ON THE OSCULATING QUARTIC OF A PLANE CURVE*

BY

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The quartic curve which has contact of the thirteenth order with a given analytic plane curve at one of its points will be called the osculating quartic. In this paper, its equation will be found in an invariant form, referred to a triangle which has a simple projective relation to the given curve. The method which we shall employ is due to Professor WILCZYNSKI.†

Let there be given three linearly independent analytic functions of \( x \),

(1) \[ y_\kappa = f_\kappa(x), \quad (\kappa = 1, 2, 3). \]

They may be interpreted as the homogeneous coordinates of a point \( P_\nu \) in a plane. As \( x \) changes, \( P_\nu \) describes an analytic plane curve \( C_\nu \). There exists a uniquely defined linear differential equation of the third order of which \( y_1, y_2, y_3 \) form a fundamental system. Let this be

(2) \[ y^{(3)} + p_1 y'' + p_2 y' + p_3 y = 0, \]

where \( p_1, p_2, p_3 \) are analytic functions of \( x \). We may, therefore, speak of \( C_\nu \) as being an integral curve of (2). But this integral curve is not unique, for every projective transformation of \( C_\nu \) is likewise an integral curve of (2). Conversely, every integral curve of (2) is a projective transformation of \( C_\nu \). Hence the properties of \( C_\nu \) determined by the coefficients (2) are common to all curves projectively equivalent to \( C_\nu \), that is, they are projective properties.

The representation of a curve in the form (1), however, involves some arbitrary elements. In the first place, since the coordinates are homogeneous, a transformation of the form

(3) \[ \eta = \lambda(x) y, \]

where \( \lambda(x) \) is an arbitrary analytic function of \( x \), does not affect the curve. Moreover, a transformation of the form

(4) \[ \xi = \xi(x), \]

where \( \xi(x) \) is an arbitrary analytic function of \( x \), merely changes the par-
metric representation, without changing the curve. Those combinations of the coefficients of (2) and of their derivatives which remain unchanged under all transformations of the form (3) and (4), the so-called invariants, therefore characterize the projective properties of the curve independently of its method of representation. If such an invariant function contains also \( y \) and its derivatives it is called a covariant.

It is convenient to consider the subgroup of the general group of transformations defined by equations (3) and (4), which is obtained by leaving the independent variable \( x \) fixed. The invariant functions for this subgroup are known as seminvariants and semicovariants. Of these we shall need

\begin{align*}
(5) &
\quad z = y' + p_1 y,
\quad \rho = y'' + 2p_1 y' + p_2 y,
\end{align*}

\begin{align*}
(6) &
\quad P_2 = p_2 - p_1^2 + p_1', \\
\quad P_5 = p_5 - 3p_1 p_2 + 2p_1^3 - p_1'..
\end{align*}

as well as the following invariants:

\begin{align*}
(7) &
\quad \theta_3 = P_3 - \frac{3}{5} P_2', \\
\quad \theta_8 = 6\theta_3 \theta_3' - 7 (\theta_3')^2 - 27 P_2 \theta_3^2.
\end{align*}

By a special transformation of the form (3) and (4), viz.,

\begin{align*}
(8) &
\quad y = T x y', \\
\quad \bar{x} = \xi(x),
\end{align*}

where \( \xi(x) \) is determined by the equations:

\begin{align*}
(9) &
\quad \xi(x) = c_1 \int e^{\int y dx} dx + c_2, \\
\quad \eta' - \frac{1}{2} \eta^2 = \frac{3}{5} P_2,
\end{align*}

in which \( c_1 \) and \( c_2 \) are arbitrary constants, equation (2) may be reduced to the Laguerre-Forsyth canonical form:

\begin{align*}
(10) &
\quad \frac{d^3 y}{dx^3} + \bar{P}_2 \bar{y} = 0.
\end{align*}

The invariant functions above mentioned have then the following exceptionally simple canonical forms:

\begin{align*}
(11) &
\quad \bar{z} = y', \\
\quad \bar{\rho} = y''', \\
\quad \bar{P}_2 = 0,
\end{align*}

\begin{align*}
(12) &
\quad \bar{\theta}_3 = \bar{P}_3, \\
\quad \bar{\theta}_8 = 6 \bar{P}_3 \bar{P}_2'' - 7 (\bar{P}_3')^2.
\end{align*}

Hereafter, we shall assume the differential equation in this form, and, for convenience in writing, omit the dashes. We shall assume also that \( P_3 \), in this equation, is not identically equal to zero. This requires only that \( C \) be not a conic.

