PROJECTIVE DIFFERENTIAL GEOMETRY OF CURVED SURFACES*

(FOURTH MEMOIR)

BY

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The theory of surfaces, as developed by the author in the preceding three memoirs, was based upon the assumption that the surface considered was referred to its asymptotic curves. The complete system of invariants and covariants was set up on that hypothesis. In order to make the theory directly applicable to a surface whose asymptotic lines are not known, as well as for the sake of formal generality, it becomes necessary to develop the theory of invariants and covariants of a system of two linear homogeneous partial differential equations of the second order with one dependent and two independent variables in its most general form. It is the purpose of the present paper to develop the essentials of this theory. The calculations, some of which are very long, will only be indicated. The final expressions for the invariants, however, are as simple and elegant as can be desired.

§ 1. *Reduction of the given system of partial differential equations to its normal form.*

Consider the system of partial differential equations

$$(1)\qquad \begin{aligned} \Omega &\equiv Ay_{uu} + 2By_{uv} + Cy_{vv} + Ly_u + My_v + Ny = 0, \\ \Omega' &\equiv A'y_{uu} + 2B'y_{uv} + C'y_{vv} + L'y_u + M'y_v + N'y = 0, \end{aligned}$$

where $A, \cdots, N'$ are functions of u and v, and where the subscripts indicate differentiations;

$$y_u = \frac{\partial y}{\partial u}, \qquad y_v = \frac{\partial y}{\partial v}, \qquad y_{uu} = \frac{\partial^2 y}{\partial u^2}, \quad \text{etc.}$$

If λ, μ, ν, ρ denote four functions of u and v whose determinant $\lambda\rho - \mu\nu$ is different from zero, the system

$$(2)\qquad \lambda\Omega + \mu\Omega' = 0, \qquad \nu\Omega + \rho\Omega' = 0$$

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is equivalent to (1). Put

$$(3)\quad AC - B^2 = I, \qquad A'C' - B'^2 = I', \qquad AC' + A'C - 2BB' = J,$$

and denote by $\bar{I}$, $\bar{I}'$, $\bar{J}$ the corresponding quantities for system (2). Then

$$(4)\qquad \bar{I} = \lambda^2 I + \lambda\mu J + \mu^2 I', \qquad \bar{I}' = \nu^2 I + \nu\rho J + \rho^2 I'.$$

Consequently, if

$$K = J^2 - 4II' \neq 0,$$

λ, μ, ν, ρ may be chosen in such a way that their determinant does not vanish and so that

$$\bar{I} = \bar{I}' = 0,$$

while $\bar{J}$ will be different from zero. In fact, if $K \neq 0$,

$$\bar{K} = (\lambda\rho - \mu\nu)K$$

will also be different from zero, so that the same will be true of $\bar{J}$. It suffices for our purpose to choose $\lambda : \mu$ and $\nu : \rho$ as the two distinct roots of the quadratic

$$\alpha^2 I + \alpha\beta J + \beta^2 I' = 0.$$

By merely solving a quadratic equation, we may therefore substitute for a system of form (1) an equivalent system for which $I = I' = 0$, while J does not vanish, provided only that $J^2 - 4II'$ is not equal to zero for the original system. We shall say that the system (1) has been reduced to its *normal form*. This reduced form of system (1) is not unique; but, the most general reduced system may be obtained from any particular one by multiplying each of the equations of the system by an arbitrary function of u and v, and by interchanging the two equations.

Assume that system (1) is in its normal form. It may be verified directly, or it may be deduced from formulæ which will be developed later, that the normal form will not be disturbed by any transformation of the form

$$\bar{y} = \lambda(u, v)y, \qquad \bar{u} = \phi(u, v), \qquad \bar{v} = \psi(u, v),$$

where λ, ϕ and ψ are arbitrary functions of u and v. We shall, therefore, always assume that the system (1) is given in its normal form, so that

$$AC - B^2 = 0, \qquad A'C' - B'^2 = 0, \qquad AC' + A'C - 2BB' \neq 0.$$

We may therefore put

$$(5)\qquad \begin{aligned} A &= h^2, & B &= hk, & C &= k^2, \\ A' &= h'^2, & B' &= h'k', & C' &= k'^2, \\ \Delta &= hk' - h'k \neq 0, \end{aligned}$$

so that (1) assumes the form

$$
(6)\qquad \begin{aligned} h^2 y_{uu} + 2hk y_{uv} + k^2 y_{vv} &= \mathrm{r} y + \mathrm{s} y_u + \mathrm{t} y_v, \\ h'^2 y_{uu} + 2h'k' y_{uv} + k'^2 y_{vv} &= \mathrm{r}' y + \mathrm{s}' y_u + \mathrm{t}' y_v. \end{aligned}
$$

The case of a system of form (1), for which

$$J^2 - 4II' = 0,$$

may be excluded from consideration for the purposes of the projective theory of surfaces. For it easily follows from formulæ given later that the integral surfaces of such a system degenerate either into curves or developables. In the latter case moreover the system is involutory; so that the most general developable integral surface of such a system is not a projective transformation of any particular one. For the projective theory of surfaces then, such systems are without interest.

§2. *The integrability conditions.*

Equation (6) may be solved for two of the second derivatives, thus expressing them homogeneously and linearly in terms of y, y_u, y_v and the remaining derivative of the second order. Thus (6) may be, in general, solved for y_{uu} and y_{vv} in terms of y, y_u, y_v, y_{uv}. But this is not always the case, in fact it cannot be done if

$$hk' + h'k = 0.$$

In that case (6) might be solved for y_{uu} and y_{uv}. In order, however, to avoid the discussion of such special cases, and so as to preserve symmetry, we shall introduce the following symmetrical combination of the second derivatives:

$$
(7)\qquad w = hh' y_{uu} + (hk' + h'k) y_{uv} + kk' y_{vv}.
$$

It is *always* possible to express y_{uu}, y_{uv}, y_{vv} as linear homogeneous functions of y, y_u, y_v and w. Put

$$
(8)\qquad \frac{h}{\Delta} = p, \qquad \frac{k}{\Delta} = q, \qquad \frac{h'}{\Delta} = p', \qquad \frac{k'}{\Delta} = q',
$$

and

$$
(9)\qquad \begin{aligned}
&a = \mathrm{r} q'^2 + \mathrm{r}' q^2, && b = \mathrm{s} q'^2 + \mathrm{s}' q^2, && c = \mathrm{t} q'^2 + \mathrm{t}' q^2, && d = -2qq', \\
&a' = -(\mathrm{r} p'q' + \mathrm{r}' pq), && b' = -(\mathrm{s} p'q' + \mathrm{s}' pq), && c' = -(\mathrm{t} p'q' + \mathrm{t}' pq), && d' = pq' + p'q, \\
&a'' = \mathrm{r} p'^2 + \mathrm{r}' p^2, && b'' = \mathrm{s} p'^2 + \mathrm{s}' p^2, && c'' = \mathrm{t} p'^2 + \mathrm{t}' p^2, && d'' = -2pp', \\
&\qquad\qquad d'^2 - dd'' = \frac{1}{\Delta^2}.
\end{aligned}
$$

Then

$$
(10)\qquad \begin{aligned} y_{uu} &= ay + by_u + cy_v + dw, \\ y_{uv} &= a'y + b'y_u + c'y_v + d'w, \\ y_{vv} &= a''y + b''y_u + c''y_v + d''w. \end{aligned}
$$

There can be no relation of the form

$$\lambda y + \mu y_u + \nu y_v + \rho w = 0, \tag{11}$$

with non-vanishing coefficients in any case which is of interest in the projective theory of surfaces. In fact, the existence of a relation of form (11), together with equations (10) and those deducible therefrom by differentiation, would imply that the most general solution of (6) could contain no more than three arbitrary constants. This is excluded because in our geometrical interpretation such a system cannot be said to have an integral surface. *The non-existence of a relation of the form* (11) *may be expressed geometrically by saying that the four points* P_y, P_{y_u}, P_{y_v}, P_w *form a non-degenerate tetrahedron.*

We proceed to express w_u and w_v in terms of y, y_u, y_v and w. From (10) we find the two expressions for y_{uuv};

$$\begin{aligned} y_{uuv} &= (a_v + a'b + a''c)y + (b_v + bb' + b''c)y_u \\ &\qquad + (c_v + a + bc' + cc'')y_v + (d_v + bd' + cd'')w + dw_v, \\ y_{uuv} &= (a'_u + ab' + a'c')y + (b'_u + a' + bb' + b'c')y_u \\ &\qquad + (c'_u + b'c + c'^2)y_v + (d'_u + b'd + c'd')w + d'w_u. \end{aligned} \tag{12}$$

Similarly

$$\begin{aligned} y_{uvv} &= (a'_v + a'b' + a''c')y + (b'_v + b'^2 + b''c')y_u \\ &\qquad + (c'_v + a' + b'c' + c'c'')y_v + (d'_v + b'd' + c'd'')w + d'w_v, \\ y_{uvv} &= (a''_u + ab'' + a'c'')y + (b''_u + a'' + bb'' + b'c'')y_u \\ &\qquad + (c''_u + b''c + c'c'')y_v + (d''_u + b''d + c''d')w + d''w_u. \end{aligned} \tag{13}$$

From these equations we deduce

$$\begin{aligned} d'w_u - dw_v &= (a_v - a'_u + a'b - ab' + a''c - a'c')y + (b_v - b'_u - a' + b''c - b'c')y_u \\ &\qquad + (c_v - c'_u + a + bc' - b'c + cc'' - c'^2)y_v \\ &\qquad + (d_v - d'_u + bd' - b'd + cd'' - c'd')w, \\ d''w_u - d'w_v &= (a'_v - a''_u + a'b' - ab'' + a''c' - a'c'')y \\ &\qquad + (b'_v - b''_u - a'' + b'^2 - bb'' + b''c' - b'c'')y_u \\ &\qquad + (c'_v - c''_u + a' + b'c' - b''c)y_v + (d'_v - d''_u + b'd' - b''d + c'd'' - c''d')w, \end{aligned} \tag{14}$$

whence

$$w_u = \alpha y + \beta y_u + \gamma y_v + \delta w, \qquad w_v = \alpha' y + \beta' y_u + \gamma' y_v + \delta' w, \tag{15}$$

where

$$\frac{1}{\Delta^2}\alpha = d'(a_v - a'_u + a'b - ab' + a''c - a'c') - d(a'_v - a''_u + a'b' - ab'' + a''c' - a'c''),$$

$$
16)\quad
\begin{aligned}
\frac{1}{\Delta^2}\beta &= d'(b_v-b'_u-a'+b''c-b'c')-d(b'_v-b''_u-a''+b'^2-bb''+b''c'-b'c''),\\
\frac{1}{\Delta^2}\gamma &= d'(c_v-c'_u+a+bc'-b'c+cc''-c'^2)-d(c'_v-c''_u+a'+b'c'-b''c),\\
\frac{1}{\Delta^2}\delta &= d'(d_v-d'_u+bd'-b'd+cd''-c'd')-d(d'_v-d''_u+b'd'-b''d+c'd''-c''d'),\\
\frac{1}{\Delta^2}\alpha' &= d''(a_v-a'_u+a'b-ab'+a''c-a'c')-d'(a'_v-a''_u+a'b'-ab''+a''c'-a'c''),\\
\frac{1}{\Delta^2}\beta' &= d''(b_v-b'_u-a'+b''c-b'c')-d'(b'_v-b''_u-a''+b'^2-bb''+b''c'-b'c''),\\
\frac{1}{\Delta^2}\gamma' &= d''(c_v-c'_u+a+bc'-b'c+cc''-c'^2)-d'(c'_v-c''_u+a'+b'c'-b''c),\\
\frac{1}{\Delta^2}\delta' &= d''(d_v-d'_u+bd'-b'd+cd''-c'd')-d'(d'_v-d''_u+b'd'-b''d+c'd''-c''d').
\end{aligned}
$$

