INFINITE DISCONTINUOUS GROUPS OF BIRATIONAL TRANSFORMATIONS WHICH LEAVE CERTAIN SURFACES INVARIANT*

BY

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It was shown by Schwarz † that no algebraic curve of genus greater than unity can remain invariant under a continuous group of birational transformations. Later Hurwitz ‡ showed that no such curve could belong to any birational group of infinite order.

The corresponding theory for surfaces is by no means complete. While those belonging to continuous groups have been determined, only a few isolated examples are known of surfaces having an infinite discontinuous group.§

All the groups which have been discussed are illustrations of two principles, the first of which refers to quartic surfaces and will be considered in two parts, the latter including the former as a particular case; the second principle is applicable to a much wider category. I propose to discuss these principles and apply the second to the determination of an extended family of new surfaces having an infinite group.

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†Ueber diejenigen algebraischen Gleichungen zwischen zwei veränderlichen Grössen, welche eine Schaar rationaler eindeutig umkehrbarer Transformationen in sich zulassen, Crelle's Journal, vol. 87 (1879), pp. 139-146.
§Humbert: Sur la décomposition des fonctions θ en facteurs, Comptes Rendus, vol. 126, (1898), pp. 394–396; Sur les fonctions abéliennes singulières, ibid., pp. 508–510, and Liouville's Journal, ser. 5, vol. 6 (1900), pp. 279–386, see page 372; Painlevé: Sur les surfaces qui admettent un groupe infini discontinu de transformations birationnelles, Comptes Rendus, vol. 126, (1898), pp. 512–514; Hutchinson: The Hessian of the cubic surface II, Bulletin of the American Mathematical Society, vol. 6 (1900), pp. 328–337, and On some birational transformations of the Kummer surface into itself, ibid., vol. 7 (1901), pp. 211–217; Fano: Sopra alcune superficie del 4° ordine rappresentabili sul piano doppio, Rendiconti del Istituto Lombardo, vol. 39 (1906), pp. 1071–1086. The first three of these articles are concerned with special forms of the Kummer surface; the treatment is entirely transcendental. The next two treat the general Kummer surface and two others into which it can be transformed; the treatment is partly transcendental and partly algebraic. The last one gives an outline of the theory of those quartic surfaces having a net of hyperelliptic curves; the treatment is purely geometric.
§1. Quartic surfaces that possess a net of hyperelliptic curves of genus two.

a. The nodal inversion (\(N\)).

1. The line joining any point \(A_x\) on a quartic surface \(F_4\) to a conical point \(N\) of the surface will meet it again in a point \(B_x\). A \((1, 1)\) correspondence exists between \(A_x, B_x\), defining a non-linear birational transformation of order 2 which will be indicated by \((N)\). If \(F_4\) has two conical points \(N_1, N_2\), these define two such transformations \((N_1), (N_2)\). Every plane section through \(N_1, N_2\) will remain invariant under the group generated by \((N_1), (N_2)\). By taking \(N_1, N_2\) as the vertices \((0, 0, 0, 1), (0, 0, 1, 0)\) of the tetrahedron of reference the equation of a plane section may be written in the form

\[x_1(a x_2^2 + bx_2 x_3 + cx_3^2) + x_2(x_2' a' x_2^2 + b' x_2 x_3 + c' x_3^2) + x_3(a'' x_2^2 + b'' x_2 x_3 + c'' x_3^2) = 0.\]

The necessary and sufficient condition that the operations \((N_1), (N_2)\) are commutative is the vanishing of the determinant *

\[
\begin{vmatrix}
a & b & c \\
a' & b' & c' \\
a'' & b'' & c''
\end{vmatrix}
\]

(a)

If now the binodal quartic be transformed into a cubic, the operations \((N_1), (N_2)\) become ordinary quadric inversions with regard to the polar conies of two ordinary points of the curve. If we still call these points \(N_1, N_2\), the geometric condition that \((N_1), (N_2)\) be commutative is that the tangents at \(N_1, N_2\) meet on the curve, i. e., that \(N_1, N_2\) be conjugate points. In general, the question whether \((N_1), (N_2)\) define a finite group is thus reduced to STEINER's celebrated "Schliessungsproblem." †