* * Proj. Diff. Geom., p. 58.
† Ibid., p. 59.
‡ Ibid., p. 25.
Let \( \alpha \) be the value of \( x \) which determines the point \( P_{\alpha} \), on the curve \( C \). The point \( P_{\alpha} \), whose coordinates are

\[
z_{\kappa} = \left[ \frac{dy_{x}}{dx} \right]_{x=\alpha}, \quad (\kappa = 1, 2, 3),
\]

is a point on the tangent to \( C \) at \( P_{\alpha} \); and, if \( P_{\alpha} \) is not a point of inflection, the point \( P_{\alpha} \), whose coordinates are

\[
\rho_{\kappa} = \left[ \frac{d^{2}y_{x}}{dx^{2}} \right]_{x=\alpha}, \quad (\kappa = 1, 2, 3),
\]

is not collinear with \( P_{\alpha} \) and \( P_{\alpha} \); so that these points determine a non-degenerate triangle, semicovariantly related to the curve \( C \). Let this be taken as a triangle of reference. We may choose the unit point of our system of homogeneous coordinates so that an expression of the form \( \uparrow \)

\[(13) \quad x_{1}y(\alpha_{0}) + x_{2}x(\alpha_{0}) + x_{3}\rho(\alpha_{0})
\]

will represent the point whose coordinates are precisely \( x_{1}, x_{2}, x_{3} \).

Let \( x = \alpha \) be an ordinary point for the function \( P_{\alpha} \); and, for convenience in writing, let \( \alpha = 0 \), since this assumption involves no loss of generality. Then for values of \(|x|\) sufficiently small, any solution of equation (10) may be expressed as a convergent power series,

\[(14) \quad G(x) = y(0) + y'(0)x + \frac{1}{2} y''(0)x^{2} + \cdots + \frac{1}{14!} y^{(14)}(0)x^{14} + \cdots.
\]

From equation (10), we find by direct differentiation:

\[
y^{(3)} = -ay, \quad y^{(4)} = -a_{1}y - az,
\]

\[
y^{(5)} = -a_{2}y - 2a_{1}z - \alpha_{1},
\]

\[
y^{(6)} = -(a_{3} - a^{2})y - 3a_{2}z - 3a_{1}\rho, etc.,
\]

where we have put \( y = y(0), z = y'(0), \rho = y''(0), \alpha = P_{\alpha}, \) and where \( a_{n} \) is an abbreviation for the \( n \)th power of the \( n \)th derivative of \( P_{\alpha} \) with respect to \( x \). Putting \( G(x) \) in the form (13), we find the following equations, which represent the curve \( C \) up to terms of the fourteenth order in the vicinity of the point \( P_{\alpha} \):

\[
y_{1} = 1 - \frac{a_{s}}{5!} x^{s} - \frac{a_{1}}{4!} x^{s} - \frac{a_{2}}{5!} x^{s} - \frac{1}{6!} (a_{3} - a^{2}) x^{s} - \frac{1}{7!} (a_{4} - 5aa_{1}) x^{7} - \frac{1}{8!} (a_{5} - 11aa_{2} - 5\alpha_{1}) x^{8} - \frac{1}{9!} (a_{6} - 21aa_{3} - 21a_{1}a_{2} + \alpha^{2}) x^{9}
\]

\( \uparrow \) Prog. Diff. Geom., p. 54.

\( \uparrow \) Prog. Diff. Geom., p. 61.
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\[ \begin{align*}
- \frac{1}{10!} (a_7 - 36a_4 - 42a_1 a_3 - 21a_1^2 + 12a_1 a_2) x^{10} \\
- \frac{1}{11!} (a_8 - 57a_3 - 84a_2 a_3 + 39a_2 a^2 + 45aa^2) x^{11} \\
- \frac{1}{12!} (a_9 - 85a_5 - 135a_0 a_3 - 182a_2 a_4 + 105a_2 a^2 - 84a_3^2 \\
+ 300aa_1 a_2 + 45a_1^2 - a_1) x^{12} \\
- \frac{1}{13!} (a_{10} - 220a_1 a_5 - 121a_2 a_7 - 30a_3 a_5 + 852aa_1 a_5 \\
+ 246a_2 a^4 + 435a_1^2 a_2 + 516aa_2 a^2 - 22a_3 a_1) x^{13} \\
- \frac{1}{14!} (a_{11} - 166aa_8 - 341a_1 a_7 - 517a_2 a_8 - 627a_3 a_5 + 519a_2 a_5 \\
- 330a_1^2 + 2124aa_1 a_4 + 1287a_1^2 a_2 + 3054aa_2 a_3 \\
+ 1386a_1 a_2^2 - 94a_3 a_2 - 177a_2 a_2) x^{14} \\
- \cdots,
\end{align*} \]

