MULTIPLE CORRESPONDENCES DETERMINED BY THE RATIONAL PLANE QUINTIC CURVE*

BY

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§ 1. *Conjugate Quintic Curves.*

The occurrence of rational curves in pairs is a well-known fact: thus, given a rational curve ρ_p^n, of order n, in a space of p dimensions, there is uniquely determined, to within a collineation, a curve ρ_{n-p-1}^n, of order n, in a space of $n-p-1$ dimensions, by requiring all hyperplane sections of either curve to be apolar to the hyperplane sections of the other.† We call two curves associated in this way *conjugate* curves.

If the rational curve ρ_p^n is regarded as the projection of the norm-curve ρ_n^n in a space S_n of n dimensions, from an S_{n-p-1}, the interpretation of this fact is immediate. An S_{n-1} in S_n meets ρ_n^n in n points which may be regarded as given by a binary form of order n: dually, a point of S_n determines on ρ_n^n a set of n points, which may be given by a second binary form. The condition of apolarity of the two forms is precisely the condition of incidence of point and S_{n-1}.

All S_{n-1}'s having n-point contact with ρ_n^n meet S_{n-p-1}^n in the hyperplanes of a curve r_{n-p-1}^n of class n. The curves obtainable by projection from S_{n-p-1} and section by S_{n-p-1} are conjugate curves.

In this paper we shall deal with the case $n=5$, $p=2$, the rational plane quintic. If our curve r_2^5 is given parametrically by

$$x_i = (\alpha_i t)^5 \qquad (i=0,1,2), \tag{1}$$

and $(b_i t)^5$ are three linearly independent quintics apolar to the α's, the conjugate quintic of (1), which we take for convenience as a curve of lines, may be given by

$$\eta_i = (b_i t)^5 \qquad (i=0,1,2). \tag{2}$$

The quintic (1) is equally the conjugate quintic of (2). Regarding (1) and (2) as situated in two independent planes π_x and π_y, we shall prove the existence

* Presented to the Society, December 28, 1910.

† W. F. MEYER, *Apolarität und rationale Kurven*, p. 9.

of certain multiple correspondences between these planes, and point out the significance of these correspondences for the curves (1) and (2).

We shall make use of the norm-curve in five dimensions to prove the existence of these correspondences and to deduce their characteristic properties.

A set of the fundamental involution of (1) is defined on (2) by tangent lines through a point y of π_y; similarly, a set of the fundamental involution of (2) is defined on (1) by its points of intersection with a line ξ of π_x.

In the first part of this paper we shall take (1) as

$$\xi_i = (\alpha_i t)^5,$$

where ξ_i are the coördinates of the line marked out on a plane π_1 by the S_4 having 5-point contact with the norm-curve at the point t, and (2) as

$$y_i = (b_i t)^5$$

where the y_i are the coördinates of the projection of the point t of the norm-curve from π_1 on a plane π_2. In the second part we shall recur to the representation given by (1) and (2) in order to regard the fundamental correspondence as one of point to point.

The point of view of this paper is closely analogous to that of STAHL,* though STAHL does not make explicit use of the correspondence which we call T. It is possible to extend many of the theorems given to rational curves in general. BERZOLARI † has obtained some of these extensions.

§ 2. *The Rational Norm-quintic.*

A norm-curve, R, in S_5 may be given parametrically as follows:

$$(3)\quad x_0 = 1, \quad x_1 = -t, \quad x_2 = t^2, \quad x_3 = -t^3, \quad x_4 = t^4, \quad x_5 = -t^5,$$

or, in S_4's,

$$(4)\quad \xi_0 = t^5, \quad \xi_1 = 5t^4, \quad \xi_2 = 10t^3, \quad \xi_3 = 10t^2, \quad \xi_4 = 5t, \quad \xi_5 = 1.$$

We shall call an S_p having $(p+1)$-point contact with R simply an S_p of R.

Any point x of S_5 carries five S_4's of R. We have from (4) that the parameters of these S_4's are given by

$$(5)\qquad (xt)^5 \equiv x_0 t^5 + 5x_i t^4 + \cdots + x_5 = 0.$$

Dually, any S_4, ξ, of S_5 meets R in five points whose parameters are given by the binary quintic

$$(6)\qquad (\xi t)^5 \equiv \xi_0 - \xi_1 t + \cdots \xi_5 t^5 = 0.$$

* W. STAHL, *Über die Fundamental-involutionen auf rationalen Curven*, Journal für die reine und angewandte Mathematik, vol. 104 (1889), pp. 43–55.

