_i$ along a closed path encircling all the finite branch points, the direction of rotation being positive. In each of the integrals $\omega_i \,(i = 2, \ldots, n)$ the integrand is reproduced multiplied by $l_1$ while the path of integration is unaltered, and accordingly

$$\omega'_i = l_1 \cdot \omega_i.$$  

For the integrals $\omega_0, \omega_1$ one of the limits of integration is variable. We therefore proceed as follows. Let a path be described from $\infty$ to $x_n$, then positively around all the finite branch points returning to $x_n$, then from $x_n$ to $\infty$. The integral along such a path is zero. This gives the relation,

$$\begin{align*}
- \omega_n + b \cdot (\omega_1 - \omega_0) + ab\omega_0 + \sum_{h=1}^{n-1} \{n + 1, h\} \cdot (\omega_{h+1} - \omega_h) + \int_{L-1}^1 \, I = 0,
\end{align*}$$

in which for brevity we introduce the notation

$$\begin{align*}
L &= ab l_1 \cdots l_n, \\
I &= \int_{x_n}^{\infty} U \, du.
\end{align*}$$
After $x_i$ has traversed the path described above, equation (2) is changed into the same relation in $\omega'_i$ and $I'$. In this replace $\omega_2, \ldots, \omega_n$ by means of (1) and $I'$ by the expression

$$I + (1 - l_1) \omega_n + b(1 - a)(1 - l_1) \omega_0 + b(l_1 - 1) \omega_1,$$

this being the result of integrating along the transformed path indicated in Fig. 3. Then replace $I$ by means of (2) and we obtain the relation

$$b(a - 1) \omega'_0 + b(1 - al_1) \omega'_1 = b(a - 1)(l_1 + L - l_1 L) \omega_0$$

$$+ b[l_1(1 - a) + L(1 - l_1)] \omega_1 + (1 - l_1) \sum_{h=2}^{n} (1 - l_h) \{ n+1, h-1 \} \omega_h.$$

Moreover, since the integral from 0 to 1 is reproduced multiplied by $l_1$, we have

$$\omega'_0 - \omega'_1 = l_1(\omega_0 - \omega_1).$$

By combining (3) and (4) we obtain the formulas for $\omega'_0, \omega'_1$ and thus we deduce the required transformation:

$$\omega'_0 = [1 + (a - 1)\{2, n+1\}]l_1 \omega_0 + \sum_{h=2}^{n} (1 - l_h) \{1, h-1\} \omega_h,$$

$$\omega'_1 = \{1, n+1\} [(a - 1) \omega_0 + \omega_1] + \sum_{h=2}^{n} (1 - l_h) \{1, h-1\} \omega_h.$$

By moving $x_i (i > 1)$ positively around all the branch points we obtain in like manner the transformations

$$\omega'_j = l_i \omega_j \quad (j = 0, 1, \ldots, i-1, i+1, \ldots, n),$$

$$\omega'_i = (a - 1) \{i, n+1\} \omega_0 + (1 - al_1) \{i, n+1\} \omega_1 + \sum_{h=2}^{n} (1 - l_h) \{i, h-1\} \omega_h.$$

If $x_i$ describes a closed path circling positively around $x_i$ (excluding all other branch points), we obtain the transformation
\[ \omega'_0 = \omega_0 + l_1 (l_i - 1) \omega_1 + l_1 (1 - l_i) \omega_i, \quad \omega'_i = [1 + l_i (l_i - 1)] \omega_i + l_i (1 - l_i) \omega_i, \]

\[ (\Sigma_{1i}) \quad \omega'_h = (1 - l_1) (1 - l_i) \omega_i + \omega_h - (1 - l_i) (1 - l_i) \omega_i \quad (h = 2, \cdots, i - 1), \]
\[ \omega'_i = (1 - l_i) \omega_i + l_i \omega_i, \quad \omega'_k = \omega_k \quad (k = i + 1, \cdots, n). \]

Finally when \( x_i \) describes a closed path about \( x_k \) we have
\[ \omega'_h = \omega_h \quad (h = 0, 1, \cdots, i - 1, k + 1, k + 2, \cdots, n), \]
\[ (\Sigma_{\omega}) \quad \omega'_i = [1 + l_i (l_k - 1)] \omega_i + l_i (1 - l_k) \omega_k, \]
\[ \omega'_j = (1 - l_i) (1 - l_k) \omega_i + \omega_j - (1 - l_i) (1 - l_k) \omega_k \quad (j = i + 1, \cdots, k - 1), \]
\[ \omega'_k = (1 - l_i) \omega_i + l_i \omega_i. \]

The above transformations evidently form a complete set of generators of the given group.