\[ y_2 = x - \frac{a}{4!} x^4 - \frac{2a}{5!} x^5 - \frac{3a}{6!} x^6 - \frac{1}{7!} (4a_2 - a^2) x^7 - \frac{1}{8!} (5a_4 - 7aa_1) x^8 \]

\[ - \frac{1}{9!} (6a_3 - 18aa_2 - 12a_1^2) x^9 - \frac{1}{10!} (7a_0 + a^3 - 39a_3 a - 63a_1 a_2) x^{10} \]

\[ - \frac{1}{11!} (8a_5 - 75aa_4 - 144a_1 a_3 - 84a_2 a^2 + 15a_2 a_1) x^{11} \]

\[ (16) \]

\[ - \frac{1}{12!} (9a_6 - 182aa_5 - 297a_1 a_4 - 396a_2 a_4 + 54a^2 a_2 + 75aa_1 a_2) x^{12} \]

\[ - \frac{1}{13!} (10a_7 - 217aa_6 - 564a_2 a_5 - 855a_2 a_4 + 159a_2 a_5 - 480a_2^2 \\
+ 558aa_1 a_2 + 120a_1^2 - a_1) x^{13} \]

\[ - \frac{1}{14!} (11a_8 - 388a_1 a_7 - 1001a_1 a_6 - 1716a_2 a_5 - 2145a_2 a_4 + 405a_2 a_4 \\
+ 1728aa_1 a_3 + 1353a_1 a_2 + 1074aa_2 a_3 - 26a_3 a_1) x^{14} \]

\[ - \cdots, \]

\[ y_3 = \frac{1}{2} x^2 - \frac{a}{5!} x^5 - \frac{3a}{6!} x^6 - \frac{v a^2}{7!} x^7 - \frac{1}{8!} (10a_3 - a_0^2) x^8 \]

\[ - \frac{1}{9!} (15a_4 - 9aa_3) x^9 - \frac{1}{10!} (21a_5 - 27aa_2 - 21a_1^2) x^{10} \]

\[ - \frac{1}{11!} (28a_6 - 66aa_5 - 132a_1 a_2 + a_3) x^{11} \]

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Let \( Q(x_1, x_2, x_3) = 0 \) be the equation of the osculating quartic. If we should substitute into its left member \( x_k = y_k \) \((k = 1, 2, 3)\), the coefficients of all powers of \( x \) up to and including the thirteenth would be equal to zero, and we should have fourteen equations for the ratios of the fifteen coefficients of \( Q \). The equation \( Q = 0 \) has been obtained by the author in this way. The details of that solution, however, will not be given here, as the equation of the osculating quartic may be obtained more easily and in a more satisfactory form by another method. We wish, however, to emphasize the fact that the equations obtained by the two different methods have been compared, for the sake of checking the results presented here, and have been found to agree.

By combining \( y_1, y_2, y_3 \) so as to eliminate the terms in \( x \) up to the fourth and eighth orders, inclusive, Professor Wilczynski has obtained as the equations of the osculating conic and cubic respectively:

\[
\begin{align*}
(17) & \quad x_2^2 - 2x_1x_3 = 0, \\
(18) & \quad 7(15P_3P_3^{(3)} - 20P_3'P_3'' - 567P_3^3)\Omega_1(x) + 20[6P_3P_3'' - 7(P_3')^2]\Omega_2(x) = 0, \\
(19) & \quad \Omega_1(x) = 5(x_2^2 - 2x_1x_3)(P_3'x_3 - 3P_3x_2) + 12P_3^2x_3^3, \\
(20) & \quad \Omega_2(x) = 5(x_2^2 - 2x_1x_3)(21P_3x_1 - P_3''x_3) - 42P_3^1x_2x_3^2 - 14P_3'P_3''x_3^2.
\end{align*}
\]

The curve \( \Omega_1(x) = 0 \) has a special significance for our problem. It is the only cubic of the pencil of cubics having eight consecutive points in common with \( C \), which has a double point at \( P_\nu \), and has therefore been called the eight-pointic nodal cubic.

The work will be simplified if we introduce a system of non-homogeneous coordinates \( X, Y \) defined by the following equations:

\[
t_1 = y_1 - \frac{a_1}{3a} y_2 + \frac{a_1^2}{18a^2} y_3,
\]

\[ t_2 = \theta_1 y_2 - \frac{\theta_1 a_1}{3a}, \]
\[ t_3 = \theta_2 y_3, \]
\[ X = \frac{t_2}{t_1}, \quad Y = \frac{t_3}{t_1}, \]

in which \( \theta_i \) is an abbreviation for one of the cube roots of \( -a/20 \). The new triangle of reference, given by (21), is characterized by the fact that if \( Y \) be developed according to powers of \( X \), the development assumes the form:

\[ Y = \frac{1}{2} X^2 + X^3 + a_4 X^7 + \cdots + a_{14} X^{14} + \cdots, \]