† L. BERZOLARI, *Sulle curve razionali di uno spazio lineare ad un numero qualunque di dimensioni*, Annali di matematica, series 2, vol. 21 (1893), p. 1.

The apolarity-condition of (5) and (6) is the incidence-condition of point and S_4:

$$(\xi x) \equiv \xi_0 x_0 + \cdots + \xi_5 x_5 = 0.$$

The binary quintics (5) and (6) we may call, briefly, the quintics x and ξ, respectively, or merely x and ξ.

The curve R has three developables: D_1, the locus of lines of R; D_2, the locus of planes of R; and D_3, the locus of spaces of R. The locus D_1 is a 2-way spread (or simply a 2-way) of order 8; the locus D_2, a 3-way of order 9; and D_3, a 4-way of order 8. To these we may add D_0, the points of R, and D_4, the S_4's of R.

The general rational quintic curve in an S_p is obtainable from R in points by projecting R on S_p from an S_{5-p-1}; in S_{p-1}'s, by cutting the S_4's of R by S_p. The points of the latter curve form the curve of intersection with S_p of D_{5-p}. The curve has stationary points (apparent cusps) at the points of intersection with S_p of D_{5-p-1}.

§ 3. *The Lines l.**

Consider a point x of S_5. Quintics apolar to the quintic x are cut out of R by S_4's on x. The ∞^2 quartics apolar to x are cut out of R by spaces on x and meeting R four times. The unique cubic apolar to x—the so-called canonizant of x—is cut out of R by a unique plane on x and trisecant to R.

From the above we have at once:

(*a*) *If R is projected from a plane π_1 on a plane π_2, sets of the fundamental involution of the curve so obtained are quintics x, where x is chosen on π_1. Line-sections of the curve are quintics ξ where ξ is made to contain π_1. The two rational quintics obtainable from R by projection from π_1 and section by π_1 are conjugate curves.*

The general line, p, of S_5 carries a unique space quadrisecant to R; in the theory of binary forms this means that there is a unique quartic apolar to all quintics of a pencil. It is evident from the fact that apolarity relations are linear that if there are two quartics apolar to all quintics of a pencil, every quartic of the pencil built on the two is apolar to every quintic of the pencil of quintics. In S_5 the corresponding theorem is that if a line carry two spaces quadrisecant to R, it carries an infinity of such spaces. Lines having this property are called by Marletta *lines l*.†

Marletta points out the following facts:

(*b*) *Any two S_3's quadrisecant to R meet in a line l.*

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* G. Marletta, *Sulle curve razionali del quinto ordine*, Rendiconti del circolo matematico di Palermo, vol. 19 (1905), p. 94.

† G. Marletta, l. c., p. 95.

It follows that there are ∞^6 lines l in S_5, a line l being determined by any selected pencil of quartics on R.

(*c*) *Lines l on a space of S_5 are in a* $(3, 1)$ *congruence.*

(*d*) *Spaces quadrisecant to R and on a line l are on a quadric* 4-*way.*

We may add to these:

(*e*) *Lines meeting R are lines l.*

Let the given line meet R in a point t. From this line R is projected upon an S_3 into a rational quartic curve ρ_3^4 : the ∞^1 lines trisecant to the latter curve are the traces on the S_3 of ∞^1 S_3's on l and meeting R in three points and the point t. The pencil of quartics determined on R by l is here a fixed point t taken with a pencil of cubics. Pencils of cubics on R are thus put into $(1, 1)$ correspondence with lines through any point t of R.

(*f*) *Lines on planes trisecant to R are lines l.*

For let a plane π meet R in the points t_1, t_2, t_3. Then an S_3 may be put on $t_1t_2t_3$ and any point, t_4, of R; thus the plane π and hence any line on it carries ∞^1 S_3's quadrisecant to R.

§4. *The Spread of Lines Bisecant to R.*

The norm-curve R has ∞^2 bisecant lines; there are ∞^1 points on each line, and therefore ∞^3 points on all lines bisecant to R. It follows that these points are on a 3-way, which we call g_6. Any space, σ, quadrisecant to R meets g_6 in 6 lines—the 6 lines of the 4-point in which σ meets R. But these 6 lines are the *complete* intersection of g_6 and σ. If not, let there be a point a on σ and not on one of these lines. Then the S_4 on σ and on the line through a bisecant to R will meet R in 6 points, but this is impossible.* Hence

(*g*) *The* 3-*way g_6 is of order six.*

The following is obvious from § 2:

(*h*) *The spread g_6 is the locus of quintics x having an apolar quadratic, i. e., whose canonizant vanishes identically.*

The coefficients of the canonizant of $(xt)^5$ represent four linearly independent cubic spreads on g_6. Also,

(*i*) *The spread g_6 is the locus of cyclic quintics x, i. e., quintics reducible to the form $\alpha t_1^5 + \beta t_2^5$*, this quintic having the apolar quadratic $t_1 t_2 = 0$.