As a consequence of these equations (15), equations (12) and (13), from which they were derived, show that the six expressions for the four third derivatives of y will be consistent. In order that the same thing may be true of all of the derivatives of order higher than three, it is necessary and sufficient that equations (15) themselves may be consistent, i. e., that

$$
(17)\qquad \frac{\partial w_u}{\partial v}=\frac{\partial w_v}{\partial u}.
$$

But it has already been remarked that there can be no relation of form (11) with non-vanishing coefficients. The condition (17) however would give rise to such a relation unless the coefficients of y, y_u, y_v and w on the left member were equal respectively to the corresponding coefficients on the right member of that equation. We thus obtain the following four relations:

$$
(18)\quad
\begin{aligned}
\alpha_v + \alpha'\beta + \alpha''\gamma + \alpha'\delta &= \alpha'_u + \alpha\beta' + \alpha'\gamma' + \alpha\delta',\\
\beta_v + b'\beta + b''\gamma + \beta'\delta &= \beta'_u + \alpha' + b\beta' + b'\gamma' + \beta\delta',\\
\gamma_v + \alpha + c'\beta + c''\gamma + \gamma'\delta &= \gamma'_u + c\beta' + c'\gamma' + \gamma\delta',\\
\delta_v + d'\beta + d''\gamma &= \delta'_u + d\beta' + d'\gamma',
\end{aligned}
$$

the integrability conditions of system (6).

The last of these four conditions may be transformed as follows. It may be written

$$
\delta_v - \delta'_u + d'\beta - d\beta' + d''\gamma - d'\gamma' = 0.
$$

But

$$
d'\beta - d\beta' = b_v - b'_u - a' + b''c - b'c',
$$

$$
d''\gamma - d'\gamma' = c'_v - c''_u + a' + b'c' - b''c,
$$

so that the condition becomes

$$\delta_v + b_v + c'_v = \delta'_u + b'_u + c''_u. \tag{19}$$

For the sake of brevity put

$$\begin{aligned} \mu &= bd' - b'd + cd'' - c'd', \\ \nu &= b'd' - b''d + c'd'' - c''d'. \end{aligned} \tag{20}$$

Then, according to (16),

$$\delta = \Delta^2[d'(d_v - d'_u) - d(d'_v - d''_u) + d'\mu - d\nu],$$
$$\delta' = \Delta^2[d''(d_v - d'_u) - d'(d'_v - d''_u) + d''\mu - d'\nu].$$

But from equations (9) we see that

$$\mu = \frac{1}{\Delta}(q'^2 s - q^2 s' - p'q't + pqt'),$$

$$\nu = \frac{1}{\Delta}(-p'q's + pqs' + p'^2 t - p^2 t'),$$

so that

$$\begin{aligned} d'\mu - d\nu &= \frac{1}{\Delta^2}(b + c'), \\ d''\mu - d'\nu &= \frac{1}{\Delta^2}(b' + c''), \end{aligned} \tag{21}$$

whence

$$\begin{aligned} \delta &= \Delta^2[d'(d_v - d'_u) - d(d'_v - d''_u)] + b + c', \\ \delta' &= \Delta^2[d''(d_v - d'_u) - d'(d'_v - d''_u)] + b' + c''. \end{aligned} \tag{22}$$

With the aid of these expressions, it will be seen that the last of the four integrability conditions (18) may be satisfied in the most general way by putting

$$\begin{aligned} \Delta^2[d'(d_v - d'_u) - d(d'_v - d''_u)] + 2b + 2c' &= \frac{\partial \epsilon'}{\partial u}, \\ \Delta^2[d''(d_v - d'_u) - d'(d'_v - d''_u)] + 2b' + 2c'' &= \frac{\partial \epsilon'}{\partial v}, \end{aligned} \tag{23}$$

where ϵ' is an arbitrary function of u and v. We may use these equations to express b and c'' in terms of ϵ' and the other coefficients of system (1).

In will be advantageous to introduce

$$D = \Delta d, \qquad D' = \Delta d', \qquad D'' = \Delta d'', \tag{24}$$

in place of d, d', d''. Then

$$D'^2 - DD'' = \Delta^2(d'^2 - dd'') = 1, \tag{25}$$

and the conditions (23) become

$$
(26)\qquad \begin{aligned} D'(D_v - D'_u) - D(D'_v - D''_u) + 2(b + c') &= \epsilon_u, \\ D''(D_v - D'_u) - D'(D'_v - D''_u) + 2(b' + c'') &= \epsilon_v, \end{aligned}
$$

where

$$\epsilon = \epsilon' - \log \Delta$$

may be any function of u and v.

§ 3. *The seminvariants.*

Let us consider the system (6), which is in its normal form. If we multiply the members of both equations by arbitrary functions ω^2 and ω'^2 of u and v, the resulting system is again in the normal form. Moreover, if the two equations of the system be interchanged, the normal form is not disturbed. The combination of these two operations gives rise to the most general system of equations equivalent to (6) and still in the normal form (cf. § 1). Only those combinations of the coefficients of (6) which remain unchanged as a result of these operations are of interest to us, because they alone represent quantities determined by the system which are identical for all equivalent systems. In other words, they are invariants of the system (1) under the transformation which consists in replacing it by any equivalent system (2). The twelve quantities, $a, b, c, D, a', b', c', D', a'', b'', c'', D''$, the *virtual coefficients* of the system, are invariants of this kind. It is for this reason that we have introduced D, D', D'' in place of d, d', d'' at the end of the preceding paragraph. These twelve quantities, however, are not independent. In fact we find the following relations between them

$$
(27)\qquad \begin{aligned} aD'' - 2a'D' + a''D &= 0, \\ bD'' - 2b'D' + b''D &= 0, \\ cD'' - 2c'D' + c''D &= 0, \\ D'^2 - DD'' &= 1, \end{aligned}
$$

from which others may be derived, of which we shall mention only the following:

$$
(28)\qquad \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} = 0.
$$

Since we may substitute equations (10) for (6), all invariants of system (6) may be expressed in terms of the twelve quantities $a, b, c, D, \cdots, D''$. Moreover, only eight of these are independent owing to the four independent relations (27), which is in harmony with the fact that system (6) involves only the eight ratios of the ten functions $h, k, h', k', \mathrm{r}, \mathrm{s}, \mathrm{t}, \mathrm{r}', \mathrm{s}', \mathrm{t}'$; but we shall retain all twelve, for the sake of symmetry.

Let us now put in (6)

$$(29)\qquad y = \lambda \bar{y},$$

and afterward divide both members of both equations by λ, where λ is an arbitrary function of u and v. The functions of the quantities, a, b, c, D, etc., and of their derivatives which remain unaltered by this transformation shall be called the *seminvariants* of system (6). The *semicovariants* have the same invariant property, but involve y and its derivatives besides the coefficients of the system.

We have from (29)

$$
(30)\qquad
\begin{aligned}
y_u &= \lambda\bar{y}_u + \lambda_u y, \qquad y_v = \lambda\bar{y}_v + \lambda_v\bar{y},\\
y_{uu} &= \lambda\bar{y}_{uu} + 2\lambda_u\bar{y}_u + \lambda_{uu}\bar{y},\\
y_{uv} &= \lambda\bar{y}_{uv} + \lambda_v\bar{y}_u + \lambda_u\bar{y}_v + \lambda_{uv}\bar{y},\\
y_{vv} &= \lambda\bar{y}_{vv} + 2\lambda_v\bar{y}_v + \lambda_{vv}\bar{y}.
\end{aligned}
$$

Substitute these values in (6), divide by λ, and denote the coefficients of the resulting system of equations by $\bar{h}$, $\bar{k}$, $\bar{\mathrm{r}}$, $\bar{\mathrm{s}}$, $\bar{\mathrm{t}}$, etc. We find

$$
\begin{aligned}
\bar{h} &= \pm h, \qquad \bar{k} = \pm k,\\
\bar{\mathrm{r}} &= \mathrm{r} + \frac{\lambda_u}{\lambda}\mathrm{s} + \frac{\lambda_v}{\lambda}\mathrm{t} - \frac{\lambda_{uu}}{\lambda}h^2 - 2\frac{\lambda_{uv}}{\lambda}hk - \frac{\lambda_{vv}}{\lambda}k^2,\\
\bar{\mathrm{s}} &= \mathrm{s} - 2h\left(\frac{\lambda_u}{\lambda}h + \frac{\lambda_v}{\lambda}k\right),\\
\bar{\mathrm{t}} &= \mathrm{t} - 2k\left(\frac{\lambda_u}{\lambda}h + \frac{\lambda_v}{\lambda}k\right),
\end{aligned}
$$

the equations of transformation for h', k', r', s', t' being of precisely the same form. Moreover, in the first two equations, either both of the upper signs or both of the lower signs must be taken.

We find consequently:

$$
(31)\qquad
\begin{aligned}
\bar{a} &= a + \frac{\lambda_u}{\lambda}b + \frac{\lambda_v}{\lambda}c - \frac{\lambda_{uu}}{\lambda}T + \frac{\lambda_{uv}}{\lambda}DD' - \frac{1}{2}\frac{\lambda_{vv}}{\lambda}D^2,\\
\bar{a}' &= a' + \frac{\lambda_u}{\lambda}b' + \frac{\lambda_v}{\lambda}c' - \frac{1}{2}\frac{\lambda_{uu}}{\lambda}D'D'' + \frac{\lambda_{uv}}{\lambda}DD'' - \frac{1}{2}\frac{\lambda_{vv}}{\lambda}DD',\\
\bar{a}'' &= a'' + \frac{\lambda_u}{\lambda}b'' + \frac{\lambda_v}{\lambda}c'' - \frac{1}{2}\frac{\lambda_{uu}}{\lambda}D''^2 + \frac{\lambda_{uv}}{\lambda}D'D'' - \frac{\lambda_{vv}}{\lambda}T,\\
\bar{b} &= b - 2\frac{\lambda_u}{\lambda}T + \frac{\lambda_v}{\lambda}DD', \qquad \bar{c} = c + \frac{\lambda_u}{\lambda}DD' - \frac{\lambda_v}{\lambda}D^2,\\
\bar{b}' &= b' - \frac{\lambda_u}{\lambda}D'D'' + \frac{\lambda_v}{\lambda}DD'', \qquad \bar{c}' = c' + \frac{\lambda_u}{\lambda}DD'' - \frac{\lambda_v}{\lambda}DD',\\
\bar{b}'' &= b'' - \frac{\lambda_u}{\lambda}D''^2 + \frac{\lambda_v}{\lambda}D'D'', \qquad \bar{c}'' = c'' + \frac{\lambda_u}{\lambda}D'D'' - 2\frac{\lambda_v}{\lambda}T,\\
\bar{D} &= D, \qquad \bar{D}' = D', \qquad \bar{D}'' = D'',
\end{aligned}
$$

where

$$T = D^2 - \tfrac{1}{2}DD''. \tag{32}$$

Therefore D, D', D'' are seminvariants. Moreover the system of equations (27) remains invariant under this transformation, although the left member of the first of these equations is not by itself a seminvariant.