in which all the coefficients are absolute invariants. Professor Wilczynski has shown that in order to obtain the canonical form (23) of the development, the triangle of reference must be chosen as follows: * "One vertex is a point on the curve and one side of the triangle is the tangent at this point. The second side is the line upon which are situated the three points of inflection of the eight-pointic nodal cubic. The third side is the polar of the intersection of the other two with respect to the osculating conic. The numerical factors, which still remain arbitrary in a projective system of coördinates after the triangle of reference has been chosen, must be determined in such a way that the coördinates of one of the three points of inflection of the eight-pointic nodal cubic shall be \((0, -\sqrt[3]{16}, 1)\), and that the coördinates of the tangent to the cubic at this point shall be \((2\sqrt[3]{16}, 3\sqrt[3]{16^2}, 48)\)." We are thus dealing with a coördinate system which has a purely projective relation to the curve \( C \).

We now proceed to obtain the development (23) explicitly up to terms of the fourteenth order. In addition to the two fundamental invariants \( \theta_3 \) and \( \theta_8 \), it will be necessary to have explicit expressions for the following system of invariants of equation (10), obtained from \( \theta_3 \) and \( \theta_8 \) by the Jacobian process:

\[
\begin{align*}
\theta_{12} &= 3\theta_3 \theta_8' - 8\theta_8 \theta_3', \\
\theta_{16} &= \theta_3 \theta_{12}' - 4\theta_{12} \theta_3', \\
\theta_{20} &= 3\theta_3 \theta_{16}' - 16\theta_{16} \theta_3', \\
\theta_{24} &= 3\theta_3 \theta_{20}' - 20\theta_{20} \theta_3', \\
\theta_{28} &= \theta_3 \theta_{24}' - 8\theta_{24} \theta_3', \\
\theta_{32} &= 3\theta_3 \theta_{28}' - 28\theta_{28} \theta_3'.
\end{align*}
\]

From (12), we find:

\[
\begin{align*}
\theta_{12} &= 2 \cdot 3^2 a^2 a_3 - 2^3 \cdot 3^3 a a_1 a_2 + 2^3 \cdot 7 a_3^3, \\
\theta_{16} &= 2 \cdot 3^2 a^2 a_4 - 2^3 \cdot 3^3 a^2 a_1 a_3 - 2^3 \cdot 3^2 a^2 a_2^2, \\
\theta_{20} &= 2 \cdot 3^3 a^4 a_5 - 2 \cdot 3^2 \cdot 5^2 a^2 a_1 a_4 + 2^3 \cdot 3^3 \cdot 31 a^2 a_1^2 a_3 - 2^3 \cdot 3^3 \cdot 7 a^2 a_2 a_3 \\
&\quad + 2^4 \cdot 3^3 \cdot 7 a^2 a_1 a_2^2 - 2^3 \cdot 3 \cdot 5 a a_1^2 a_2 + 2^3 \cdot 7 a_1^3, \\
\end{align*}
\]

\[ \theta_2 = 2 \cdot 3^4 a^6 a_6 - 2 \cdot 3^4 \cdot 11 a^4 a_4 a_6 - 2 \cdot 3^3 \cdot 67 a^4 a_2 a_4 + 2 \cdot 3^3 \cdot 647 a^3 a_4^2 a_4 \]
\[ - 2 \cdot 3^4 \cdot 7 a^5 a_2^2 + 2 \cdot 3^3 \cdot 41 a^3 a_4 a_2 a_3 - 2 \cdot 3^2 \cdot 13 \cdot 29 a^2 a_3^2 a_3 \]
\[ - 2 \cdot 3^2 \cdot 7^2 \cdot 11 a^4 a_4 a_4 - 2 \cdot 3^2 \cdot 31 a^2 a_4 a_4 + 2 \cdot 3^2 \cdot 7 \cdot 193 a^2 a_2 a_4 \]
\[ + 2 \cdot 3 \cdot 7^3 \cdot 7 a^6 a_2 a_2 - 2 \cdot 3^2 \cdot 3 \cdot 7 \cdot 893 a^3 a_3 a_3 \]
\[ + 2 \cdot 3^3 \cdot 7 \cdot 253 a^2 a_3 a_3 + 2 \cdot 3^2 \cdot 3 \cdot 2441 a^2 a_3 a_3^2 - 2 \cdot 3^3 \cdot 7^2 a^2 a_3 a_3 \]
\[ - 2 \cdot 3^1 \cdot 5 \cdot 7 a a_3 a_2 + 2 \cdot 3^1 \cdot 32 \cdot 7 a a_3 a_2 \]
\[ + 2 \cdot 3^2 \cdot 3^2 \cdot 1521 a^2 a_3 a_2 a_3 + 2 \cdot 3^2 \cdot 7^2 a^2 a_3 a_2 a_3 \]
\[ + 2 \cdot 3 \cdot 3 \cdot 10799 a^4 a_4 a_3 a_3 - 2 \cdot 3^2 \cdot 3 \cdot 7 \cdot 4063 a^2 a_3 a_3 a_3 - 2 \cdot 3 \cdot 3 \cdot 16417 a^2 a_3 a_3 \]
\[ + 2 \cdot 3^3 \cdot 13697 a^2 a_3 a_2 a_2 - 2 \cdot 3^2 \cdot 3 \cdot 23 a^3 a_2 a_2 + 2 \cdot 3 \cdot 7 \cdot 5 a a_6 a_2 - 2 \cdot 3 \cdot 5 \cdot 7 a a_6 a_2 \]
\[ + 2 \cdot 3 \cdot 7 a a_6 a_2 \]