The intersection of two spreads in S_5 will be indicated by writing them consecutively: thus, α being a space in S_5, $g_6\alpha$ is the sextic curve in which g_6 meets α, and $g_6\pi$ is six points in a plane π.

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* A similar argument may be used to show that the spread of lines bisecant to the rational norm-curve in S_n is a 3-way of order $\frac{1}{2}(n-1)(n-2)$, which is the number of nodes of the rational plane n-ic, as it should be. Cf. CASTELNUOVO, *Studio dell' involuzione sulle curve rationali mediante la loro curva normale dello spazio a n dimensioni*, Atti del R. Istituto Veneto, series 6, vol. 4 (1886), p. 1167.

(*j*) *The sextic $g_6\alpha$ admits* ∞^1 *inscribed* 5-*planes.*

For any one of the ∞^1 S_4's on α meets R in five points; the ten lines joining these five points two and two meet α in the ten points of a 5-plane inscribed in $g_6\alpha$. The ten lines of this 5-plane are given by planes on three of the five points, and the planes, by spaces on four of the five points.

(*k*) *Any line trisecant to $g_6\alpha$ is the trace on α of a plane trisecant to R.*

For, let a line p meet g_6 three times. p then meets three lines, p_1, p_2, p_3, each bisecant to R — let these three lines meet R in the points a_1, b_1; a_2, b_2; a_3, b_3, respectively. On p and p_i we may then put an S_4 meeting R six times, if a_i, b_i are all distinct points. It follows that they cannot be all distinct. Let say $b_1 = b_2$. Then p_1 and p_2 meet and p_1, p_2, p must be on a plane trisecant to R, and the theorem is proved.

By (*j*), $g_6\alpha$ is a sextic curve in space admitting ∞^1 inscribed 5-planes. Such a curve is known to possess the following properties:*

(l_1) *It is a curve of genus* 3.

(l_2) *Planes of a group of* 5 *inscribed in the curve are planes of a fixed norm-curve,* r^3.

(l_3) *The groups of inscribed* 5-*planes mark on* r^3_α *sets of an involution* $I_{1,4}$.†

From theorems (*f*) and (l_2) we have

(*m*) *The* (3, 1) *congruence of lines l in a space α is the congruence of axes of* r^3_α.

The curve R determines on any space α the following:

(n_1) *The rational curve $D_2\alpha$ of order nine and class five, having stationary point at the eight points $D_1\alpha$.*

The planes of $D_2\alpha$ are the planes $D_4\alpha$.

The curve $D_2\alpha$, of class five, given in α is sufficient to determine all geometrical forms in D obtainable from R and α; R and α completely characterize $D_2\alpha$ and all its concomitant forms. If an invariant of $D_2\alpha$ vanishes α must have some special position with reference to R.

(n_2) *The curve $g_6\alpha$,* covariantly associated with $D_2\alpha$ in a manner which will be pointed out in the following section. This curve $g_6\alpha$ is obviously the locus of a point x of α such that the five parameters of planes of $D_2\alpha$ through x form a cyclic quintic.

(n_3) *The curve r^3_α, the locus of* 5-*planes inscribed in $g_6\alpha$.*

(n_4) *The locus of lines trisecant to $g_6\alpha$; a ruled surface of order eight having $g_6\alpha$ as a triple curve.*

We shall call this surface Φ_α.

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* A. C. Dixon, *On systems of three quaternary quadrics*, etc., Proceedings of the London Mathematical Society, series 2, vol. 7 (1909), p. 150.

† $I_{1,4}$: An involution of five things, one of which determines four, a pencil of quintics.

§ 5. *A Cremona Transformation Between Two Spaces.*

It is of interest to characterize briefly a cremona transformation determined by R between two spaces α and α' of S_5. Points of $g_6\alpha$ and $g_6\alpha'$, sextic curves of genus 3, are singular points of this transformation—a property which it possesses in common with the cubo-cubic cremona transformation determined by three bilinear forms.