In general, if \((N_1 N_2)^k = 1, k\) must satisfy a certain relation which can be most easily expressed in terms of elliptic functions. Since any binodal quartic or non-singular cubic can be birationally derived from a space quartic curve of the first kind \(c_1\), the \((2, 2)\) correspondence between the points in which the lines through \(N_1, N_2\) meet the curve again can be defined by means of the generators of a quadric surface passing through the quartic curve. The quartic curve may be defined by the equations

\[x_1 = \rho \varphi''(u), \quad x_2 = \rho \varphi'(u), \quad x_3 = \rho \varphi(u), \quad x_4 = \rho.\]

Four points corresponding to the values \(u_1, u_2, u_3, u_4\) of the parameter \(u\) will lie in the same plane if, and only if,

\[u_1 + u_2 + u_3 + u_4 \equiv 0 \pmod{2\omega_1, 2\omega_2}.
\]

If the cycle is closed for one point, it will be for every point.
Given any two points $u$ and $v$ on $c_i$. Through the line joining them can be passed a quadric surface of the pencil having $c_i$ for basis curve. If $u + v \equiv c$, then the parameters of the points on any generator of the second system must satisfy the relation $u + v \equiv -c$. Starting with the point $u$, we can obtain $v$ by the equation $u + v \equiv c$, then from $v$ we can find $u_1$ on the other generator by $v + u_1 \equiv -c$. Similarly $u_1 + v_1 \equiv c$, $v_1 + u_2 \equiv -c$, etc. If $u_k = u$, by eliminating the intermediate terms we obtain the condition

$$2kc \equiv 0.$$ 

For an arbitrary $F_4$ with two or more double points this condition will not be satisfied for any finite integer $k$, hence:

Quartic surfaces having $m\ (1 < m \equiv 16)$ double points exist which are invariant under a noncontinuous group of birational transformations of infinite order.

2. This result is interesting in view of Klein's* question regarding operations which leave the 16-nodal Kummer surface invariant. By means of properly chosen operations of the linear $G_{16}$, any $(N_i)$ can be transformed into any other $(N_k)$. This gives rise to at least thirty-two operations. By duality we have at once 32 more. The question is whether the surface is invariant under any other than these 64 operations. This question was answered in the affirmative by Humbert† by means of hyperelliptic theta functions.

On applying condition (a) to the equation of the surface referred to a Göpel tetrad, this question can be answered in the affirmative immediately. Similarly, by making use of the invariants whose vanishing expresses that $(N_iN_k)$ is finite it is seen that the surface has an infinite group, as the condition for finiteness would impose a relation among the three essential constants of the surface. By means of the known $G_{32}$ the generators of this group can now be easily determined.

3. Another interesting case is furnished by the Weddle surface, the locus of the vertex of a quadric cone through six given points. Since this surface can be birationally transformed into the Kummer surface, it must have a $G_{32}$. This group was shown by Baker to be defined by the six operations $(N_i)$.‡

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‡_This same result can be obtained from the parameters belonging to two points collinear with $N$ as given in Humbert's third paper cited above, p. 470, but this fact is not there mentioned._ Baker: _Elementary note on the Weddle quartic surface_, Proceedings of the London Mathematical Society, ser. 2, vol. 1 (1903), pp. 247–261, gives an algebraic proof. A much more extensive discussion of the transformations of both surfaces is given by Baker: _An introduction to the theory of multiply periodic functions_ (1907); see pp. 69–82.
By the preceding principles this theorem can be obtained at once as follows. The equation of the surface may be written in the form

\[
\begin{vmatrix}
\frac{a_1}{x_1} & x_1 & a_1 & 1 \\
\frac{a_2}{x_2} & x_2 & a_2 & 1 \\
\frac{a_3}{x_3} & x_3 & a_3 & 1 \\
\frac{a_4}{x_4} & x_4 & a_4 & 1
\end{vmatrix} = 0,
\]

the six nodes being at the four vertices of the tetrahedron of reference and the two points \((1, 1, 1, 1), (a_1, a_2, a_3, a_4)\). In a plane containing any pair of nodes, as \(k_4x_3 = k_3x_4\), the two points on \(x_1 = 0\) are \((0, 0, k_3, k_4), [0, k_3(a_2 - a_4) + k_4(a_3 - a_4), k_3(a_3 - a_4), k_4(a_3 - a_4)]\). Similarly the two points on \(x_2 = 0\) are \((0, 0, k_3, k_4), [k_2(a_4 - a_4) + k_4(a_3 - a_4), 0, k_3(a_3 - a_4), k_4(a_3 - a_4)]\).