Since we are supposing the integrability conditions to be satisfied, the coefficients of our system will satisfy equations (26) where ϵ is some function of u and v. Let $\bar{\epsilon}$ denote the value of this function after the transformation. We find, from (26) and (31),

$$\bar{\epsilon}_u = \epsilon_u - 4\frac{\lambda_u}{\lambda}, \qquad \bar{\epsilon}_v = \epsilon_v - 4\frac{\lambda_v}{\lambda}.$$

Consequently $\bar{\epsilon}_u$ and $\bar{\epsilon}_v$ may be reduced to zero in the most general way by putting

$$\lambda = ke^{\frac{1}{4}\epsilon},$$

where k is an arbitrary constant. We shall then have

$$\begin{gathered} \frac{\lambda_u}{\lambda} = \tfrac{1}{4}\epsilon_u, \qquad \frac{\lambda_v}{\lambda} = \tfrac{1}{4}\epsilon_v, \\ \frac{\lambda_{uu}}{\lambda} = \tfrac{1}{4}\epsilon_{uu} + \tfrac{1}{16}\epsilon_u^2, \qquad \frac{\lambda_{uv}}{\lambda} = \tfrac{1}{4}\epsilon_{uv} + \tfrac{1}{16}\epsilon_u\epsilon_v, \qquad \frac{\lambda_{vv}}{\lambda} = \tfrac{1}{4}\epsilon_{vv} + \tfrac{1}{16}\epsilon_v^2. \end{gathered} \tag{33}$$

The substitution

$$y = ke^{\frac{1}{4}\epsilon}\bar{y}$$

will reduce system (6) to another one of the same form. The quantities A, B, C, D, etc., which correspond to a, b, c, D, etc. in this new system, are obtained by substituting the values (33) for λ_u/λ, λ_v/λ, etc., in equations (31). These quantities satisfy the relations

$$\begin{gathered} D'(D_v - D'_u) - D(D'_v - D''_u) + 2(B + C') = 0, \\ D''(D_v - D'_u) - D'(D'_v - D''_u) + 2(B' + C'') = 0, \end{gathered} \tag{34}$$

obtained from (26) on account of the vanishing of $\bar{\epsilon}_u$ and $\bar{\epsilon}_v$.

We have called the quantities a, b, c, D, $\cdots$ the *virtual coefficients* of system (6). We have seen that there exists for every such system a uniquely determined transformed system whose virtual coefficients satisfy the conditions (34). This transformed system shall be said to be in its *semi-canonical form.*

The virtual coefficients of the semi-canonical form, A, B, C, D, *etc., are seminvariants of system* (6). They are the values of $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{D}$, etc., obtained from (31) by substituting for λ_u/λ, λ_v/λ, etc., the values (33).

The seminvariance of these quantities is a consequence of the uniqueness of the semi-canonical form, and may moreover be tested directly by means of equations (31).

As a consequence of (27) *the seminvariants satisfy the relations*

$$
\begin{aligned}
&AD'' - 2A'D' + A''D = 0, \\
(35)\qquad &BD'' - 2B'D' + B''D = 0, \\
&CD'' - 2C'D' + C''D = 0, \\
&D'^2 - DD'' = 1,
\end{aligned}
$$

as well as (34), *so that only seven of them are independent. The complete system of seminvariants obviously consists of these quantities and their derivatives with respect to the two independent variables u and v.*

In fact, every seminvariant S is a function of the virtual coefficients of the system considered and of their derivatives, which does not change its value when any transformation of the form

$$y = \lambda \bar{y}$$

is made. But if the system be reduced to the semi-canonical form, every such function becomes a function of *its* virtual coefficients alone, i. e., of A, B, C, D, etc., and of their derivatives. Since S does not change its value as a result of this reduction, S is therefore a function of these variables only.

§ 4. *The invariants.*

We now proceed to investigate the effect of a transformation of the independent variables upon the seminvariants. Let new variables be introduced by means of the equations

$$(36)\qquad \bar{u} = \phi(u, v), \qquad \bar{v} = \psi(u, v),$$

where ϕ and ψ denote arbitrary functions of u and v. We find

$$
\begin{aligned}
&y_u = y_{\bar{u}}\phi_u + y_{\bar{v}}\psi_u, \qquad y_v = y_{\bar{u}}\phi_v + y_{\bar{v}}\psi_v, \\
&y_{uu} = y_{\bar{u}\bar{u}}\phi_u^2 + 2y_{\bar{u}\bar{v}}\phi_u\psi_u + y_{\bar{v}\bar{v}}\psi_u^2 + y_{\bar{u}}\phi_{uu} + y_{\bar{v}}\psi_{uu}, \\
(37)\qquad &y_{uv} = y_{\bar{u}\bar{u}}\phi_u\phi_v + y_{\bar{u}\bar{v}}(\phi_u\psi_v + \phi_v\psi_u) + y_{\bar{v}\bar{v}}\psi_u\psi_v + y_{\bar{u}}\phi_{uv} + y_{\bar{v}}\psi_{uv}, \\
&y_{vv} = y_{\bar{u}\bar{u}}\phi_v^2 + 2y_{\bar{v}\bar{u}}\phi_v\psi_v + y_{\bar{v}\bar{v}}\psi_v^2 + y_{\bar{u}}\phi_{vv} + y_{\bar{v}}\psi_{vv}.
\end{aligned}
$$

Substitute these values in system (6) and denote the coefficients of the transformed system by $\bar{h}$, $\bar{k}$, $\bar{\mathrm{r}}$, $\bar{\mathrm{s}}$, $\bar{\mathrm{t}}$, etc. We shall find

$$
\begin{aligned}
&\bar{h} = \pm(\phi_u h + \phi_v k), \qquad \bar{k} = \pm(\psi_u h + \psi_v k), \qquad \bar{\mathrm{r}} = \mathrm{r}, \\
(38)\qquad &\bar{\mathrm{s}} = \phi_u \mathrm{s} + \phi_v \mathrm{t} - \phi_{uu}h^2 - 2\phi_{uv}hk - \phi_{vv}k^2, \\
&\bar{\mathrm{t}} = \psi_u \mathrm{s} + \psi_v \mathrm{t} - \psi_{uu}h^2 - 2\psi_{uv}hk - \psi_{vv}k^2,
\end{aligned}
$$

and similar expressions for $\bar{h}'$, $\bar{k}'$, etc. The two upper or the two lower signs

may be chosen, in (38), at will; for the virtual coefficients and consequently the seminvariants contain only such combinations of h, k, h', k' as are unaffected by the sign chosen.

Introduce infinitesimal transformations by putting

$$(39)\qquad \bar{u} = \phi(u, v) = u + \kappa(u, v)\delta t, \qquad \bar{v} = \psi(u, v) = v + \lambda(u, v)\delta t,$$

where δt is an infinitesimal. We find the following infinitesimal transformations for h, k, etc.:

$$\delta h = \pm(\kappa_u h + \kappa_v k)\delta t, \qquad \delta k = \pm(\lambda_u h + \lambda_v k)\delta t, \qquad \delta \mathrm{r} = 0,$$

$$(40)\qquad \delta \mathrm{s} = (\kappa_u \mathrm{s} + \kappa_v \mathrm{t} - \kappa_{uu} h^2 - 2\kappa_{uv} hk - \kappa_{vv} k^2)\delta t,$$

$$\delta \mathrm{t} = (\lambda_u \mathrm{s} + \lambda_v \mathrm{t} - \lambda_{uu} h^2 - 2\lambda_{uv} hk - \lambda_{vv} k^2)\delta t.$$

The equations for $\delta h', \cdots \delta t'$ are of exactly the same form. Making use of the equations (9) and (24) which define a, b, c, D, etc., we find:

$$\begin{aligned}
\delta a &= -(2a\kappa_u + 2a'\lambda_u)\delta t,\\
\delta a' &= -(a'\kappa_u + a\kappa_v + a''\lambda_u + a'\lambda_v)\delta t,\\
\delta a'' &= -(2a'\kappa_v + 2a''\lambda_v)\delta t,\\
\delta b &= [-b\kappa_u + c\kappa_v - 2b'\lambda_u - T\kappa_{uu} + DD'\kappa_{uv} - \tfrac{1}{2}D^2\kappa_{vv}]\delta t,\\
\delta b' &= [(c' - b)\kappa_v - b''\lambda_u - b'\lambda_v - \tfrac{1}{2}D'D''\kappa_{uu} + DD''\kappa_{uv} - \tfrac{1}{2}DD'\kappa_{vv}]\delta t,\\
\delta b'' &= [b''\kappa_u + (c'' - 2b')\kappa_v - 2b''\lambda_v - \tfrac{1}{2}D''^2\kappa_{uu} + D'D''\kappa_{uv} - T\kappa_{vv}]\delta t,\\
\delta c &= [-2c\kappa_u + (b - 2c')\lambda_u + c\lambda_v - T\lambda_{uu} + DD'\lambda_{uv} - \tfrac{1}{2}D^2\lambda_{vv}]\delta t,\\
\delta c' &= [-c'\kappa_u - c\kappa_v + (b' - c'')\lambda_u - \tfrac{1}{2}D'D''\lambda_{uu} + DD''\lambda_{uv} - \tfrac{1}{2}DD'\lambda_{vv}]\delta t,\\
\delta c'' &= [-2c'\kappa_v + b''\lambda_u - c''\lambda_v - \tfrac{1}{2}D''^2\lambda_{uu} + D'D''\lambda_{uv} - T\lambda_{vv}]\delta t,\\
\delta D &= (-D\kappa_u - 2D'\lambda_u + D\lambda_v)\delta t,\\
\delta D' &= (-D\kappa_v - D''\lambda_u)\delta t,\\
\delta D'' &= (D''\kappa_u - 2D'\kappa_v - D''\lambda_v)\delta t.
\end{aligned}\tag{41}$$

The infinitesimal transformations of the derivatives of these quantities may be obtained by noticing that, for any function Φ, the following equations hold:

$$(42)\qquad \begin{aligned}
\delta\Phi_u &= \frac{\partial}{\partial u}(\delta\Phi) - (\kappa_u\Phi_u + \lambda_u\Phi_v)\delta t,\\
\delta\Phi_v &= \frac{\partial}{\partial v}(\delta\Phi) - (\kappa_v\Phi_u + \lambda_v\Phi_v)\delta t.
\end{aligned}$$

Thus we find in particular:

$$
(43)\quad
\begin{aligned}
\delta D_u &= [-2D_u\kappa_u - (2D'_u + D_v)\lambda_u + D_u\lambda_v - D\kappa_{uu} - 2D'\lambda_{uu} + D\lambda_{uv}]\delta t,\\
\delta D_v &= [D_v\kappa_u - D_u\kappa_v - 2D'_u\lambda_u - D\kappa_{uv} - 2D'\lambda_{uv} + D\lambda_{vv}]\delta t,\\
\delta D'_u &= [-D'_u\kappa_u - D_u\kappa_v - (D''_u + D'_v)\lambda_u - D\kappa_{uv} - D''\lambda_{uu}]\,\delta t,\\
\delta D'_v &= [-(D_v + D'_u)\kappa_v - D''_v\lambda_u - D'_v\lambda_v - D\kappa_{vv} - D''\lambda_{uv}]\,\delta t,\\
\delta D''_u &= [-2D'_u\kappa_v - D''_v\lambda_u - D''_u\lambda_v + D''\kappa_{uu} - 2D'\kappa_{uv} - D''\lambda_{uv}]\,\delta t,\\
\delta D''_v &= [D''_v\kappa_u - (2D'_v + D''_u)\kappa_v - 2D''_v\lambda_v + D''\kappa_{uv} - 2D'\kappa_{vv} - D''\lambda_{vv}]\delta t.
\end{aligned}
$$