In S_5 a plane meets a space, in general, in a point. Given R and the two spaces α and α', we have through the general point x of α a unique plane π_x trisecant to R.* π_x meets α' in a point x', which is unique when x is given. We have thus a cremona transformation (x, x') between the spaces α and α'. Let us call this transformation W.

Any point x, of $g_6\alpha$ is on a line bisecant to R. The point x then carries ∞^1 planes trisecant to R. The curve R is projected from the bisecant line on x by planes of a cubic 3-way cone which meets α' in a cubic curve. Hence to the points of $g_6\alpha$ correspond cubic curves in α', and, similarly, to the points of $g_6\alpha'$ correspond cubic curves in α.

Points of $g_6\alpha$ and $g_6\alpha'$ *are triple singular points of* W.

Through a line of Φ_α, there is a plane trisecant to R; hence to any point of such a line corresponds the unique point, x, in which this plane meets α.

There are ∞^1 such planes meeting α' in the ∞^1 lines of $\Phi_{\alpha'}$. Their locus is a 3-way which meets α in a curve K_α of simple singular points of W. This 3-way is met by any S_4 on α' in $\Phi_{\alpha'}$ and in 10 planes, a total intersection of order $10 + 8 = 18$. Hence

The curve K_α is of order 18. *Its genus is* 3, *since it is in one-one correspondence with the lines of Φ_α, and these, in turn, are by the* 5-*planes inscribed to* $g_6\alpha'$, *in one-one correspondence with the latter curve.*

A line of Φ_α is a fundamental line in α, i. e., a line such that all of its points have only one correspondent. It contains three triple singular points, its intersections with $g_6\alpha$. Hence it is represented in α' by a point, taken with three fundamental cubic curves, of toal order 9. Hence

The transformation W is of order 9.

Let us call the locus of cubic curves in α' corresponding to points of $g_6\alpha$, $G_{\alpha'}$ and the similarly determined surface in α, G_α. We desire to know the order of these surfaces. A plane π_α in α has as correspondent a nonic surface $\pi_{\alpha'}$ in α'. $\pi_{\alpha'}$ has $K_{\alpha'}$ as a simple curve, $g_6\alpha'$ as a triple curve. $\pi_{\alpha'}$ is sent by W into:

the plane π_α of degree 1,
the surface Φ_α of degree 8,
the surface G_α, taken 3 times, of degree 72,
making a total of degree 81.

* See § 3, first paragraph.

It follows that G_a is of order 24.

Certain general deductions can be made from the above observations, namely,

Planes trisecant to R and meeting a given line are on a 3-way of order 9. *Planes trisecant to R and meeting a given plane are on a 4-way of order* 9. *Planes trisecant to R and meeting a given space in a line are on a 3-way of order* 18.

§6. *Osculants.*

First osculants of R as a curve of points are the curves $D_1 \xi$, where ξ is any S_4 of R.* Dually, first osculants of R as a curve of S_4's are the perspections of the spaces of R from point x on R. Similarly, mixed cubic osculants of R are perspections of its planes from its bisecant lines; mixed conic osculants, perspections of its lines from its trisecant planes.†

For a rational curve, projections of osculants are osculants of projections; dually, sections of developables of osculants are osculants of sections.

Let π be any plane of S_5, α, any space on π, and ξ, any S_4 on α. There is determined on π the rational curve $D_3\,\pi$ of class 5. The 6 points $g_6\,\pi$ may be called the *cyclic points* of $D_3\,\pi$, since tangents to the latter curve from one of these points touch in five points whose parameters form a quintic reducible to the form $\alpha t_1^5 + \beta t_2^5 = 0$. The $S_4\,\xi$ meets R in five points $1, 2, 3, 4, 5$—a set of the fundamental involution of $D_3\,\pi$.

On $1, 2, 3, 4, 5$ there are the ten lines 12, ten planes 123, and five spaces 1234. The osculant 1 of $D_2\,\alpha$ has stationary points at 12α, 13α, 14α, 15α. These are four points of $g_6\,\alpha$. We have then

The curve $g_6\,\alpha$ is the locus of stationary points of quartic osculants of $D_2\,\alpha$.

The 3-way r_a^3 is the locus of planes of tetrahedra of stationary points of quartic osculants of $D_2\,\alpha$.