The necessary and sufficient condition that \((N_1), (N_2)\) be commutative is that the point \([k_3(a_1 - a_4) + k_4(a_3 - a_4), k_3(a_3 - a_4), k_3(a_3 - a_4), k_4(a_3 - a_4)]\) lie on the surface. By direct substitution we find that it does lie on the surface; similarly for the other pairs of nodes.

4. Moreover the result can be obtained geometrically. Let \(c_3\) be the space cubic curve passing through the six points \(N_i\). It lies on the Weddle surface defined by these points. The quadric cone having its vertex at \(N_i\) and passing through the five other points \(N_k\) contains \(c_3\) and is the tangent cone at the conical point \(N_i\) of the Weddle surface.*

Any plane section through \(N_i N_k\) will contain the line \(N_i N_k\) and a cubic curve passing through \(N_i N_k\). The tangent to the plane cubic at \(N_i\) is the generator of the tangent cone at \(N_i\) lying in the given plane and not passing through \(N_k\). Similarly for \(N_k\). Since these generators meet on \(c_3\), it follows that \(N_i, N_k\) are conjugate points with regard to every plane cubic section passing through them, hence \((N_i), (N_k)\) are commutative.

It is known that through any given plane quartic curve a Kummer surface can be passed, while all the plane sections of a Weddle surface are restricted, the relation among the constants being expressed by the equation

\[A^2 + 144B = 0,\]

wherein \(A\) is the cubic and \(B\) the sextic invariant of a quartic curve.†

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The preceding results furnish a new geometric interpretation of this important theorem for the case in which the plane passes through two conical points of the surface.

§ 1b. The Involutions (I).

5. A quartic surface \( F_4 \) of genus one \( (p_n = p_g = p = 1) \) which contains one curve of genus 2 will contain a net of such curves, any two of which will meet in two points, forming a rational involution \( I \). (Fano, 1. c.) The curves of order 4 are plane sections, and the surface has a double point. These curves define the inversions \( (N) \). In case the curves are of order 5 they are cut from \( F_4 \) by a net of quadrics, all having for residual the same twisted cubic.

Given a curve \( c_6 \) of order six and genus 2 on \( F_4 \). It is then one of a net \( \sigma_1 \), defining an involution \( I_1 \). Each \( c_6 \) of \( \sigma_1 \) will be cut in \( \infty^1 \) pairs of points of \( I_1 \), forming the canonical \( g_2^1 \). Through each \( c_6 \) pass \( \infty^2 \) cubic surfaces \( F_3 \). The residual intersection will be another sextic \( c_6' \), also of genus 2, and belonging to a net \( \sigma_2 \), distinct from \( \sigma_1 \) provided no \( F_3 \) of the net can be found which touches \( F_4 \) in every point of \( c_6 \). The net \( \sigma_2 \) will define an involution \( I_2 \) on \( F_4 \).

The \( \infty^1 \) lines determined by the pairs of points in \( g_2^1 \) on \( c_6 \) define a ruled cubic surface \( R_3 \) whose double directrix is a quadrisecant of \( c_6 \). The other points of intersection of each generator with \( F_4 \) will belong to the canonical \( g_2^1 \) on the residual \( c_6' \). The curves \( c_6, c_6' \) have four points of intersection on the double directrix. Every line joining a pair of points in \( I_1 \) will also join a pair of points in \( I_2 \). The lines joining any point to its conjugate will therefore define a congruence of order 2, one of the lines joining a given point to its conjugate in \( I_1 \) and the other joining the same point to its conjugate in \( I_2 \). Starting with any point on the surface we can first find its conjugate in \( I_1 \) by a birational transformation of order 2, then the conjugate of the latter as to \( I_2 \), also a transformation of order 2.\(^*\) If a line of \( I_1 \) be given, there are two lines, one through each of the points of this line belonging to \( I_1 \), which connect it with its conjugate in \( I_2 \). Hence between the lines of \( I_1, I_2 \) there is a \( (2, 2) \) correspondence, and the condition for periodicity is reduced to that of the preceding case.\(^*\)

The determination of these hyperelliptic curves by algebraic processes is in general a very difficult problem.