From (26) we now find

$$
(44)\quad
\begin{aligned}
\delta\epsilon_u &= -(\epsilon_u\kappa_u + \epsilon_v\lambda_u + 2\kappa_{uu} + 2\lambda_{uv})\delta t,\\
\delta\epsilon_v &= -(\epsilon_u\kappa_v + \epsilon_v\lambda_v + 2\kappa_{uv} + 2\lambda_{vv})\delta t,
\end{aligned}
$$

whence

$$
(45)\quad
\begin{aligned}
\delta\epsilon_{uu} &= -(2\epsilon_{uu}\kappa_u + 2\epsilon_{uv}\lambda_u + \epsilon_u\kappa_{uu} + \epsilon_v\lambda_{uu} + 2\kappa_{uuu} + 2\lambda_{uuv})\delta t,\\
\delta\epsilon_{uv} &= -(\epsilon_{uv}\kappa_u + \epsilon_{uu}\kappa_v + \epsilon_{vv}\lambda_u + \epsilon_{uv}\lambda_v + \epsilon_u\kappa_{uv} + \epsilon_v\lambda_{uv} + 2\kappa_{uuv} + 2\lambda_{uvv})\delta t,\\
\delta\epsilon_{vv} &= -(2\epsilon_{uv}\kappa_v + 2\epsilon_{vv}\lambda_v + \epsilon_u\kappa_{vv} + \epsilon_v\lambda_{vv} + 2\kappa_{uvv} + 2\lambda_{vvv})\delta t.
\end{aligned}
$$

The quantities ϵ_u, ϵ_v, etc., enter into the composition of the seminvariants. Making use of equations (31), (33), (41), (44),(45), we obtain following expressions for their infinitesimal transformations:

$$
(46)\quad
\begin{aligned}
\delta A &= [-2A\kappa_u - 2A'\lambda_u - \tfrac{1}{2}B(\kappa_{uu} + \lambda_{uv}) - \tfrac{1}{2}C(\kappa_{uv} + \lambda_{vv})\\
&\qquad + \tfrac{1}{2}T(\kappa_{uuu} + \lambda_{uuv}) - \tfrac{1}{2}DD'(\kappa_{uuv} + \lambda_{uvv}) + \tfrac{1}{4}D^2(\kappa_{uvv} + \lambda_{vvv})]\delta t,\\
\delta A' &= [-A'\kappa_u - A\kappa_v - A''\lambda_u - A'\lambda_v - \tfrac{1}{2}B'(\kappa_{uu} + \lambda_{uv}) - \tfrac{1}{2}C'(\kappa_{uv} + \lambda_{vv})\\
&\qquad + \tfrac{1}{4}D'D''(\kappa_{uuu} + \lambda_{uuv}) - \tfrac{1}{2}DD''(\kappa_{uuv} + \lambda_{uvv}) + \tfrac{1}{4}DD'(\kappa_{uvv} + \lambda_{vvv})]\delta t,\\
\delta A'' &= [-2A'\kappa_v - 2A''\lambda_v - \tfrac{1}{2}B''(\kappa_{uu} + \lambda_{uv}) - \tfrac{1}{2}C''(\kappa_{uv} + \lambda_{vv})\\
&\qquad + \tfrac{1}{4}D''^2(\kappa_{uuu} + \lambda_{uuv}) - \tfrac{1}{2}D'D''(\kappa_{uuv} + \lambda_{uvv}) + \tfrac{1}{2}T(\kappa_{uvv} + \lambda_{vvv})]\delta t,
\end{aligned}
$$

and further,

$$
(47)\quad
\begin{aligned}
\delta B &= [-B\kappa_u + C\kappa_v - 2B'\lambda_u + \tfrac{1}{2}DD'(\kappa_{uv} - \lambda_{vv}) - \tfrac{1}{2}D^2\kappa_{vv} + T\lambda_{uv}]\,\delta t,\\
\delta B' &= [(C' - B)\kappa_v - B''\lambda_u - B'\lambda_v + \tfrac{1}{2}DD''(\kappa_{uv} - \lambda_{vv}) - \tfrac{1}{2}DD'\kappa_{vv} + \tfrac{1}{2}D'D''\lambda_{uv}]\delta t,\\
\delta B'' &= [B''\kappa_u + (C'' - 2B')\kappa_v - 2B''\lambda_v + \tfrac{1}{2}D'D''(\kappa_{uv} - \lambda_{vv}) - T\kappa_{vv} + \tfrac{1}{2}D''^2\lambda_{uv}]\delta t,\\
\delta C &= [-2C\kappa_u + (B - 2C')\lambda_u + C\lambda_v - \tfrac{1}{2}DD'(\kappa_{uu} - \lambda_{uv}) + \tfrac{1}{2}D^2\kappa_{uv} - T\lambda_{uu}]\delta t,\\
\delta C' &= [-C'\kappa_u - C\kappa_v + (B' - C'')\lambda_u - \tfrac{1}{2}DD''(\kappa_{uu} - \lambda_{uv})\\
&\qquad + \tfrac{1}{2}DD'\kappa_{uv} - \tfrac{1}{2}D'D''\lambda_{uu}]\delta t,\\
\delta C'' &= [-2C'\kappa_v + B''\lambda_u - C''\lambda_v - \tfrac{1}{2}D'D''(\kappa_{uu} - \lambda_{uv}) + T\kappa_{uv} - \tfrac{1}{2}D''^2\lambda_{uu}]\delta t.
\end{aligned}
$$

Let us put

$$
(48)\qquad
\begin{aligned}
M &= D'D_v - DD'_v, & N &= D'D_u - DD'_u,\\
M' &= D''D_v - DD''_v, & N' &= D''D_u - DD''_u,\\
M'' &= D''D'_v - D'D''_v, & N'' &= D''D'_u - D'D''_u,
\end{aligned}
$$

Then will

$$
(49)\qquad
\begin{aligned}
\delta M &= [-M\kappa_u - N\kappa_v - M'\lambda_u - DD'(\kappa_{uv} - \lambda_{vv}) + D^2\kappa_{vv} - 2T\lambda_{uv}]\delta t,\\
\delta M' &= [-(N' + 2M)\kappa_v - 2M''\lambda_u - M'\lambda_v - 2DD''(\kappa_{uv} - \lambda_{vv})\\
&\qquad + 2DD'\kappa_{vv} - 2D'D''\lambda_{uv}]\,\delta t,\\
\delta M'' &= [M''\kappa_u - (M' + N'')\kappa_v - 2M''\lambda_v - D'D''(\kappa_{uv} - \lambda_{vv}) + 2T\kappa_{vv} - D''^2\lambda_{uv}]\delta t,\\
\delta N &= [-2N\kappa_u - (M + N')\lambda_u + N\lambda_v - DD'(\kappa_{uu} - \lambda_{uv}) + D^2\kappa_{uv} - 2T\lambda_{uu}]\delta t,\\
\delta N' &= [-N'\kappa_u - 2N\kappa_v - (M' + 2N'')\lambda_u - 2DD''(\kappa_{uu} - \lambda_{uv})\\
&\qquad + 2DD'\kappa_{uv} - 2D'D''\lambda_{uu}]\,\delta t,\\
\delta N'' &= [-N'\kappa_v - M''\lambda_u - N''\lambda_v - D'D''(\kappa_{uu} - \lambda_{uv}) + 2T\kappa_{uv} - D''^2\lambda_{uu}]\,\delta t.
\end{aligned}
$$

If we put now

$$
(50)\qquad
\begin{aligned}
\mathfrak{B} &= B + \tfrac{1}{2}M, & \mathfrak{C} &= C - \tfrac{1}{2}N,\\
\mathfrak{B}' &= B' + \tfrac{1}{4}M', & \mathfrak{C}' &= C' - \tfrac{1}{4}N',\\
\mathfrak{B}'' &= B'' + \tfrac{1}{2}M'', & \mathfrak{C}'' &= C'' - \tfrac{1}{2}N'',
\end{aligned}
$$

we shall therefore find

$$
(51)\qquad
\begin{aligned}
\delta\mathfrak{B} &= [-\mathfrak{B}\kappa_u + \mathfrak{C}\kappa_v - 2\mathfrak{B}'\lambda_u]\delta t, & \delta\mathfrak{C} &= [-2\mathfrak{C}\kappa_u + (\mathfrak{B} - 2\mathfrak{C}')\lambda_u + \mathfrak{C}\lambda_v]\delta t,\\
\delta\mathfrak{B}' &= [(\mathfrak{C}' - \mathfrak{B})\kappa_v - \mathfrak{B}''\lambda_u - \mathfrak{B}'\lambda_v]\delta t, & \delta\mathfrak{C}' &= [-\mathfrak{C}'\kappa_u - \mathfrak{C}\kappa_v + (\mathfrak{B}' - \mathfrak{C}'')\lambda_u]\delta t,\\
\delta\mathfrak{B}'' &= [\mathfrak{B}''\kappa_u + (\mathfrak{C}'' - 2\mathfrak{B}')\kappa_v - 2\mathfrak{B}''\lambda_v]\delta t, & \delta\mathfrak{C}'' &= [-2\mathfrak{C}'\kappa_v + \mathfrak{B}''\lambda_u - \mathfrak{C}''\lambda_v]\delta t.
\end{aligned}
$$

From (34), (35), (48) and (50) we find the following relations:

$$
(52)\qquad
\begin{aligned}
&\mathfrak{B}D'' - 2\mathfrak{B}'D' + \mathfrak{B}''D = 0, \qquad \mathfrak{C}D'' - 2\mathfrak{C}'D' + \mathfrak{C}''D = 0\\
&\mathfrak{B} + \mathfrak{C}' = 0, \qquad \mathfrak{B}' + \mathfrak{C}'' = 0, \qquad D'^2 - DD'' = 1,
\end{aligned}
$$

which system of equations remains invariant under all of the transformations here considered.