We shall also need \( \theta_3 = 3 \theta_3 \theta' - 32 \theta_3 \theta' \). These Jacobians, together with \( \theta_3 \) and \( \theta_3' \), form a complete system (\( \Sigma \) ) of invariants of equation (10), in the sense that any rational invariant whatever, involving \( \theta_3 \equiv a \), and its derivatives, \( \theta_3(i) \equiv a(i = 1, 2 \dots 9) \), may be expressed rationally in terms of the invariants of the system.* Equations (25) show that \( a, a_2, a_3, \ldots, a_9 \) may be expressed rationally in terms of the members of \( \Sigma \) and of \( q_1 \). Since the coefficients \( a \) in the development (23) are absolute invariants, when \( a \) and its derivatives have been replaced by the members of this system, the terms in \( a \), involving \( a_i \) must in the aggregate disappear. We may therefore neglect, at the outset, all terms involving \( a_i \); and it will be understood that terms involving \( a_i \) have been omitted from the right hand members of all the equations which follow, up to and including (31). Equations (22) now take the simple form:

\[ (26) \]
\[ X = \frac{\theta_1 y_2}{y_1}, \quad Y = \frac{\theta_2 y_3}{y_1}, \]

or more explicitly:

\[ \theta_1^{-1} X = x + \frac{3}{4} a x^4 + \frac{3}{6} a^2 x^4 + \frac{3}{7} (33 a^2 + a_3) x^7 + \frac{3}{8} a_4 x^8 \]

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\[
\frac{3}{91}(183a_2 + a_5)x^9 + \frac{3}{101}(a_6 + 273a_2 + 375a^3)x^{10}
\]

\[
+ \frac{3}{111}(a_7 + 388a_4 + 418a_2^2)x^{11} + \frac{3}{121}(a_8 + 531a_4 + 1512a_2a_3
\]

\[\quad + 60165a^2a_2)x^{12} + \frac{3}{181}(a_9 + 705a_6 + 2586a_2a_4
\]

\[\quad + 118377a^3a_2 + 1512a_6 + 101744a_4)x^{13} + \cdots,
\]

\[
\theta_1^{-5}Y = \frac{1}{2}x^2 + \frac{9}{51}ax^5 + \frac{15}{71}a_2x^7 + \frac{9}{81}(2a_3 + 58a^2)x^9 + \frac{21}{91}a_4x^{10}
\]

\[+ \frac{24}{101}(a_5 + 150a_2a_3)x^{11} + \frac{9}{111}(3a_6 + 671a_5 + 8289a^3)x^{12}
\]

\[
+ \frac{3}{121}(10a_7 + 3171a_4 + 3570a_2^2)x^{13} + \frac{3}{181}(11a_8 + 4758a_6 + 6867a_4 + 27195a_2a_4 + 1069173a^3a_2 + 16020a_6 + 8580087a^4)_x^{14} + \cdots.
\]

Reverting the series (27), we have:

\[
x = \theta_1^{-1}X - \frac{3}{41}a_1\theta_1^{-4}X^4 - \frac{3}{61}a_2\theta_1^{-5}X^5 + \frac{3}{71}(72a^2 - a_3)\theta_1^{-7}X^7
\]

\[\quad - \frac{3}{81}a_4\theta_1^{-8}X^8 + \frac{3}{91}(447a_2 - a_6)\theta_1^{-9}X^9 + \frac{3}{101}(717a_4 - 2305a_5)
\]

\[\quad - a_8)\theta_1^{-10}X^{10} + \frac{3}{111}(1097a_4 + 973a_2^2 - a_7)\theta_1^{-11}X^{11} + \frac{3}{121}(1614a_6 + 3636a_2a_3
\]

\[\quad - 985815a^5a_5 + 3636a_2^2 + 1831440a^4 - a_9)\theta_1^{-13}X^{13} + \cdots.
\]