§ 2. Systems of bitangents (T).

6. Let \( a = \Sigma a_i x_i = 0, b = 0 \) be the equations of a straight line, the coefficients \( a_i, b_i \) being rational functions of two non-homogeneous parameters \( \kappa, \tau \). If the coordinates of a fixed point \( y \) be substituted for \( x_i \), the number of roots in \( \kappa, \tau \) defines the order of the congruence \( T \), that is, the number of lines

\(^*\) Fano, 1. c., showed by a different method that the operation \( (I_1, I_4) \) is in general of infinite order.
belonging to the system which pass through the point. The locus of the point $y$ for which two of the lines coincide is the focal surface of the congruence. Thus, except for particular cases, the line of a given congruence of any order which touches the focal surface $F$ at a given point $P$ can be rationally separated from the other lines passing through $P$ by the process of partial elimination. We shall be concerned only with such congruences as have a single line (counted twice) passing through $P$ on $F$, and lying in its tangent plane. The other lines of $T$ passing through $P$ have one point of intersection with $F$ at $P$.*

Congruences of the first order can have no focal surface; for those of the second order there is no residual line, while in the case of cubic congruences points on the focal surface are characterized by one double and one single root in $\kappa, \tau$.

Every line of $T$ is a bitangent to $F$, the points of tangency being $P_1, P_2$. The operation of interchanging $P_1, P_2$ is a birational transformation of order 2 which will be denoted by $(T)$. If $F$ is the complete focal surface for more than one congruence, we have two or more operations of order 2, and their product will be in general of infinite order. The necessary and sufficient condition that the transformations $(T_1), (T_2)$ be birational is that the two congruences $T_1, T_2$ be rationally separable.

7. A number of important surfaces having this property can be obtained as the apparent contour of Segre's cubic variety $\Gamma$ in four dimensional space $S_4$,† quartics by projecting from a point on $\Gamma$, and sextics by projecting from any point in $S_4$. In particular, if $\Gamma$ contains a plane and can be generated by trilinear systems, the contour will be the focal surface of at least two congruences, and the equations of the transformations $(T_1), (T_2)$ can be determined.

Let

\[ \Gamma \equiv \begin{vmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & 0 \\ u_1 & u_2 & u_3 \end{vmatrix} = 0 \]

be the equation of a cubic variety containing the plane $x_4 = 0, x_5 = 0$. A space $\kappa x_1 - \kappa x_5 = 0$ passing through this plane will cut from $\Gamma$ a quadric surface, one such quadric passing through any point of $\Gamma$.

The two systems of generators are rationally separable, being

\[ \frac{u_3}{u_1 k_2 - u_2 k_1} = \frac{x_3}{x_1 k_2 - x_2 k_1} = \tau_1, \quad \frac{u_3}{x_3/x_1 k_2 - x_2 k_1} = \tau_2, \]

* An example of congruences of another kind is furnished by a family of quadrics having one variable parameter, when the equations of the two systems of generators cannot be rationally separated.

the second one being defined by the pencil of spaces passing through the other plane \( u_3 = 0, x_3 = 0 \) on \( \Gamma \).

If now a plane be passed through any line \( \tau_1 \) and a fixed point \( K \), the section of \( \Gamma \) made by this plane will consist of \( \tau_1 \) and a conic cutting \( \tau_1 \) in two points \( P', P'' \). The locus of the points \( P_1, P_2 \) in which the lines \( KP', KP'' \) are cut by an \( S_3 \) not passing through \( K \) is the focal surface of the congruence \( T_1 \), the points \( P_1, P_2 \) being the points of contact. If \( k = (c_1, c_2, \ldots) \), and \( S_3 \) be defined by \( \Sigma e_i x_i = 0 \), the equations take the form

\[
\frac{u_3 - \tau_1(u_1 k_2 - u_2 k_1)}{u_3(c) - \tau_1[u(c) k_2 - u(c) k_1]} = \frac{x_3 - \tau_1(x_1 k_2 - x_2 k_1)}{c_3 - \tau_1(c_1 k_2 - c_2 k_1)} = \frac{k_2 x_4 - k_1 x_5}{k_2 c_4 - k_1 c_5}
\]

from which \( x_5 \), for example, can be eliminated by means of \( \Sigma e_i x_i = 0 \). If \( \tau_1 \) be eliminated, a cubic in \( k_1 : k_2 \) results, whose envelop is the focal surface.