§ 2. The invariant Hermitian form.

Our next problem is to deduce the Hermitian form which is invariant for this group. Assuming the form to be
\[ \sum_{j, k = 0}^{n} c_{jk} \omega_j \bar{\omega}_k \quad (c_{jk} = \bar{c}_{kj}), \]
we first apply the transformation \( T_i \) \((i > 1)\) and equate the coefficients of \( \omega_k \bar{\omega}_i \) on both sides of the identity
\[ \sum c_{jk} \omega_j \bar{\omega}_k = \sum c_{jk} \omega_j \bar{\omega}_k \]
after substituting in the left member for \( \omega'_j, \bar{\omega}'_k \) the expressions given in \((T_i)\). This leads to the formulas
\[ c_{ii} = c_{ii} \frac{(1 - \alpha) \{ i, n + 1 \}}{l_i - L}, \quad c_{ii} = c_{ii} \frac{(al_i - 1) \{ i, n + 1 \}}{l_i - L}, \]
\[ c_{ki} = c_{ii} \frac{(l_k - 1) \{ i, k - 1 \}}{l_i - L}. \]

The coefficients of other corresponding terms are identical. Taking \( k = 2 \) in the last of these formulas and replacing \( i \) by \( k \) we obtain
\[ c_{2k} = c_{kk} \frac{(l_2 - 1) \{ k, 1 \}}{l_k - L}. \]

Since \( c_{2k} = \bar{c}_{k2} \) we deduce from this last equation
\[ c_{k2} = c_{kk} \frac{(l_2 - 1) \{ k, 1 \}}{l_k - L} = c_{kk} \frac{(1 - l_k) \{ 2, k \}}{(l_k - L)}. \]
Equate this to the result of putting \( i = 2 \) in the last formula of (6) and we have

\[
(7) 
\]

\[
c_{kk} = c_{22} \frac{(1 - l_k)(l_k - L)}{(1 - l_2)(l_2 - L)}. 
\]

By applying \( \sum_0 \) to (5) we obtain the additional relation

\[
(8) 
\]

\[
c_{01} = c_{11} \frac{a - 1}{1 - a l_1}. 
\]

Also, by using \( \sum_i \) we find the new condition

\[
b(1 - l_i) \sum_{k=1}^{i-1} c_{0k} + (1 - b l_i) c_{0i} + (1 - l_i) \sum_{k=i+1}^{n} c_{0k} = 0. 
\]

In this we substitute \( c_{0i} \) from (8), \( c_{0k} \) from (6), and then \( c_{hk}, c_{kk} \) from (7). After using the identities

\[
\sum_{k=2}^{i-1} (1 - l_k) \{ h, n + 1 \} = \{ 2, n + 1 \} - \{ i, n + 1 \}, 
\]

\[
\sum_{k=i+1}^{n} (1 - l_k) \{ k, n + 1 \} = \{ i + 1, n + 1 \} - b, 
\]

we obtain the relation

\[
c_{22} = c_{11} \frac{(1 - l_2)(l_2 - L)}{(1 - a l_1) (a l_1 - L)}. 
\]

By applying \( \sum_i \) to the Hermitian form we find the additional relation

\[
c_{00}(1 - b) + c_{01}(1 - b l_1) + \sum_{k=2}^{n} c_{0k}(1 - l_k) = 0, 
\]

from which we deduce, with the aid of the preceding results,

\[
c_{11} = c_{00} \frac{1 - a l_1}{(1 - a)(a - L)}, 
\]

and hence

\[
c_{kk} = c_{00} \frac{(1 - l_k)(l_k - L)}{(1 - a)(a - L)}, 
\]

\[
c_{01} = c_{00} \frac{l_1 L - a}{a - L}, 
\]

\[
c_{0i} = c_{00} \frac{i}{a - L}, 
\]

\[
c_{ii} = c_{00} \frac{i}{(1 - a)(a - L)}, 
\]

\[
c_{ki} = c_{00} \frac{i - l_i}{(1 - a)(a - L)}, \quad (k + i, i > 1). 
\]