Substituting this value of \(x\) in equation (28), we find the following symbolic expression for the development (28):

\[Y = \frac{1}{2}X^2 + X^3 - \frac{6}{71}a_2\theta_1^{-3}X^7 + \frac{6}{81}(105a^2 - a_3)\theta_1^{-5}X^5 - \frac{6}{91}a_4\theta_1^{-7}X^9
\]

\[+ \frac{6}{101}(630a_2 - a_6)\theta_1^{-8}X^{10} + \frac{6}{111}(990a_3 - 43848a^3 - a_9)\theta_1^{-9}X^{11}
\]
\[
(30) \quad + \frac{6}{121} (1485a_4a_2 + 1286a_2^2 - a_5) \theta_1^{10} X^{13} + \frac{6}{181} (2145a_5a_6 + 5148a_4a_4
- 806922a^2 a_2 - a_6) \theta_1^{11} X^{13} + \frac{6}{142} (300a_6 + 9009a_2a_4
- 174128a^2 a_6 + 5148a_4^2 + 46540494a^4 - a_6) \theta_1^{12} X^{14} + \ldots.
\]

It remains to replace the quantities \(a_i\) by the members of the system of invariants \(\Sigma\) of the differential equation (10). Solving equations (25) for these quantities, we find:

\[
\begin{align*}
\alpha = \theta_1, & \quad 2 \cdot 3 \theta_2 \cdot a_2 = \theta_5, & \quad 2 \cdot 3 \theta_4 \cdot a_5 = \theta_20 + 7 \theta_{12} \theta_3, \\
2 \cdot 3 \theta_3 \cdot a_5 &= \theta_24 + 67 \theta_{10} \theta_5 + 2 \cdot 7 \theta_{12} + 2 \cdot 5 \theta_6^3, \\
2 \cdot 5 \theta_3 \cdot a_5 &= 2 \cdot 8 \theta_5 + 2 \cdot 5^2 \theta_3 \theta_5 + 151 \theta_{10} \theta_3 + 3 \cdot 127 \theta_{12} \theta_3, \\
(31) \quad 2 \cdot 8 \theta_5 \cdot a_5 &= 2 \cdot 8 \theta_3 + 2 \cdot 71 \theta_4 \theta_5 + 2 \cdot 8 \cdot 5 \cdot 17 \theta_{10} \theta_5 + 251 \theta_{20} \theta_3 + 3 \cdot 151 \theta_{12}^2 + 2 \cdot 3 \cdot 11 \theta_{12} \theta_5 + 2 \cdot 3 \cdot 5 \cdot 7 \theta_5^3, \\
2 \cdot 8 \theta_3 \cdot a_5 &= 2 \cdot 3 \cdot 5 \cdot 7 \theta_5 + 2 \cdot 3 \cdot 97 \theta_3 \theta_5 + 3 \cdot 131 \theta_4 \theta_5 + 3 \cdot 7 \cdot 79 \theta_{20} \theta_5 + 2 \cdot 5 \cdot 103 \theta_{20} \theta_5 + 2 \cdot 7 \cdot 8 \theta_5 \theta_3 + 2 \cdot 3 \cdot 2851 \theta_{12} \theta_5, \\
\end{align*}
\]

where we still neglect to write the terms involving \(a_i\). The required coefficients \(a_i\) are therefore the following absolute invariants:

\[
\begin{align*}
\alpha_7 &= 2 \cdot 8 \cdot 3 \cdot 7 \cdot 1 \cdot \theta_5^2 \theta_3 + 2 \cdot 8 \theta_7, \\
\alpha_9 &= 2 \cdot 8 \cdot 3 \cdot 5 \cdot 7 \cdot 1 \cdot \theta_5 \theta_3 + 2 \cdot 8 \cdot 3 \cdot 5 \cdot 7 \theta_5^2, \\
\alpha_9 &= 2 \cdot 8 \cdot 3 \cdot 5 \cdot 7 \cdot 1 \cdot \theta_5 \theta_3 + 2 \cdot 8 \cdot 3 \cdot 5 \cdot 7 \theta_5^2, \\
\alpha_{10} &= 2 \cdot 8 \cdot 3 \cdot 7 \cdot 1 \cdot \theta_5 \theta_3 - 2 \cdot 8 \cdot 3 \cdot 5 \cdot 7 \theta_5 \theta_3, \\
\alpha_{11} &= 2 \cdot 8 \cdot 3 \cdot 7 \cdot 1 \cdot 11 \cdot 1 \cdot \theta_3 \theta_5 + 2 \cdot 7 \theta_4 \theta_5 + 2 \cdot 7 \theta_4 \theta_5 + 2 \cdot 8 \cdot 3 \cdot 5 \cdot 11 \theta_3 \theta_5 + 2 \cdot 8 \cdot 3 \cdot 5 \cdot 7 \theta_5 \theta_3 + 2 \cdot 8 \cdot 3 \cdot 5 \cdot 7 \theta_5 \theta_3, \\
\alpha_{12} &= 2 \cdot 8 \cdot 3 \cdot 7 \cdot 1 \cdot 11 \cdot 1 \cdot \theta_5 \theta_3 + 2 \cdot 8 \cdot 3 \cdot 5 \cdot 11 \theta_3 \theta_5 + 151 \theta_{10} \theta_3 + 2 \cdot 8 \cdot 3 \cdot 5 \cdot 11 \theta_3 \theta_5 + 2 \cdot 8 \cdot 3 \cdot 5 \cdot 7 \theta_5 \theta_3, \\
\end{align*}
\]

