A NOTE CONCERNING VEBLEN'S AXIOMS FOR GEOMETRY*

BY

ROBERT L. MOORE

VEBLEN has given a set of axioms for geometry.† The first eight of these are as follows:‡

AXIOM I. There exist at least two distinct points.

AXIOM II. If points A, B, C are in the order ABC, they are in the order CBA.

AXIOM III. If points A, B, C are in the order ABC, they are not in the order BCA.

AXIOM IV. If points A, B, C are in the order ABC, then A is distinct from C.

AXIOM V. If A and B are any two distinct points, there exists a point C such that A, B, C are in the order ABC.

DEF. I. The *line* AB ($A \neq B$) consists of A and B and all points X in one of the possible orders ABX, AXB, XAB. The points X in the order AXB constitute the *segment* AB. A and B are the *end-points* of the segment.

AXIOM VI. If points C and D ($C \neq D$) lie on the line AB, then A lies on the line CD.

AXIOM VII. If there exist three distinct points, there exist three distinct points A, B, C not in any of the orders ABC, BCA, or CAB.

AXIOM VIII. If three distinct points A, B and C do not lie on the same line, and D and E are two points in the orders BCD and CEA, then a point F exists in the order AFB and such that D, E, F lie on the same line.

Axiom II may be divided into two parts which I will call Axiom II_1 and Axiom II_2.

AXIOM II_1. If A, B, C are three distinct points in the order ABC, they are in the order CBA.

AXIOM II_2. If the points A, B, C are not all distinct, then, if they are in the order ABC, they are in the order CBA.

Though Veblen's Axioms I–XII, as they stand, are mutually independent,

* Presented to the Society, in a somewhat different form, October 26, 1907.

† These Transactions, vol. 5 (1904), pp. 343–384.

‡ Except that I have taken the liberty of inserting the word "distinct" between the words "three" and "points" in the statement of Axiom VII. Without this slight modification (or a corresponding *understanding*) Axioms I–XII are not sufficient for Euclidean geometry.

I wish to show, in the present paper, that Axiom II_1 (a *part* of Axiom II) is a consequence of Axioms III and VI and Def. I while Axiom II_2 (the *remainder* of Axiom II) and Axiom IV are* both consequences of Axioms I, III, V′, VI, VII, VIII and Def. I, where Axiom V′ is the following modification of Axiom V.

AXIOM V′. If A and B are any two distinct points, there exists a point C, *different from A and from B*, such that A, B, C are in the order ABC.

THEOREM I. *Axiom II_1 is a consequence of Axioms III and VI and Def. I.*

Proof. By hypothesis, ABC.† Hence, by Def. I, C and B are on AB. Therefore, by Axiom VI, A is on CB. Hence, by Def. I, ACB, CAB or CBA. The second of these alternatives is ruled out by Axiom III and the hypothesis ABC. Of the two remaining alternatives, if ACB were true then, by Def. I, A and B would lie on the line CB and therefore, by Axiom VI, C would be on the line BA; and thus, by Def. I, it would be true that either CBA, BCA or BAC. But BCA is ruled out by ABC and Axiom III, while BAC is ruled out by ACB and Axiom III. Hence it must be true that CBA.

LEMMA I. From Axiom VI and Def. I it follows that if the points C, D, E are on the line AB and $C \neq D$, then E is on the line CD.

Proof. If $C \neq E$ then, by hypothesis and Axiom VI, A is on CE and, by Def. I, E is on CE. Hence, by Axiom VI, if $A \neq E$, then C is on EA. Similarly if $A \neq E$ and $D \neq E$ then D is on EA. Hence if $A \neq E$, $C \neq E$, $D \neq E$, then, by Axiom VI, E is on CD. If $E = C$ or $E = D$ then, by Def. I, E is on CD. If $E = A$ then, by Axiom VI, E is on CD.

LEMMA II. From Axiom III, VI, Def. I and Axiom VII it follows that if A and B are two distinct points, then there exists a point C such that A, B, C are not on the same line.

Proof. If there exists no such point C then, by Lemma I, every point C is on the line AB. Hence, by another application of Lemma I, if D, E, F are any three distinct points, then F is on DE and also on ED — which is contrary to Theorem I, Axiom VII and Def. I.

THEOREM II. *From Axiom I, III, V′, VI, VII, VIII and Def. I it follows that, whatever points A and B may be, the order ABB is impossible, the order AAB is impossible and the order ABA is impossible.*

Proof. If $B = A$ then ABB is impossible by Axiom III.

If $B \neq A$, then by Lemma II there exists a point C such that A, B and C are not on the same line. By Axiom V′ there exists a point D, different from B, such that BCD. By Def. I, C is on the line BD. If A, B and D were col-

* Here Axiom II_2 is to be proved by showing that its hypothesis cannot be satisfied in the presence of Axioms I, III, V′, VI, VII, VIII and Def. I.

† "ABC," understood as a sentence, means that A, B, C are in the order ABC.

linear then, by Lemma I, A also would be on the line BD and then, by Axiom VI, B would be on the line CA — which would be contrary to hypothesis. Hence A, B and D are not collinear. Suppose now that ABB. Then, by Axiom VIII, there would exist a point E such that DEA and such that E, C, B lie on the same line (which, by Lemma I, must be the line BC). But, since A, B and D are not collinear, therefore, by Lemma I, D is the only point common to the lines AD and BC. Hence E must be D, and therefore DDA. Now, by Axiom V′, there exists a point F, distinct from D and from B, such that BDF. By Theorem I, FDB. Moreover, since F lies on BD and is different from B and from D, and A, B and D are non-collinear, therefore, by Lemma I, A, D, F are not on the same line. Hence, by Axiom VIII, since FDB and DDA, there exists a point K such that AKF and such that B, D and K are on the same line. It may easily be seen that K is F. Hence AFF. Now A, F and B are non-collinear and furthermore AFF and FDB. Hence, by Axiom VIII, there exists a point T such that BTA and such that F, D and T lie on the same line. Clearly T must be B. Hence BBA. Thus if ABB, then BBA — which is an impossible combination according to Axiom III. Hence ABB is impossible.

Suppose secondly that $A \neq B$ and AAB. Then by hypothesis, Lemma II, Def. I and Axiom V′ there exists a point C such that A, B, C are not on the same line and a point D different from A and from C such that CAD. By Axiom VIII there exists a point H such that BHC and such that D, A and H are on the same line. Then clearly H is C and thus BCC. But this has been shown to be impossible. Thus AAB is impossible.

Similarly it may be shown that if $A \neq B$ and ABA then there exists a point D such that DAA. Thus ABA is impossible.

THEOREM III. *Axioms II and IV are consequences of Axioms I, III, V′, VI, VII, VIII and Def. I.*

Theorem III is a corollary of Theorems I and II.

In the demonstration of Theorem II advantage was taken of what might perhaps be called a sort of transitivity or cyclical arrangement among the orders BCD, CEA, AFB of Axiom VIII. If, in this axiom, "AFB" should be replaced by "BFA," no other change being made in Axioms I–XII, then, for the set of axioms, thus slightly modified, Theorem II would no longer hold. To see this consider the independence example in which points and order are those of ordinary Euclidean geometry except that, *in addition* to the ordinary ordering, points A, B, C are in the order ABC whenever $A \neq B$ and $B = C$.

THE UNIVERSITY OF PENNSYLVANIA,
PHILADELPHIA, PENNSYLVANIA.