The Hermitian form is now completely determined, and it can be verified that all the remaining conditions are satisfied in order that \( H \) may be invariant for the generating substitutions of the group. I have carried out this verification, but the details are too long to reproduce here.
For certain cases in three homogeneous variables Le Vavasseur has calculated the numerical value of the determinant \(|c_{ik}|\) of \(H\) and also of its minors. It is easy to evaluate this determinant for the general case. Every element contains the factor \(C = c_{\infty}/(1 - \tilde{a})(a - L)\). We remove, then, the factor \(C^{n+1}\) and divide the elements of the first column by \(1 - \tilde{a}\), those of the third by \(1 - \tilde{l}_2\), those of the fourth by \(1 - \tilde{l}_3\), \ldots, those of the \((n+1)\)th by \(1 - \tilde{l}_n\), and treat the resulting determinant as follows. Multiply the \(n\)th column by \(-\tilde{l}_{n-1}\) and add it to the last. Then multiply the \((n-1)\)th column by \(-\tilde{l}_{n-2}\) and add to the \(n\)th, and so on; finally, multiply the first column by \(\tilde{a}\tilde{l}_1\) and add to the third, and multiply the first column by \(1 - \tilde{a}\tilde{l}_1\) and add to the second. In the determinant thus formed, add the last row to the preceding, then the \(n\)th row to the \((n-1)\)th, and so on; finally, add the third row to the second. Except for the first row and column, the elements outside the main diagonal of the resulting determinant are all zero and the evaluation is immediately effected. The result is \(-L(1 - \tilde{b})(1 - \tilde{l}_1)(1 - L)^n\). Combining this with the factors previously removed, we have

\[|c_{ik}| = -(a-1)(b-1)(l_1-1)(l_2-1) \cdots (l_n-1)(1-L)^n \left[\left(1-\tilde{a}\right)(a-L)\right]^n.\]

The determinant \(|c_{ik}|\) vanishes of rank 1 when \(L = 1\). For it is easy to verify that every minor of order 2 vanishes identically when it contains no element from the main diagonal, while if it does contain such an element, the minor has \(L - 1\) as a factor. From this it follows that in general \(H\) may be written \(A_0y_0\tilde{y}_0 + (L - 1)H_1\), where \(H_1\) is an Hermitian form in \(n\) variables \(y_1, y_2, \ldots, y_n\).

In the case of three variables \(\omega_1, \omega_2, \omega_3\) we are able by means of this theorem to reduce \(H\) at once to a simple canonical form. Giving to \(c_{\infty}\) the value \((1 - \tilde{a})(a - L)(1 - \tilde{a}l_1)(\tilde{a}l_1 - L)\) and introducing for brevity

\[\eta = (1 - \tilde{a}l_1)(1 - bl_2)\omega_1 - (1 - l_2)(1 - al_1)\omega_2 - (1 - a)(1 - bl_2)\omega_3,\]

we obtain

\[H = \eta\tilde{y} + (L - 1)[(1 - \tilde{a})(1 - bl_2)(1 - l_2)\omega_0\tilde{y}_0 + (1 - \tilde{b})(1 - l_2)(1 - \tilde{a}l_1)\omega_2\tilde{y}_2].\]

§ 3. Conditions for uniform inversion.

We now introduce as non-homogeneous variables \(x_i\) the ratios of the hypergeometric integrals, \(x_i = \omega_i/\omega_0\), and proceed to consider their developments in the vicinity of singular values of the variables \(x_i\). For the sake of brevity the notation is made symmetrical by replacing the points \(u = 0, 1\) by \(x_0, x_{n+1}\) respectively and by using the integrals

\[\eta_i = \int^{x_i}_{x_0} U\,du\]
In place of the $\omega_i$ of which they are linear functions. Take now the singular point $x_h = x_k (h \neq k, h, k = 0, 1, \ldots, n + 1)$, $x_i = a_i$ (the $a_i$ being arbitrarily chosen but different from any of the branch points, $i \neq h, k$). Then all of the hypergeometric integrals are regular in the vicinity of such a point with the exception of

$$\eta_j - \eta_k = \int_{x_k}^{x_h} U du.$$  

By substituting $x_h = x_k$, $u - x_k = v$, expanding $U$ in ascending powers of $v$, and integrating between the limits 0 and $x$ we find that $\eta_j - \eta_k$ has the form

$$(9) \quad \eta_j - \eta_k = x_h^{-k + \lambda_k - 1} R,$$

where $R$ is regular in the vicinity of the given point.

We next examine the singularity $x_i = \infty$. By substituting $x_i = 1/x$, $u = 1/v$ it is readily seen that all of the integrals are expressible in the form $x^1 - \lambda_i Q$, in which $Q$ is regular, with the exception of the integral

$$\int_{x_i}^{\infty} U du$$

which has the form $x^\mu P$, in which $P$ is regular and $\mu$ is

$$1 + \sum_{j=0}^{n+1} (1 - \lambda_j).$$

The quotients of these integrals are accordingly all regular with the exception of one which has the form

$$(10) \quad x^{\mu + \lambda_i - 1} S,$$

$S$ being a regular function.