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Let the equation of the osculating quartic be assumed in the form:

\[ Q_x = \delta_{3333} Y^4 + \delta_{3332} Y^3 X + \delta_{3322} Y^2 X^2 + \delta_{3222} Y X^3 + \delta_{333} Y^3 + \delta_{332} Y^2 X + \delta_{322} Y X^2 + \delta_{32} Y X + \delta_{2} X + \delta = 0. \]  

If we substitute the development (23) for \( Y \) into the left member of this equation, the coefficients of all powers of \( X \) up to and including the thirteenth must be equal to zero, that is, we must have:

\[
\begin{align*}
\delta &= \delta_3 = 0, \\
\delta_2 &+ \frac{1}{2} \delta_3 = 0, \\
\delta_2 &+ \delta_3 = 0, \\
\delta_3 &+ \delta_3 = 0, \\
\alpha_7 \delta_3 &+ \alpha_3 \delta_3 = 0, \\
\alpha_6 \delta_3 &+ \alpha_7 \delta_3 = 0, \\
\alpha_5 \delta_3 &+ \alpha_6 \delta_3 = 0, \\
\alpha_4 \delta_3 &+ \alpha_5 \delta_3 = 0, \\
\alpha_3 \delta_3 &+ \alpha_4 \delta_3 = 0, \\
\alpha_2 \delta_3 &+ \alpha_3 \delta_3 = 0, \\
\alpha_1 \delta_3 &+ \alpha_2 \delta_3 = 0, \\
\alpha_0 \delta_3 &+ \delta_3 = 0.
\end{align*}
\]

If we substitute the development (23) for \( Y \) into the left member of this equation, the coefficients of all powers of \( X \) up to and including the thirteenth must be equal to zero, that is, we must have:

\[
\begin{align*}
\delta &= \delta_2 = 0, \\
\delta_2 &+ \frac{1}{2} \delta_3 = 0, \\
\delta_2 &+ \delta_3 = 0, \\
\delta_3 &+ \delta_3 = 0, \\
\alpha_7 \delta_3 &+ \alpha_3 \delta_3 = 0, \\
\alpha_6 \delta_3 &+ \alpha_7 \delta_3 = 0, \\
\alpha_5 \delta_3 &+ \alpha_6 \delta_3 = 0, \\
\alpha_4 \delta_3 &+ \alpha_5 \delta_3 = 0, \\
\alpha_3 \delta_3 &+ \alpha_4 \delta_3 = 0, \\
\alpha_2 \delta_3 &+ \alpha_3 \delta_3 = 0, \\
\alpha_1 \delta_3 &+ \alpha_2 \delta_3 = 0, \\
\alpha_0 \delta_3 &+ \delta_3 = 0.
\end{align*}
\]

Solving these equations, we find:

\[
\begin{align*}
\delta_{22} &= -2, \\
\delta_{23} &= 2, \\
\delta_{33} - 4 \delta_{322} &= 8, \\
\Delta_{3222} &= 2.
\end{align*}
\]
\[
\begin{bmatrix}
  \alpha_7 & \alpha_8 - 4 & \alpha_9 & \alpha_{10} \\
  d_{11} & d_{21} & d_{31} & d_{41} \\
  d_{12} & d_{22} & d_{32} & d_{42} \\
  d_{13} & d_{23} & d_{33} & d_{43}
\end{bmatrix}
\]
\[\delta_{222} = -4\Delta_{222} - 32\delta_{22},\]