If then we start with a point \( (x'_1, \ldots, x'_r) \) or \( x' \) on \( \Gamma \), the associated \( S_3 \) and the line \( \tau_1 \) are known and the points \( P' = (x'), P'' = (x'') \) are at once defined as the roots of a quadratic equation, one of which is known. The other root is expressed as a rational function of the given one, and the equations of \( (T_1) \) are determined. In the same way we determine \( (T_2) \).

If the system \( T_1 \) belongs to a linear complex, the points \( P_1, P_2 \) are poles of the tangent planes at \( P_2, P_1 \) respectively, so that \( (T_1) \) may be regarded as the product of the two commutative operations, duality as to the surface and duality as to the complex. When \( T_1, T_2 \) both belong to linear complexes, the group generated by \( (T_1), (T_2) \) may be finite. In particular, the necessary and sufficient condition that \( (T_1), (T_2) \) are commutative is that the complexes to which \( T_1, T_2 \) belong are in involution.

The lines of the congruence \( T_1 \) can be arranged on a system of \( \infty^1 \) quadric surfaces, the congruence \( T_2 \) being composed of the other system of generators of these quadrics. Any line \( l \) of \( T_1 \) will therefore determine two lines \( m_1, m_2 \) of \( T_2 \), so that a \( (2, 2) \) correspondence exists between \( l, m \). Thus the locus of \( P_1, P_2 \) corresponding to the quadric defined by \( l, m \) is a space curve \( c_4 \) of order 4 of the first kind. By the operations \( (T_1), (T_2) \) the curve \( c_4 \) remains invariant, so that each quadric of the system goes into itself. The points \( P_1, P_2 \) define a \( (2, 2) \) correspondence upon \( c_4 \), hence the discussion of the periodicity of the operation \( (T_1T_2) \) can be reduced to that of the preceding case.*

When the focal surface is of order 6 the residual section made by any quadric of bitangents is another \( c_4 \) which is also the curve of contact of another quadric. If the line \( P_1P_2 \) cuts \( F_6 \) in \( Q_1, Q_2 \) the operation of interchanging \( Q_1, Q_2 \) is also birational, since only a single line of each congruence passes through \( Q_1 \).

*For the literature concerning the \((2, 2)\) correspondence on \( c_4 \), see O. Staude: Flächen zweiten Grades und ihre Systeme und Durchdringungskurven, Encyklopädie der mathematischen Wissenschaften, III C2., no. 123; in particular, footnote 513.
apart from the one (counted twice) which touches $F_1$ at $Q_1$. All the theorems regarding periodicity can be applied immediately to this case also.

8. An interesting illustration is furnished by the variety $\Gamma$ having ten nodes. The contour from any point on $\Gamma$ is the general Kummer surface. The six systems of bitangents belong to six linear complexes mutually in involution, hence the operations $(T_i)$ generate the well-known $G_{32}$ first found by Klein.*

The product of any two operations $(T_i)(T_k)$ is a linear transformation, and thus the linear group of order 16 is defined. Many of the properties of the Kummer surface and of its systems of bitangents can be deduced readily from this starting point.

9. An illustration of a different kind, wherein the group generated by the $(T_i)(T_k)$ is finite, is obtained from the variety

\[(1) \quad \Gamma \equiv x_1x_2x_3 + \lambda x_4x_5x_6 = 0, \quad \Sigma x_i = 1, \quad \lambda \neq 1,
\]

the center of projection being $P \equiv (1, 1, 1, -1, -1, -1)$. The details of this case will sufficiently illustrate the general procedure. The line of the system $I$ which passes through the point $x'$ may be expressed by the equations

\[(2) \quad \frac{x'_1x'_4 - x'_5x'_2}{x'_1 + x'_4} = \frac{x'_2x'_6 - x'_5x'_3}{x'_2 + x'_3} = \frac{x'_3x'_6 - x'_5x'_3}{x'_3 + x'_6} = l.
\]