By means of the relations (52), five of the nine variables $\mathfrak{B}$, $\mathfrak{B}'$, $\mathfrak{B}''$, $\mathfrak{C}$, $\mathfrak{C}'$, $\mathfrak{C}''$, D, D', D'' may be expressed in terms of the other four. For instance, if

$$DD'' - 4D'^2 \neq 0,$$

we shall find

$$\mathfrak{C}' = -\mathfrak{B} = \frac{\mathfrak{B}''D^2 - 2\mathfrak{C}D'D''}{DD'' - 4D'^2},$$

$$\mathfrak{B}' = -\mathfrak{C}'' = \frac{-2\mathfrak{B}''DD' + \mathfrak{C}D''^2}{DD'' - 4D'^2},$$

$$D' = \sqrt{1 + DD''}.$$

Any invariant, depending only upon these nine seminvariants, therefore becomes a function of D, D'', $\mathfrak{B}''$ and $\mathfrak{C}$ alone. From (41) and (51) we see that every absolute invariant of this kind must satisfy a complete system of four linear homogeneous partial differential equations of the first order with four independent variables. These equations are found to be independent, a fact which may also be expressed by saying that the determinant Δ_4 of the fourth order made up of the sixteen coefficients of these four equations is not identically equal to zero. Therefore *no such absolute invariant exists.* To be sure, the curves $u =$ const. and $v =$ const. upon the integral surface might be such as to make $DD'' - 4D'^2$ vanish, so as to destroy the validity of this argument. But this condition

$$DD'' - 4D'^2 = 0$$

is not invariant, and may always be avoided by a transformation of the independent variables. Such a transformation cannot, however, affect the existence or non-existence of an invariant. We have shown therefore that there exists no absolute invariant which involves only the above nine seminvariants. Therefore there can be at most one *relative* invariant which is a function of these nine variables alone. Such a one actually exists and, owing to the relations (52), may be written in various forms. We proceed to obtain it.

Let

(53) $$P = \mathfrak{B}'\mathfrak{C}'' - \mathfrak{B}''\mathfrak{C}', \qquad R = \mathfrak{B}\mathfrak{C}'' - \mathfrak{B}''\mathfrak{C}, \qquad S = \mathfrak{B}\mathfrak{C}' - \mathfrak{B}'\mathfrak{C}.$$

Then wil

(54) $$\begin{aligned} \delta P &= -(R\kappa_v + 2P\lambda_v)\,\delta t, \\ \delta R &= -(R\kappa_u + 2S\kappa_v + 2P\lambda_u + R\lambda_v)\,\delta t, \\ \delta S &= -(2S\kappa_u + R\lambda_u)\,\delta t. \end{aligned}$$

If these equations be combined with the last three equations of system (41), which give the infinitesimal transformations of D, D', D'', it will be seen that *θ is a relative invariant, where*

(55) $$\theta = \begin{vmatrix} \mathfrak{B} & \mathfrak{B}' & \mathfrak{B}'' \\ \mathfrak{C} & \mathfrak{C}' & \mathfrak{C}'' \\ D & D' & D'' \end{vmatrix} = PD - RD' + SD''.$$

In fact we find

$$\delta\theta = -(\kappa_u + \lambda_v)\theta\delta t. \tag{56}$$

We wish to determine next those invariants which involve the first derivatives of the quantities $\mathfrak{B}$, $\mathfrak{C}$ and D, as well as these quantities themselves. We shall see that they must be solutions of a complete system of ten partial differential equations with twelve independent variables. There must exist, consequently, two absolute, or three relative invariants of this kind including θ; therefore there will be two new relative invariants to be determined.

The expressions for δD_u, etc., have already been deduced [cf. eq. (43)]. From (51) and (42) we find the following formulae:

$$\begin{aligned}
\delta\mathfrak{B}_u &= [-2\mathfrak{B}_u\kappa_u + \mathfrak{C}_u\kappa_v - (2\mathfrak{B}'_u + \mathfrak{B}_v)\lambda_u - \mathfrak{B}\kappa_{uu} + \mathfrak{C}\kappa_{uv} - 2\mathfrak{B}'\lambda_{uu}]\delta t,\\
\delta\mathfrak{B}_v &= [-\mathfrak{B}_v\kappa_u + (\mathfrak{C}_v - \mathfrak{B}_u)\kappa_v - 2\mathfrak{B}'_v\lambda_u - \mathfrak{B}_v\lambda_v - \mathfrak{B}\kappa_{uv} + \mathfrak{C}\kappa_{vv} - 2\mathfrak{B}'\lambda_{uv}]\delta t,\\
\delta\mathfrak{B}''_u &= [(\mathfrak{C}''_u - 2\mathfrak{B}'_u)\kappa_v - \mathfrak{B}''_v\lambda_u - 2\mathfrak{B}''_u\lambda_v + \mathfrak{B}''\kappa_{uu} + (\mathfrak{C}'' - 2\mathfrak{B}')\kappa_{uv} - 2\mathfrak{B}''\lambda_{uv}]\delta t,\\
\delta\mathfrak{B}''_v &= [\mathfrak{B}''_v\kappa_u + (\mathfrak{C}''_v - 2\mathfrak{B}'_v - \mathfrak{B}''_u)\kappa_v - 3\mathfrak{B}''_v\lambda_v + \mathfrak{B}''\kappa_{uv} + (\mathfrak{C}'' - 2\mathfrak{B}')\kappa_{vv} - 2\mathfrak{B}''\lambda_{vv}]\delta t,\\
\delta\mathfrak{C}_u &= [-3\mathfrak{C}_u\kappa_u + (\mathfrak{B}_u - 2\mathfrak{C}'_u - \mathfrak{C}_v)\lambda_u + \mathfrak{C}_u\lambda_v - 2\mathfrak{C}\kappa_{uu} + (\mathfrak{B} - 2\mathfrak{C}')\lambda_{uu} + \mathfrak{C}\lambda_{uv}]\delta t,\\
\delta\mathfrak{C}_v &= [-2\mathfrak{C}_v\kappa_u - \mathfrak{C}_u\kappa_v + (\mathfrak{B}_v - 2\mathfrak{C}'_v)\lambda_u - 2\mathfrak{C}\kappa_{uv} + (\mathfrak{B} - 2\mathfrak{C}')\lambda_{uv} + \mathfrak{C}\lambda_{vv}]\delta t,\\
\delta\mathfrak{C}''_u &= [-\mathfrak{C}''_u\kappa_u - 2\mathfrak{C}'_u\kappa_v + (\mathfrak{B}''_u - \mathfrak{C}''_v)\lambda_u - \mathfrak{C}''_u\lambda_v - 2\mathfrak{C}'\kappa_{uv} + \mathfrak{B}''\lambda_{uu} - \mathfrak{C}''\lambda_{uv}]\delta t,\\
\delta\mathfrak{C}''_v &= [-(2\mathfrak{C}'_v + \mathfrak{C}''_u)\kappa_v + \mathfrak{B}''_v\lambda_u - 2\mathfrak{C}''_v\lambda_v - 2\mathfrak{C}'\kappa_{vv} + \mathfrak{B}''\lambda_{uv} - \mathfrak{C}''\lambda_{vv}]\delta t.
\end{aligned} \tag{57}$$

The expressions for $\delta\mathfrak{B}'_u$, $\delta\mathfrak{B}'_v$, $\delta\mathfrak{C}'_u$, $\delta\mathfrak{C}'_v$ have been omitted because the relations

$$\mathfrak{B} + \mathfrak{C}' = 0, \qquad \mathfrak{B}' + \mathfrak{C}'' = 0$$

render them unnecessary. These relations may also be used for the purpose of simplifying some of the terms in (57).

If we assume again, for a moment, that $DD'' - 4D'^2$ is different from zero, we may think of D, D'', $\mathfrak{B}''$, $\mathfrak{C}$ and the eight derivatives of the first order of these quantities as being the twelve independent variables of the complete system which determines the invariants under consideration. The first four equations of the system, obtained by equating to zero the coefficients of κ_u, κ_v, λ_u, λ_v in the general expression for δf, where f is an arbitrary function of these variables, are known to be independent. In fact, the matrix of the 4×12 coefficients of these four equations contains the determinant of the fourth order, denoted above by Δ_4, which is known to be different from zero. The elements of this determinant are the coefficients of $\partial f/\partial D$, $\partial f/\partial D''$, $\partial f/\partial\mathfrak{B}''$, $\partial f/\partial\mathfrak{C}$ in the first four equations of the complete system. The matrix of the coefficients of $\partial f/\partial D_u$,

$\partial f/\partial D_v, \cdots \partial f/\partial \mathfrak{C}_v$ in the remaining six equations is

$$
(58) \quad \begin{vmatrix}
-D & 0 & +D'' & 0 & +\mathfrak{B}'' & 0 & -2\mathfrak{C} & 0 \\
0 & -D & -2D' & +D'' & +3\mathfrak{C}'' & +\mathfrak{B}'' & 0 & -2\mathfrak{C} \\
0 & 0 & 0 & -2D' & 0 & +3\mathfrak{C}'' & 0 & 0 \\
-2D' & 0 & 0 & 0 & 0 & 0 & +3\mathfrak{B} & 0 \\
+D & -2D' & -D'' & 0 & -2\mathfrak{B}'' & 0 & +\mathfrak{C} & +3\mathfrak{B} \\
0 & +D & 0 & -D'' & 0 & -2\mathfrak{B}'' & 0 & +\mathfrak{C}
\end{vmatrix}
$$

The independent variables u and v of system (6) may be so chosen that $D = D'' = 0$, $D' = 1$, in which case of course $DD'' - 4D'^2$ does not vanish. This is equivalent to the choice of the asymptotic curves of the integral surface as parameter lines. In that case the determinant of the sixth order Δ_6 obtained from the matrix (58) by suppressing the last two columns becomes equal to $\mathfrak{B}''^3$ except for a numerical factor. It does not vanish, therefore, unless the surface is ruled. In the matrix of all of the 10×12 coefficients of the ten equations of the complete system there occurs therefore a determinant of the tenth order which may be represented as follows:

$$
(59) \qquad \Delta_{10} = \begin{vmatrix} \Delta_4 & \Delta' \\ 0 & \Delta_6 \end{vmatrix}
$$

where Δ' represents a matrix of four rows and six columns, and where 0 represents a matrix of six rows and four columns, all of whose elements are equal to zero. Obviously

$$
\Delta_{10} = \Delta_4 \cdot \Delta_6
$$

and is therefore different from zero. Consequently the ten equations of the complete system are independent; and, as announced before, there exist two new relative invariants, distinct from θ, which involve no higher derivatives than the first of the quantities $\mathfrak{B}$, $\mathfrak{C}$, D.

In order to find these new invariants, we first determine from (56) a set of auxiliary quantities whose infinitesimal transformations depend only upon the first derivatives of κ and λ. This may be done as follows. In each of the eight equations (57) there occur only three of the second derivatives of κ and λ, and these may be eliminated by combining properly with the expressions for the infinitesimal transformations of P_u, P_v, R_u, R_v, S_u, S_v. For instance we have, from (53) and (57),

$$
\delta\mathfrak{B}_u = [-2\mathfrak{B}_u\kappa_u + \mathfrak{C}_u\kappa_v + (2\mathfrak{C}''_u - \mathfrak{B}_v)\lambda_u - \mathfrak{B}\kappa_{uu} + \mathfrak{C}\kappa_{uv} + 2\mathfrak{C}''\lambda_{uu}]\,\delta t,
$$

$$
\delta P_u = [-P_u\kappa_u - R_u\kappa_v - P_v\lambda_u - 2P_u\lambda_v - R\kappa_{uv} - 2P\lambda_{uv}]\,\delta t,
$$

$$
(60)\quad
\begin{aligned}
\delta R_u &= [-2R_u\kappa_u - 2S_u\kappa_v - (2P_u + R_v)\lambda_u - R_u\lambda_v - R\kappa_{uu} - 2S\kappa_{uv} - 2P\lambda_{uu} \\
&\qquad - R\lambda_{uv}]\,\delta t, \\
\delta S_u &= [-3S_u\kappa_u - (R_u + S_v)\lambda_u - 2S\kappa_{uu} - R\lambda_{uu}]\,\delta t, \\
\delta S_v &= [-2S_v\kappa_u - S_u\kappa_v - R_v\lambda_u - S_v\lambda_v - 2S\kappa_{uv} - R\lambda_{uv}]\,\delta t.
\end{aligned}
$$

We determine the ratios of five quantities a, b, c, d, e, in such a way that

$$a\delta\mathfrak{B}_v + b\delta P_u + c\delta R_u + d\delta S_u + e\delta S_v$$

shall be free from the second derivatives of κ and λ. Since κ_{vv} and λ_{vv} do not occur in any of the expressions (60), this condition gives us just four linear homogeneous equations for the four ratios. Their solution gives

$$a = \theta^2, \qquad b = \mathfrak{C}R, \qquad c = -(\mathfrak{B}R + 4\mathfrak{C}''S),$$

$$d = 2(\mathfrak{B}P + \mathfrak{C}''R), \qquad e = \mathfrak{B}R - 2\mathfrak{C}P + 4\mathfrak{C}''S.$$

If now

$$a\mathfrak{B}_u + bP_u + cR_u + dS_u + eS_v$$

be denoted by $\binom{\mathfrak{B}}{u}$, it is certain that the expression for $\delta\binom{\mathfrak{B}}{u}$ will contain no terms which contain the second derivatives of κ or λ as factors. In precisely similar fashion, seven other auxiliary quantities are introduced.