where

\[
\begin{align*}
  d_{11} &= \alpha_5 - 6, & d_{14} &= \alpha_6 - \alpha_9 \alpha_7 - 8\alpha_7, \\
  d_{12} &= \alpha_6 - 2\alpha_9^2, & d_{22} &= \alpha_{11} - 2\alpha_9 \alpha_7 - \alpha_7^2 - 4\alpha_9 + 12, \\
  d_{13} &= \alpha_{10} - 2\alpha_9 \alpha_7 - 4\alpha_7, & d_{33} &= \alpha_{12} - 3\alpha_9 \alpha_7 - 8\alpha_7^2, \\
  d_{21} &= \alpha_9 - \alpha_9^2, & d_{41} &= \alpha_{11} - \alpha_9 \alpha_7 - 2\alpha_9 - 32, \\
  d_{22} &= \alpha_9 - \alpha_9^2, & d_{42} &= \alpha_{12} - 2\alpha_{10} \alpha_9 \alpha_7 - 2\alpha_9 - 6\alpha_9^2, \\
  d_{32} &= \alpha_{11} - \alpha_9 \alpha_7 - 2\alpha_9^2 + 6\alpha_9 - 12, & d_{43} &= \alpha_{13} + 6\alpha_{10} - 2\alpha_{10} \alpha_9 - \alpha_9^2 - 14\alpha_9 \alpha_7 - 28\alpha_7;
\end{align*}
\]

so that the equation of the osculating quartic is

\[
(\delta_{33} - 4\delta_{222})(2Y - X^2 - 8XY^2)Y - \delta_{333}(X^3 + 16Y^3 - 2XY)Y
\]

\[+ 2\delta_{222}(X^2 - 2Y)^2 + 2\delta_{222}(X^2 - 2Y)X + 2\delta_{33}(X^2 - 2Y)
\]

\[+ [2\alpha_7(\delta_{33} - 4\delta_{222}) - 8\alpha_9 \delta_{33} - 8(\alpha_3 - 4)\delta_{222}](X^2 - 2Y)Y^2
\]

\[+ 64[(\alpha_3 - 2)\delta_{23} + \alpha_7 \delta_{222}]Y^4 + 8(4\alpha_7 \delta_{33} Y^2 + 2\delta_{222} XY + \delta_{22} X^2)XY = 0,
\]

where all of the coefficients are absolute invariants. This is easily reduced to homogeneous form, if desired, by equations (22). Referred to the same triangle, the equations of the osculating conic and cubic are respectively:

\[
X^2 - 2Y = 0,
\]

\[
\alpha_7(2Y - X^2 - 8XY^2) + 2\alpha_7^2(X^2 - 2Y)Y
\]

\[+ (\alpha_3 - 4)(X^3 + 16Y^3 - 2XY) = 0.
\]

When \(\alpha_7 = 0\), (39) reduces to the equation of the eight-pointic nodal cubic.

If we substitute the canonical development (23) of the curve \(C_\gamma\) into the left member of the equation (33) of the osculating quartic, we find that the coefficient of \(X^{14}\) is

\[
\alpha_{14} \delta_5 + \alpha_{13} \delta_{23} + (\alpha_{12} + 2\alpha_9 + \alpha_7^2) \delta_{33} + \alpha_{11} \delta_{333} + \alpha_{11} + 2\alpha_9 \delta_{33} + (\frac{3}{2}\alpha_{10} + 3\alpha_7) \delta_{333}
\]

\[+ \alpha_{11} \delta_{333} + (\alpha_{10} + 2\alpha_9) \delta_{333} + \frac{3}{2} \alpha_9 \delta_{333} + (\frac{3}{2} \alpha_9 + \frac{1}{2}) \delta_{333}.
\]

When this expression vanishes, the osculating quartic has fifteen consecutive
points in common with $C_y$ at $P_y$, or, as we may say, hyperosculates $C_y$ at that point. This condition may be put into the form:

$$K = \begin{vmatrix} d_{11} & d_{21} & d_{31} & d_{41} \\ d_{12} & d_{22} & d_{32} & d_{42} \\ d_{13} & d_{23} & d_{33} & d_{43} \\ d_{14} & d_{24} & d_{34} & d_{44} \end{vmatrix} = 0,$$

(40)

where

$$d_{14} = a_1 - 2a_9 a_7 - 4a_9 - a_7^2 - 24,$$

$$d_{24} = a_2 - a_1 a_7 - 2a_9 a_8 + 6a_9 - a_8 a_7^2 - 2a_7^2,$$

(41)

$$d_{34} = a_3 - a_2 a_7 - 2a_9 a_8 + 6a_9 - a_8 a_7^2 - 16a_8 a_7,$$

$$d_{44} = a_4 + 6a_11 - 3a_1 a_7 - 14a_9 a_7 - a_11 a_7^2 - 8a_9^2 - 40 a_9 - 144.$$  

If the invariant equation (40) is satisfied for all values of $x$, that is, at all points of the curve $C_y$, this curve is itself a quartic.

University of Illinois,
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