Solve these equations for $x'_4$, $x'_5$, $x'_6$ in terms of $l$ and substitute the results in (1). The equations of the conic and of the line may be written in the form

\[(3) \quad x'_0x'_6(x'_1 + x'_4)x'_2x'_3 + x'_4x'_6(x'_2 + x'_5)x'_1x'_3 + x'_4x'_6(x'_3 + x'_6)x'_1x'_2 = 0,
\]

\[(4) \quad x'_2x'_4(x'_1 + x'_4)x'_1 + x'_1x'_5(x'_2 + x'_5)x'_2 + x'_1x'_2(x'_3 + x'_6)x'_3 = 0.
\]

If now $x'$ be taken on the first polar of $\Gamma$ as to $P$,

\[(5) \quad H \equiv x_1x_2 + x_1x_3 + x_2x_3 - \lambda(x_4x_5 + x_5x_6 + x_6x_4) = 0,
\]

it will be one of the points of intersection of the line and the conic. Making (3), (4) simultaneous and making use of (2), we obtain the coördinates of $x''$, the second point of intersection. The results are

\[(6) \quad x'_1 = \frac{1}{x'_4}, \quad x'_2 = \frac{1}{x'_5}, \quad x'_3 = \frac{1}{x'_6}, \quad x'_4 = \frac{1}{x'_1}, \quad x'_5 = \frac{1}{x'_2}, \quad x'_6 = \frac{1}{x'_3}.
\]


† For this notation and the discussion of this variety, see my paper: Surfaces derived from the cubic variety having nine double points in four dimensional space, these Transactions, vol. 10 (1909), pp. 71–78.
Let the lines joining $x'$ to $P$ be cut by the polar $S_3$ of $P$ as to $\Gamma$

\[ x_1 + x_2 + x_3 + \lambda(x_4 + x_5 + x_6) = 0. \]

To distinguish between points $x$ on $\Gamma$ and points on the $S_3$ defined by (7), coordinates in the latter will be denoted by $y_i$. It is defined by

\[ y_1 + y_2 + y_3 = 0, \quad y_4 + y_5 + y_6 = 0. \]

From a point $x$ on $\Gamma(x)$ we obtain the corresponding point in $S_3$ by means of the equations

\[ y_1 = x_2 + x_3 - 2x_1, \quad y_4 = x_5 + x_6 - 2x_4, \]
\[ y_2 = x_1 + x_3 - 2x_2, \quad y_5 = x_4 + x_6 - 2x_5, \]
\[ y_3 = x_1 + x_2 - 2x_3, \quad y_6 = x_4 + x_5 - 2x_6, \]

and the reciprocal relations, giving $x$ when $y$ is known, are

\[ x_1 = 2y_1H(y) - 3\Gamma(y), \quad x_4 = 2y_4H(y) + 3\Gamma(y), \]
\[ x_2 = 2y_2H(y) - 3\Gamma(y), \quad x_5 = 2y_5H(y) + 3\Gamma(y), \]
\[ x_3 = 2y_3H(y) - 3\Gamma(y), \quad x_6 = 2y_6H(y) + 3\Gamma(y). \]

Now by means of equations (6), (9), (10) we can obtain the equations of the birational transformation in $S_3$. The results are

\[ \rho y_1' = -\lambda \left[ 2H(y)(y_4y_5 + y_5y_6 - 2y_4y_6) + 3\Gamma(y)(y_5 - 2y_4 - 2y_6) \right], \]
\[ \rho y_2' = -\lambda \left[ 2H(y)(y_4y_6 + y_5y_6 - 2y_4y_5) + 3\Gamma(y)(y_6 - 2y_4 - 2y_5) \right], \]
\[ \rho y_3' = -\lambda \left[ 2H(y)(y_4y_6 + y_4y_5 - 2y_6y_6) + 3\Gamma(y)(y_4 - 2y_5 - 2y_6) \right], \]
\[ \rho y_4' = 2H(y)(y_1y_2 + y_2y_3 - 2y_1y_3) - 3\Gamma(y)(y_2 - 2y_1 - 2y_3), \]
\[ \rho y_5' = 2H(y)(y_1y_3 + y_2y_3 - 2y_1y_2) - 3\Gamma(y)(y_3 - 2y_1 - 2y_2), \]
\[ \rho y_6' = 2H(y)(y_1y_2 + y_1y_3 - 2y_2y_3) - 3\Gamma(y)(y_1 - 2y_2 - 2y_3), \]

the equations of the surface being

\[ y_1 + y_2 + y_3 = 0, \quad y_4 + y_5 + y_6 = 0, \quad 27(1 - \lambda)\Gamma^2(y) + 4H^3(y) = 0. \]

The transformation defined by (11) is not a Cremona transformation, being birational for points of the surface only.