Consequently let

$$
(61)\quad
\begin{aligned}
\binom{\mathfrak{B}}{u} &= \theta^2\mathfrak{B}_u + bP_u + cR_u + dS_u + eS_v, \\
\binom{\mathfrak{B}}{v} &= \theta^2\mathfrak{B}_v + bP_v + (c+e)R_v + dS_v - eP_u, \\
\binom{\mathfrak{B}''}{u} &= \theta^2\mathfrak{B}''_u + b''P_u + c''R_u + d''S_u + e''S_v, \\
\binom{\mathfrak{B}''}{v} &= \theta^2\mathfrak{B}''_v + b''P_v + (c''+e'')R_v + d''S_v - e''P_u, \\
\binom{\mathfrak{C}}{u} &= \theta^2\mathfrak{C}_u + mP_u + nR_u + pS_u + qS_v, \\
\binom{\mathfrak{C}}{v} &= \theta^2\mathfrak{C}_v + mP_v + (n+q)R_v + pS_v - qP_u, \\
\binom{\mathfrak{C}''}{u} &= \theta^2\mathfrak{C}''_u + m''P_u + n''R_u + p''S_u + q''S_v, \\
\binom{\mathfrak{C}''}{v} &= \theta^2\mathfrak{C}''_v + m''P_v + (n''+q'')R_v + p''S_v - q''P_u,
\end{aligned}
$$

where

$$b = \mathfrak{C}R, \qquad c = -(\mathfrak{B}R + 4\mathfrak{C}''S), \qquad d = 2(\mathfrak{B}P + \mathfrak{C}''R),$$

$$e = \mathfrak{B}R - 2\mathfrak{C}P + 4\mathfrak{C}''S, \qquad b'' = 3\mathfrak{C}''R + 4\mathfrak{B}''S, \qquad c'' = \mathfrak{B}''R,$$

$$
(62)\qquad
\begin{gathered}
d'' = -2\mathfrak{B}''P, \qquad e'' = -3(\mathfrak{B}''R + 2\mathfrak{C}''P),\\
m = -2\mathfrak{C}S, \qquad n = -2(\mathfrak{C}R + 3\mathfrak{B}S), \qquad p = 3\mathfrak{B}R + 4\mathfrak{C}P,\\
q = 3(\mathfrak{C}R + 2\mathfrak{B}S), \qquad m'' = 2(\mathfrak{B}R + \mathfrak{C}''S),\\
n'' = -2\mathfrak{B}''S, \qquad p'' = \mathfrak{B}''R, \qquad q'' = -\mathfrak{C}''R - 4\mathfrak{B}P + 2\mathfrak{B}''S.
\end{gathered}
$$

Then will

$$
(63)\qquad
\begin{aligned}
\delta\begin{pmatrix}\mathfrak{B}\\ u\end{pmatrix} &= \left[-4\begin{pmatrix}\mathfrak{B}\\ u\end{pmatrix}\kappa_u + \begin{pmatrix}\mathfrak{C}\\ u\end{pmatrix}\kappa_v + \left\{2\begin{pmatrix}\mathfrak{C}''\\ u\end{pmatrix} - \begin{pmatrix}\mathfrak{B}\\ v\end{pmatrix}\right\}\lambda_u - 2\begin{pmatrix}\mathfrak{B}\\ u\end{pmatrix}\lambda_v\right]\delta t,\\
\delta\begin{pmatrix}\mathfrak{B}''\\ u\end{pmatrix} &= \left[-2\begin{pmatrix}\mathfrak{B}''\\ u\end{pmatrix}\kappa_u + 3\begin{pmatrix}\mathfrak{C}''\\ u\end{pmatrix}\kappa_v - \begin{pmatrix}\mathfrak{B}''\\ v\end{pmatrix}\lambda_u - 4\begin{pmatrix}\mathfrak{B}''\\ u\end{pmatrix}\lambda_v\right]\delta t,\\
\delta\begin{pmatrix}\mathfrak{C}\\ u\end{pmatrix} &= \left[-5\begin{pmatrix}\mathfrak{C}\\ u\end{pmatrix}\kappa_u + \left\{3\begin{pmatrix}\mathfrak{B}\\ u\end{pmatrix} - \begin{pmatrix}\mathfrak{C}\\ v\end{pmatrix}\right\}\lambda_u - \begin{pmatrix}\mathfrak{C}\\ u\end{pmatrix}\lambda_v\right]\delta t,\\
\delta\begin{pmatrix}\mathfrak{C}''\\ u\end{pmatrix} &= \left[-3\begin{pmatrix}\mathfrak{C}''\\ u\end{pmatrix}\kappa_u + 2\begin{pmatrix}\mathfrak{B}\\ u\end{pmatrix}\kappa_v + \left\{\begin{pmatrix}\mathfrak{B}''\\ u\end{pmatrix} - \begin{pmatrix}\mathfrak{C}''\\ v\end{pmatrix}\right\}\lambda_u - 3\begin{pmatrix}\mathfrak{C}''\\ u\end{pmatrix}\lambda_v\right]\delta t,\\
\delta\begin{pmatrix}\mathfrak{B}\\ v\end{pmatrix} &= \left[-3\begin{pmatrix}\mathfrak{B}\\ v\end{pmatrix}\kappa_u + \left\{\begin{pmatrix}\mathfrak{C}\\ v\end{pmatrix} - \begin{pmatrix}\mathfrak{B}\\ u\end{pmatrix}\right\}\kappa_v + 2\begin{pmatrix}\mathfrak{C}''\\ v\end{pmatrix}\lambda_u - 3\begin{pmatrix}\mathfrak{B}\\ v\end{pmatrix}\lambda_v\right]\delta t,\\
\delta\begin{pmatrix}\mathfrak{B}''\\ v\end{pmatrix} &= \left[-\begin{pmatrix}\mathfrak{B}''\\ v\end{pmatrix}\kappa_u + \left\{3\begin{pmatrix}\mathfrak{C}''\\ v\end{pmatrix} - \begin{pmatrix}\mathfrak{B}''\\ u\end{pmatrix}\right\}\kappa_v - 5\begin{pmatrix}\mathfrak{B}''\\ v\end{pmatrix}\lambda_v\right]\delta t,\\
\delta\begin{pmatrix}\mathfrak{C}\\ v\end{pmatrix} &= \left[-4\begin{pmatrix}\mathfrak{C}\\ v\end{pmatrix}\kappa_u - \begin{pmatrix}\mathfrak{C}\\ u\end{pmatrix}\kappa_v + 3\begin{pmatrix}\mathfrak{B}\\ v\end{pmatrix}\lambda_u - 2\begin{pmatrix}\mathfrak{C}\\ v\end{pmatrix}\lambda_v\right]\delta t,\\
\delta\begin{pmatrix}\mathfrak{C}''\\ v\end{pmatrix} &= \left[-2\begin{pmatrix}\mathfrak{C}''\\ v\end{pmatrix}\kappa_u + \left\{2\begin{pmatrix}\mathfrak{B}\\ v\end{pmatrix} - \begin{pmatrix}\mathfrak{C}''\\ u\end{pmatrix}\right\}\kappa_v + \begin{pmatrix}\mathfrak{B}''\\ v\end{pmatrix}\lambda_u - 4\begin{pmatrix}\mathfrak{C}''\\ v\end{pmatrix}\lambda_v\right]\delta t.
\end{aligned}
$$

We denote by Φ and Ψ the following functions;

$$
(64)\qquad
\begin{aligned}
\Phi &= \left[\begin{pmatrix}\mathfrak{C}''\\ u\end{pmatrix} + \begin{pmatrix}\mathfrak{B}\\ v\end{pmatrix}\right]^2 - \left[\begin{pmatrix}\mathfrak{B}\\ u\end{pmatrix} + \begin{pmatrix}\mathfrak{C}\\ v\end{pmatrix}\right]\left[\begin{pmatrix}\mathfrak{B}''\\ u\end{pmatrix} + \begin{pmatrix}\mathfrak{C}''\\ v\end{pmatrix}\right],\\
\Psi &= -2\left[\begin{pmatrix}\mathfrak{C}''\\ u\end{pmatrix} - \begin{pmatrix}\mathfrak{B}\\ v\end{pmatrix}\right]^3 + \begin{pmatrix}\mathfrak{B}''\\ v\end{pmatrix}\left[3\begin{pmatrix}\mathfrak{B}\\ u\end{pmatrix} - \begin{pmatrix}\mathfrak{C}\\ v\end{pmatrix}\right]^2 - \begin{pmatrix}\mathfrak{C}\\ u\end{pmatrix}\left[\begin{pmatrix}\mathfrak{B}''\\ u\end{pmatrix} - 3\begin{pmatrix}\mathfrak{C}''\\ v\end{pmatrix}\right]^2\\
&\quad + \left[\begin{pmatrix}\mathfrak{C}''\\ u\end{pmatrix} - \begin{pmatrix}\mathfrak{B}\\ v\end{pmatrix}\right]\left[3\begin{pmatrix}\mathfrak{B}\\ u\end{pmatrix} - \begin{pmatrix}\mathfrak{C}\\ v\end{pmatrix}\right]\left[\begin{pmatrix}\mathfrak{B}''\\ u\end{pmatrix} - 3\begin{pmatrix}\mathfrak{C}''\\ v\end{pmatrix}\right]\\
&\quad - 8\begin{pmatrix}\mathfrak{C}\\ u\end{pmatrix}\begin{pmatrix}\mathfrak{B}''\\ v\end{pmatrix}\left[\begin{pmatrix}\mathfrak{C}''\\ u\end{pmatrix} - \begin{pmatrix}\mathfrak{B}\\ v\end{pmatrix}\right].
\end{aligned}
$$

These quantities, Φ and Ψ, are invariants. In fact

$$
(65)\qquad \delta\Phi = -6(\kappa_u + \lambda_v)\Phi\delta t, \qquad \delta\Psi = -9(\kappa_u + \lambda_v)\Psi\delta t,
$$

so that

$$\frac{\Phi}{\theta^6} \quad \textit{and} \quad \frac{\Psi}{\theta^9}$$

are absolute invariants. These are the two whose existence was demonstrated above.