Replacing $x_0, x_{n+1}$ by 0, 1, we now consider the variables $x_i$ as functions of the $\xi_i (i = 1, 2, \ldots, n)$. It is at once evident from what precedes that they are automorphic functions, being absolutely invariant for the group of linear transformations already determined. They are not in general uniform functions. It is interesting to determine under what conditions they are uniform. Since the expressions (9) and (10) are linear fractional functions of the $\xi_i$, it is evidently necessary that the exponents of the $x$, namely,

$$\lambda_i + \lambda_k - 1, \quad n + 2 - (\lambda_0 + \lambda_1 + \cdots + \lambda_i - 1 + \lambda_{i+1} + \cdots + \lambda_{n+1}),$$

be each the reciprocal of an integer. There are

$$\frac{1}{2} (n + 2)(n + 1) + n + 2 = \frac{1}{2} (n + 2)(n + 3)$$

such conditions, except in the case $n = 1$ when the number is 3. The number of different solutions of the problem of determining the exponents $\lambda_i$ so as to satisfy these conditions is infinite for $n = 1$ and finite for $n = 2$.*

number of conditions increases rapidly with increasing \( n \), it may be presumed that the number of solutions is finite in each case excepting \( n = 1 \).

\section*{§ 4. Case of equal exponents.}

When any two or more exponents in the hypergeometric integral are equal, the group may be extended. For this purpose let the branch points 0, 1 be replaced by \( x_0, x_{n+1} \) as in the preceding section. Take the case \( \lambda_0 = \lambda_1 \) and consider first a variation of the points \( x_0 \) and \( x_1 \) by which they become permuted, the paths described by these two points forming a closed region not containing any of the other branch points. It will be assumed in this and the following cases that the paths are described positively with reference to the region they enclose. The following method, which might also have been used in § 2, is convenient for the case in hand. Take an arbitrary point \( u_0 \) of the \( u \)-plane and denote by \( I_0, I_1 \) the result of integrating \( \int U \, du \) along a convenient path from \( u_0 \) to \( x_0, x_1 \) respectively. We will denote these paths by \( p_0, p_1 \). When \( x_0 \) and \( x_1 \) are interchanged these paths will be changed continuously into new paths \( p_0', p_1' \) (Fig. 4). Then we have

\[ I_0' = I_0 + l_0 \lambda_0, \quad I_1' = I_0, \quad I_k' = I_k \quad (k = 2, \ldots, n+1), \]

and hence

\[ \lambda_0' = I_1' - I_0' = l_0 \lambda_0, \quad \lambda_1' = I_1' - I_{n+1}' = \lambda_1 - \lambda_0, \quad \lambda_j' = \lambda_j \quad (j = 2, \ldots, n). \]

In general, if \( \lambda_1 = \lambda_k \), we deduce in like manner the following results in which the symbol \( (ik) \) is used to indicate the substitution which results from the permutation of the branch points \( x_i, x_k \):

\[
\begin{align*}
\lambda_0' &= \lambda_1 - \lambda_0, \quad \lambda_h' &= (1 - l_0) (\lambda_1 - \lambda_0 - \lambda_i) + \lambda_h \quad (h = 1, 2, \ldots, i-1; i = 2, \ldots, n), \\
\lambda_i' &= \lambda_1 - \lambda_0, \quad \lambda_j' &= \lambda_j \quad (j = i+1, \ldots, n); \\
\lambda_0' &= \lambda_0 + l_0 (\lambda_i - \lambda_1), \quad \lambda_i' = \lambda_1 + l_0 (\lambda_0 - \lambda_1) \quad (i = 1, \ldots, n); \\
\lambda_h' &= (1 - l_1) (\lambda_i - \lambda_1) + \lambda_h \quad (h = 2, 3, \ldots, i-1), \\
\lambda_i' &= \lambda_i, \quad \lambda_j' = \lambda_j \quad (j = i+1, \ldots, n); 
\end{align*}
\]
\[(1, n + 1) \quad \omega_0' = \omega_0 - l_1 \omega_1, \quad \omega_1' = -l_1 \omega_1, \quad \omega_i' = \omega_i - l_1 \omega_1 \quad (i = 2, \cdots, n); \]
\[\omega_h' = \omega_h, \quad \omega_k' = \omega_i \quad (h = 1, 2, \cdots, i - 1, k + 1, \cdots, n),\]
\[\omega_i' = (1 - l_i) \omega_i + l_i \omega_k, \quad (j = i + 1, \cdots, k - 1),\]
\[\omega_j' = (1 - l_i)(\omega_i - \omega_k) + \omega_j \quad (1 < i < k < n + 1).\]

Cornell University, Ithaca, N. Y.