In the same manner we may obtain the systems $(T_4), (T_5)$, defined by $\Pi, \Pi_1, \Pi_2$. If we use the notation $(x_1x_2x_3)$ to indicate the cyclic substitution...
\( x_i = \rho x_i', \) etc., the results may be written in the form

\[ T_2 = T_1 l_2, \quad T_3 = T_1 l_2', \quad T_4 = T_1 l_4, \quad T_5 = T_1 l_5, \quad T_6 = T_1 l_4 l_5 l_4', \]

wherein

\[ l_2 = (y_1 y_2 y_3) (y_4 y_5 y_6); \quad l_4 = (y_2 y_3) (y_5 y_6); \quad l_5 = (y_1 y_2) (y_4 y_6). \]

Moreover

\[ T_1 l_2 = l_2 T_1, \quad T_1 l_4 = l_4 T_1, \quad T_1 l_5 = l_5 T_1, \]

from which it follows that \( T_1, \ldots, T_6 \) generate a group of order 36 which contains a linear subgroup of order 18. The surface is also invariant under the odd substitutions \( t = (y_2 y_3) \) etc., making another linear group of order 18. The operations \( (T_i) \) combine with these in the same manner as with the preceding, making a total group of order 72.

10. From any point \( P_1 \) on \( F_6 \) can be drawn just one line of \( I \) touching \( F_6 \) at \( P_1 \). Let \( P_2 \) be the second point of contact of this line. Similarly, the line \( P_2 P_3 \) belongs to \( IV \) and touches \( F_6 \) at \( P_2 \) and at \( P_3 \). The line \( P_3 P_4 \) belongs to \( I \). Since \( (T_4 T_1)^2 = 1 \) it follows that \( P_4 \) belongs to \( IV \) and has its points of contact at \( P_4 \) and at \( P_1 \). The vertices \( P_1, P_2, P_3, P_4 \) define a tetrahedron inscribed in \( F_6 \) and the planes \( P_1 P_2 P_3 \), etc. are all tangent planes. The transformation \( (T_1 T_4) \) transforms \( P_1 \) into \( P_3 \) and \( P_2 \) into \( P_4 \); it is the axial involution \( l_4 \). We have the following theorem:

There are nine systems of \( \infty^2 \) tetrahedra which are inscribed in and circumscribed about \( F_6 \). Two opposite edges of the tetrahedra of each system always meet two fixed lines.

11. When the line \( \sigma \) touches the residual conic, section of \( \Gamma \), \( P_1 = P_2 \) and the corresponding line in \( S_3 \) has four coincident points in common with \( F_6 \). The locus of the point of contact in \( I \) is a curve defined as the intersection of \( \Gamma (x) \) with \( H(x) \) and the variety defined by the equation

\[
\frac{x_1 x_2 x_3 x_4 x_5 x_6}{x_1^2 x_4^2} + \frac{(x_2 + x_5)^4}{x_2^2 x_5^2} + \frac{(x_3 + x_6)^4}{x_3^2 x_6^2}
- 2(x_1 + x_4)^2(x_2 + x_5)^2(x_3 + x_6)^2\left[\frac{x_1 x_4}{(x_1 + x_4)^2} + \frac{x_2 x_5}{(x_2 + x_5)^2} + \frac{x_3 x_6}{(x_3 + x_6)^2}\right] = 0.
\]

By means of equations (6), (9), (10) the equations of the curve in \( S_3 \) can be obtained. Let \( C_i \) denote the curve at the points of which the lines of \( I \) have four point contact. From any point \( P_1 \) of this curve draw the line of \( IV \). From the preceding theorem the second point of contact must also lie on \( C_i \). The ruled surfaces belonging to \( IV, V, VI \) which have \( C_1 \) for directrix curve are such that every generator of each is a bisection of the curve. If the line of \( II \) touching \( F_6 \) at \( P_1 \) also touches it at \( P' \), and the line \( P' P'' \) belongs to \( I \), then \( P'' \) is a point of \( C_2 \). The six curves \( C_i \) are birationally equivalent.

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