The three invariants θ, Φ and Ψ are independent of the seminvariants A, A', A''. We shall now proceed to calculate invariants involving also these variables.

We again begin by forming from A, A', A'' a set of three quantities $\mathfrak{A}$, $\mathfrak{A}'$, $\mathfrak{A}''$, such that $\delta\mathfrak{A}$, $\delta\mathfrak{A}'$, $\delta\mathfrak{A}''$ shall contain only the first derivatives of κ and λ. Put

$$(66) \qquad E = \frac{\theta_{uu}}{\theta} - \frac{3}{2}\frac{\theta_u^2}{\theta^2}, \qquad F = \frac{\theta_{uv}}{\theta} - \frac{3}{2}\frac{\theta_u\theta_v}{\theta^2}, \qquad G = \frac{\theta_{vv}}{\theta} - \frac{3}{2}\frac{\theta_v^2}{\theta^2},$$

and let

$$(67) \qquad \begin{aligned} \mathfrak{A} &= A - \frac{1}{2}\left(B\frac{\theta_u}{\theta} + C\frac{\theta_v}{\theta}\right) + \tfrac{1}{2}TE - \tfrac{1}{2}DD'F + \tfrac{1}{4}D^2G, \\ \mathfrak{A}' &= A' - \frac{1}{2}\left(B'\frac{\theta_u}{\theta} + C'\frac{\theta_v}{\theta}\right) + \tfrac{1}{4}D'D''E - \tfrac{1}{2}DD''F + \tfrac{1}{4}DD'G, \\ \mathfrak{A}'' &= A'' - \frac{1}{2}\left(B''\frac{\theta_u}{\theta} + C''\frac{\theta_v}{\theta}\right) + \tfrac{1}{4}D''^2E - \tfrac{1}{2}D'D''F + \tfrac{1}{2}TG, \end{aligned}$$

where, as before

$$T = D'^2 - \tfrac{1}{2}DD''.$$

Then

$$(68) \qquad \begin{aligned} \delta\mathfrak{A} &= [-2\mathfrak{A}\kappa_u - 2\mathfrak{A}'\lambda_u]\,\delta t, \\ \delta\mathfrak{A}' &= [-\mathfrak{A}'\kappa_u - \mathfrak{A}\kappa_v - \mathfrak{A}''\lambda_u - \mathfrak{A}'\lambda_v]\,\delta t, \\ \delta\mathfrak{A}'' &= [-2\mathfrak{A}'\kappa_v - 2\mathfrak{A}''\lambda_v]\,\delta t. \end{aligned}$$

Notice that $\mathfrak{A}$, $\mathfrak{A}'$, $\mathfrak{A}''$ satisfy the same relation

$$(69) \qquad D''\mathfrak{A} - 2D'\mathfrak{A}' + D\mathfrak{A}'' = 0,$$

which is also satisfied by $\mathfrak{B}$, $\mathfrak{B}'$, $\mathfrak{B}''$ and $\mathfrak{C}$, $\mathfrak{C}'$, $\mathfrak{C}''$.

Equations (68) show that

$$(70) \qquad \Sigma = \mathfrak{A}'^2 - \mathfrak{A}\mathfrak{A}''$$

is an invariant for which

$$\delta\Sigma = -2(\kappa_u + \lambda_v)\Sigma\delta t.$$

Another invariant may be constructed as follows: Let

$$(71) \qquad (\mathfrak{A}, \mathfrak{B}) = \mathfrak{A}\mathfrak{B}'' - 2\mathfrak{A}'\mathfrak{B}' + \mathfrak{A}''\mathfrak{B}, \qquad (\mathfrak{A}, \mathfrak{C}) = \mathfrak{A}\mathfrak{C}'' - 2\mathfrak{A}'\mathfrak{C}' + \mathfrak{A}''\mathfrak{C}.$$

Then

$$(72) \qquad \begin{aligned} \delta(\mathfrak{A}, \mathfrak{B}) &= [-(\mathfrak{A}, \mathfrak{B})\kappa_u + (\mathfrak{A}, \mathfrak{C})\kappa_v - 2(\mathfrak{A}, \mathfrak{B})\lambda_v]\,\delta t, \\ \delta(\mathfrak{A}, \mathfrak{C}) &= [-2(\mathfrak{A}, \mathfrak{C})\kappa_u + (\mathfrak{A}, \mathfrak{B})\lambda_u - (\mathfrak{A}, \mathfrak{C})\lambda_v]\,\delta t. \end{aligned}$$

Consequently, if we put

$$(73)\qquad \mathfrak{A}(\mathfrak{A},\mathfrak{B})+\mathfrak{A}'(\mathfrak{A},\mathfrak{C})=\mathfrak{F},\qquad \mathfrak{A}'(\mathfrak{A},\mathfrak{B})+\mathfrak{A}''(\mathfrak{A},\mathfrak{C})=\mathfrak{G},$$

we shall find

$$(74)\quad \delta\mathfrak{F}=[-3\mathfrak{F}\kappa_u-\mathfrak{G}\lambda_u-2\mathfrak{F}\lambda_v]\,\delta t,\quad \delta\mathfrak{G}=[-2\mathfrak{G}\kappa_u-\mathfrak{F}\kappa_v-3\mathfrak{G}\lambda_v]\,\delta t,$$

so that

$$(75)\quad \Omega=(\mathfrak{A},\mathfrak{B})\mathfrak{F}+(\mathfrak{A},\mathfrak{C})\mathfrak{G}=\mathfrak{A}(\mathfrak{A},\mathfrak{B})^2+2\mathfrak{A}'(\mathfrak{A},\mathfrak{B})(\mathfrak{A},\mathfrak{C})+\mathfrak{A}''(\mathfrak{A},\mathfrak{C})^2$$

is another invariant (*of weight four*), *since*

$$(76)\qquad \delta\Omega=-4(\kappa_u+\lambda_v)\,\Omega\delta t.$$

From the invariants already obtained, others may be derived by differentiation processes which are very closely related to the differential parameters of the metrical theory of surfaces.

In fact, *let I and J be any two absolute invariants. Then*

$$\Delta_1(I)=\frac{1}{\theta}[D''I_u^2-2D'I_uI_v+DI_v^2],$$

$$\Delta_1(J)=\frac{1}{\theta}[D''J_u^2-2D'J_uJ_v+DJ_v^2],$$

$$(77)\qquad \Delta(I,J)=\frac{1}{\theta}[D''I_uJ_u-D'(I_uJ_v+I_vJ_u)+DI_vJ_v],$$

$$\Delta_2(I)=\frac{1}{\theta}\left[\frac{\partial}{\partial u}(D''I_u-D'I_v)+\frac{\partial}{\partial v}(DI_v-D'I_u)\right],$$

$$\Delta_2(J)=\frac{1}{\theta}\left[\frac{\partial}{\partial u}(D''J_u-D'J_v)+\frac{\partial}{\partial v}(DJ_v-D'J_u)\right]$$

are again absolute invariants, as is also the Jacobian

$$(78)\qquad \Gamma(I,J)=\frac{1}{\theta}(I_uJ_v-I_vJ_u).$$

It is easy to verify that $\mathfrak{A}/\theta$, $\mathfrak{A}'/\theta$, $\mathfrak{A}''/\theta$ are cogredient with D, D', D''. Consequently *the quantities*

$$\Lambda_1(I)=\frac{1}{\theta^2}[\mathfrak{A}''I_u^2-2\mathfrak{A}'I_uI_v+\mathfrak{A}I_v^2],$$

$$\Lambda_1(J)=\frac{1}{\theta^2}[\mathfrak{A}''J_u^2-2\mathfrak{A}'J_uJ_v+\mathfrak{A}J_v^2],$$

$$(79)\qquad \Lambda(I,J)=\frac{1}{\theta^2}[\mathfrak{A}''I_uJ_u-\mathfrak{A}'(I_uJ_v+I_vJ_u)+\mathfrak{A}I_vJ_v],$$

$$\Lambda_2(I) = \frac{1}{\theta}\left[\frac{\partial}{\partial u}\left(\frac{\mathfrak{A}''I_u - \mathfrak{A}'I_v}{\theta}\right) + \frac{\partial}{\partial v}\left(\frac{\mathfrak{A}I_v - \mathfrak{A}'I_u}{\theta}\right)\right],$$

$$\Lambda_2(J) = \frac{1}{\theta}\left[\frac{\partial}{\partial u}\left(\frac{\mathfrak{A}''J_u - \mathfrak{A}'J_v}{\theta}\right) + \frac{\partial}{\partial v}\left(\frac{\mathfrak{A}J_v - \mathfrak{A}'J_u}{\theta}\right)\right]$$

are also absolute invariants.

Each of these differentiation processes may be repeated, or combined with each of the others so as to produce a large number of invariants. But not all of the invariants obtained in this way are functionally independent. In order to obtain the relations between them, it is convenient to write down the simplified form which they assume if system (6) is assumed to be in its canonical form, for which

$$D = 0 \qquad D' = 1 \qquad D'' = 0, \tag{80}$$

so that

$$\mathfrak{A}' = 0, \qquad \mathfrak{B} = \mathfrak{B}' = 0, \qquad \mathfrak{C}' = \mathfrak{C}'' = 0, \tag{81}$$

and consequently further

$$\begin{gathered} A' = 0, \qquad B = B' = 0, \qquad C' = C'' = 0, \\ \mathfrak{B}'' = B'', \qquad \mathfrak{C} = C, \qquad \theta = -B''C, \\ \mathfrak{A} = A - \tfrac{1}{2}C\frac{\theta_v}{\theta} + \frac{1}{2}\left(\frac{\theta_{uu}}{\theta} - \frac{3}{2}\frac{\theta_u^2}{\theta}\right), \\ \mathfrak{A}'' = A'' - \tfrac{1}{2}B''\frac{\theta_u}{\theta} + \frac{1}{2}\left(\frac{\theta_{vv}}{\theta} - \frac{3}{2}\frac{\theta_v^2}{\theta^2}\right). \end{gathered} \tag{82}$$

The above derived differential invariants of the first order become

$$\Delta_1(I) = -\frac{2}{\theta}I_uI_v, \quad \Delta_1(J) = -\frac{2}{\theta}J_uJ_v, \quad \Delta(I, J) = -\frac{1}{\theta}(I_uJ_v + I_vJ_u),$$

$$\Lambda_1(I) = \frac{1}{\theta^2}(\mathfrak{A}''I_u^2 + \mathfrak{A}I_v^2), \qquad \Lambda_1(J) = \frac{1}{\theta^2}(\mathfrak{A}''J_u^2 + \mathfrak{A}J_v^2). \tag{83}$$

$$\Lambda(I, J) = \frac{1}{\theta^2}(\mathfrak{A}''I_uJ_u + \mathfrak{A}I_vJ_v), \qquad \Gamma(I, J) = \frac{1}{\theta}(I_uJ_v - I_vJ_u),$$

while those of the second order reduce to

$$\begin{gathered} \Delta_2(I) = -\frac{2}{\theta}I_{uv}, \qquad \Delta_2(J) = -\frac{2}{\theta}J_{uv}, \\ \Lambda_2(I) = \frac{1}{\theta}\left[\frac{\partial}{\partial u}\left(\frac{\mathfrak{A}''I_u}{\theta}\right) + \frac{\partial}{\partial v}\left(\frac{\mathfrak{A}I_v}{\theta}\right)\right], \\ \Lambda_2(J) = \frac{1}{\theta}\left[\frac{\partial}{\partial u}\left(\frac{\mathfrak{A}''J_u}{\theta}\right) + \frac{\partial}{\partial v}\left(\frac{\mathfrak{A}J_v}{\theta}\right)\right]. \end{gathered} \tag{84}$$

From (83) *we see at once that*

$$\Delta(I, J)^2 - \Gamma(I, J)^2 - \Delta_1(I)\Delta_1(J) = 0. \tag{85}$$

Further we find

$$\begin{aligned} -\Delta(I, J)^2\Gamma(I, J)^2\mathfrak{A} &= -J_u^2\Lambda_1(I) + I_u^2\Lambda_1(J), \\ -\Delta(I, J)^2\Gamma(I, J)^2\mathfrak{A}'' &= +J_v^2\Lambda_1(I) - I_v^2\Lambda_1(J), \end{aligned} \tag{86}$$

which gives, upon substitution into the expression for $\Lambda(I, J)$, *the further relation*

$$2\Delta(I, J)\Lambda(I, J) - \Delta_1(I)\Lambda_1(J) - \Delta_1(J)\Lambda_1(I) = 0. \tag{87}$$

We shall not, in the present paper, investigate the relations between the operators of the second order and those obtained by repetition of the operators of the first order; nor shall we inquire into the question whether all invariants of the given system of differential equations can be obtained in this way. Let us confine our attention to those invariants which contain no derivatives of higher than the second order of the quantities $\mathfrak{B}$, $\mathfrak{B}'$, $\mathfrak{B}''$, $\mathfrak{C}$, $\mathfrak{C}'$, $\mathfrak{C}''$, D, D', D'', and which involve the quantities $\mathfrak{A}$, $\mathfrak{A}'$, $\mathfrak{A}''$ themselves but none of their derivatives. We now proceed to show that we are already in possession of a functionally complete set of invariants of this kind.

The invariants contemplated depend upon 26 independent arguments, viz.: four independent variables among the nine quantities $\mathfrak{B}, \cdots, D''$, their eight first, their twelve second derivatives, and two independent variables among the three $\mathfrak{A}$'s.

The infinitesimal transformations of these 26 variables involve terms containing the first, second, and third derivatives of κ and λ as factors. The absolute invariants of the class considered must therefore satisfy a complete system of 18 equations and 26 independent variables. If these equations are all independent, *there will be* $26 - 18 = 8$ *such invariants. They may be chosen as follows:*

$$\begin{gathered} \frac{\Phi}{\theta^6}, \quad \frac{\Psi}{\theta^9}, \\ \Delta_1\left(\frac{\Phi}{\theta^6}\right), \quad \Delta_1\left(\frac{\Psi}{\theta^9}\right), \quad \Delta\left(\frac{\Phi}{\theta^6}, \frac{\Psi}{\theta^9}\right), \\ \Lambda_1\left(\frac{\Phi}{\theta^6}\right), \quad \Lambda_1\left(\frac{\Psi}{\theta^9}\right), \quad \frac{\Omega}{\theta^4}. \end{gathered} \tag{88}$$

It remains to prove that the eighteen equations just mentioned are independent, and that the same is true of the eight invariants (88).

In the matrix formed by the coefficients of the first ten equations of our complete system, there occurs the determinant Δ_{10} of equation (59), which is known to be different from zero. The last eight equations, obtained by equating to

zero the coefficients of the third derivatives of κ and λ in the general expression for δf, contain only the twelve independent second derivatives of the $\mathfrak{B}$'s, $\mathfrak{C}$'s, and D's. If we assume again for the moment that $DD'' - 4D'^2$ does not vanish, we may think specifically of D, D'', $\mathfrak{B}''$ and $\mathfrak{C}$ together with their derivatives as the independent variables. The matrix of the 12×8 coefficients of the last eight equations will then be as follows:

$$\begin{matrix}
-D & 0 & 0 & +D'' & 0 & 0 & +\mathfrak{B}'' & 0 & 0 & -2\mathfrak{C} & 0 & 0 \\
0 & -D & 0 & -2D' & +D'' & 0 & +3\mathfrak{C}'' & +\mathfrak{B}'' & 0 & 0 & -2\mathfrak{C} & 0 \\
0 & 0 & -D & 0 & -2D' & +D'' & 0 & +3\mathfrak{C}'' & +\mathfrak{B}'' & 0 & 0 & -2\mathfrak{C} \\
0 & 0 & 0 & 0 & 0 & -2D' & 0 & 0 & +3\mathfrak{C}'' & 0 & 0 & 0 \\
-2D' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & +3\mathfrak{B} & 0 & 0 \\
+D & -2D' & 0 & -D'' & 0 & 0 & -2\mathfrak{B}'' & 0 & 0 & +\mathfrak{C} & +3\mathfrak{B} & 0 \\
0 & +D & -2D' & 0 & -D'' & 0 & 0 & -2\mathfrak{B}'' & 0 & 0 & +\mathfrak{C} & +3\mathfrak{B} \\
0 & 0 & +D & 0 & 0 & -D'' & 0 & 0 & -2\mathfrak{B}'' & 0 & 0 & +\mathfrak{C}
\end{matrix}$$

If the system (6) is in its canonical form, so that equations (80), (81) and (82) are satisfied, the determinant of the eighth order, obtained from the above matrix by omitting the eighth, tenth, eleventh, and twelfth columns, becomes equal to B''^2 multiplied by a power of 2. This determinant Δ_8 is therefore not identically equal to zero.

If we now consider the whole system of eighteen equations there will be in the matrix of its coefficients a determinant of order eighteen of the form

$$\Delta_{18} = \begin{vmatrix} \Delta_{10} & \Delta'' \\ 0 & \Delta_8 \end{vmatrix} = \Delta_{10} \cdot \Delta_8, \tag{89}$$

where Δ'' denotes a rectangular matrix of ten rows and eight columns, while 0 denotes a rectangular matrix of eight rows and ten columns all of whose elements are equal to zero. This determinant Δ_{18} does not vanish identically; consequently the eighteen equations of the complete system are independent.

The eight invariants (88) are obviously members of the class here considered. It only remains to show that they are independent.

In order to do this we again make use of the canonical form. According to (80), (81), (82), the canonical forms of the eight invariants are as follows:

$$\begin{aligned}
\frac{\Phi}{\theta^6} &= \frac{(2CB''_u + B''C_u)(2B''C_v + CB''_v)}{B''^3 C^3}, \\
\frac{\Psi}{\theta^9} &= -\frac{2CB''_u + B''C_u}{B''^4 C^5} + \frac{2B''C_v + CB''_v}{B''^5 C^4},
\end{aligned} \tag{90}$$

$$\Delta_1\left(\frac{\Phi}{\theta^6}\right)=\frac{2}{B''C}\frac{\partial}{\partial u}\left(\frac{\Phi}{\theta^6}\right)\frac{\partial}{\partial v}\left(\frac{\Phi}{\theta^6}\right),$$

$$\Delta_1\left(\frac{\Psi}{\theta^9}\right)=\frac{2}{B''C}\frac{\partial}{\partial u}\left(\frac{\Psi}{\theta^9}\right)\frac{\partial}{\partial v}\left(\frac{\Psi}{\theta^9}\right),$$

$$(90)\qquad \Delta\left(\frac{\Phi}{\theta^6},\frac{\Psi}{\theta^9}\right)=\frac{1}{B''C}\left[\frac{\partial}{\partial u}\left(\frac{\Phi}{\theta^6}\right)\frac{\partial}{\partial v}\left(\frac{\Psi}{\theta^9}\right)+\frac{\partial}{\partial v}\left(\frac{\Phi}{\theta^6}\right)\frac{\partial}{\partial u}\left(\frac{\Psi}{\theta^9}\right)\right],$$

$$\Lambda_1\left(\frac{\Phi}{\theta^6}\right)=\frac{1}{(B''C)^2}\left[\mathfrak{A}''\left\{\frac{\partial}{\partial u}\left(\frac{\Phi}{\theta^6}\right)\right\}^2+\mathfrak{A}\left\{\frac{\partial}{\partial v}\left(\frac{\Phi}{\theta^6}\right)\right\}^2\right],$$

$$\Lambda_1\left(\frac{\Psi}{\theta^9}\right)=\frac{1}{(B''C)^2}\left[\mathfrak{A}''\left\{\frac{\partial}{\partial u}\left(\frac{\Psi}{\theta^9}\right)\right\}^2+\mathfrak{A}\left\{\frac{\partial}{\partial v}\left(\frac{\Psi}{\theta^9}\right)\right\}^2\right],$$

$$\frac{\Omega}{\theta^4}=\frac{\mathfrak{A}^3B''^2+\mathfrak{A}''^3C^2}{(B''C)^4}.$$

In the canonical form, our system of differential equations (6) becomes

$$(91)\qquad \begin{aligned} y_{uu}&=Ay+Cy_v\,,\\ y_{vv}&=A''y+B''y_u,\end{aligned}$$

If we compare this with the system as written in the FIRST and SECOND MEMOIRS, viz.: *

$$(92)\qquad \begin{aligned} y_{uu}+2by_v+fy&=0,\\ y_{vv}+2a'y_u+gy&=0,\end{aligned}$$

we see that the quantities A, A'', B'', C correspond to $-f$, $-g$, $-2a'$, $-2b$ respectively. Let us remember further that in the SECOND MEMOIR † we introduced two absolute invariants, I and J, defined by the equations:

$$(93)\qquad I=\frac{a'b_u+2a'_ub}{4a'b\sqrt[3]{a'b^2}},\qquad J=\frac{a'_vb+2a'b_v}{4a'b\sqrt[3]{a'^2b}}.$$

We thus find

$$(94)\qquad \frac{\Phi}{4\theta^6}=IJ,\qquad \frac{\Psi}{8\theta^9}=J^3-I^3.$$

Thus the first two invariants of (90) are functions of I and J alone. The next three are independent combinations of the first derivatives of I and J, containing besides the single combination $B''C$ of B'' and C. The following two invariants

$$\Lambda_1\left(\frac{\Phi}{\theta^6}\right)\qquad\text{and}\qquad\Lambda_1\left(\frac{\Psi}{\theta^9}\right)$$

* These Transactions, vol. 8 (1907), p. 246.

† These Transactions, vol. 9 (1908), p. 103.

depend upon the same quantities, but contain the two further variables $\mathfrak{A}$ and $\mathfrak{A}''$ in two independent combinations. Finally Ω/θ^4 is independent of all the rest because it contains B'' and C in a combination independent of $B''C$. This completes the proof that the eight functions (89) form a functionally complete system of invariants of the order considered.

We have found, moreover, the relations (94) between the rational invariants θ, Φ, Ψ and the irrational invariants I and J which present themselves in the canonical development

$$z = xy + \tfrac{1}{6}(x^3 + y^3) + \tfrac{1}{24}(Ix^4 + Jy^4) + \cdots$$

of a surface in the vicinity of an ordinary point.

The relation of the other invariants obtained in this paper to those considered in the previous memoirs, the question in regard to the completeness of our system of invariants and differential parameters, the determination of the covariants and the introduction of the adjoined system in its general form must be left for a later occasion.

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