The second osculant 12 of $D_3\,\pi$ has stationary points at the points 123π, 124π, 125π. For $D_3\pi$ we have

(*p*) *Any set of the fundamental involution of $D_3\,\pi$, say $1, 2, 3, 4, 5$, determines ten mixed cubic osculants O_{ijk} of $D_3\,\pi$. There exist five lines of π, $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5$, such that the 3-points $\mu_i\,\mu_j$, $\mu_j\,\mu_k$, $\mu_k\,\mu_i$ are the stationary points of the osculant O_{ijk}. That is, the stationary points of cubic osculants of pairs of points selected from a set of the fundamental involution of $D_3\,\pi$ are ten points of a 5-line.*

§7. *The Curves $r_a^3 \cdot \pi$* ‡ *and $\Phi_a\,\pi$.*

The $(3, 1)$ congruence of lines l in a space α is the congruence of axes of r_a^3. The class of this congruence being one, there is a unique line l on any plane π. Hence,

* Cf. Berzolari, l. c., p. 7.

† Let $\xi_i = (\alpha_i t)^n$ be the parametric equations of a rational curve: $\xi_i = (\alpha_i t_1) \cdots (\alpha_i t_k)(\alpha_i t)^{n-k}$ is a *mixed* osculant; $\xi_i = (\alpha_i t_1)^k (\alpha_i t)^{n-k}$ is a *pure* osculant.

‡ We indicate by $r_a^3 \cdot \pi$ the curve of lines in which planes of r_a^3 meet π.

The curve $r_\alpha^3 \cdot \pi$ has the unique line l on the π as double tangent.

Any two spaces α and α' on π are contained in a common S_4, ξ. Hence,

The two curves $r_\alpha^3 \cdot \pi$ and $r_{\alpha'}^3 \cdot \pi$ have, besides the line l on π, five lines $\mu_i^{(\xi)}$ in common, $\mu_i^{(\xi)}$ being the five lines in which spaces on four out of the five points $R\xi$ meet π.

The pencil of S_4's, ξ, on α, a space containing π, cut out of R a pencil of the fundamental involution of $D_3\,\pi$. This leads to the following

Given the curve $D_3\,\pi$ and any set, f, of the fundamental involution on this curve, stationary points of cubic osculants of pairs of points chosen from the roots of f are ten points of a 5-line. If f vary in a pencil the locus of the 5-lines so determined is a cubic curve having the covariant line l on π as double tangent.

Given two lines p and p' of π, there is a unique space σ quadrisecant to R on p, and on σ and π there is a unique S_4, ξ. Similarly p' determines a space, σ', and an S_4, ξ'. ξ and ξ' meet in an S_3, α, containing π. The curve $r_\alpha^3 \cdot \pi$ is thus uniquely determined by requiring it to touch p and p'. Hence *

The curves $r_\alpha^3 \cdot \pi$ are in a linear system of ∞^2 cubics having the double tangent l.

Again, given two points x and x' of π, there are on x and x' respectively two planes π_1 and π_1' trisecant to R. On π_1 and π there is an S_4, ξ, and on π_1' and π, an S_4, ξ'. The two S_4's, ξ and ξ' meet in a space α on π. The surface Φ_α contains the lines $\pi_1\alpha$ and $\pi_1'\alpha$, and $\Phi_\alpha\,\pi$ is hence on x and x'. There is thus a unique curve $\Phi_\alpha\,\pi$ on two general points of π, and we have this theorem:†

The curves $\Phi_\alpha\,\pi$ are in a linear two-fold system. All curves of this system have as triple points the six cyclic points $g_6\,\pi$ of $D_3\,\pi$.

§ 8. *The Correspondences T and U.*

Projecting R from a plane π_ξ on a plane π_η we obtain a rational quintic curve ρ_2^5. The S_4's on π_ξ cut out of R line-sections of ρ_2^5 and points of π_ξ determine on R sets of the fundamental involution of ρ_2^5, i. e. the curves $D_3\,\pi_\xi$ and ρ_2^5 are conjugate quintics.

We determine a correspondence T between the lines ξ of π_ξ and the lines η of π_η in the following manner: A line ξ of π_ξ carries in general a unique space σ_ξ quadrisecant to R; the only case in which σ_ξ is not uniquely determinate is when ξ is the line l on π_ξ. On σ_ξ and π_ξ there is an S_4 which meets π_η in a line η. T is defined as the correspondence (ξ, η).

Conversely, given a line η of π_η there is a unique S_4 on η and π_ξ. This S_4 meets R in five points; on any four of these five points there is an S_3, σ, quadrisecant to R. Each of these five spaces meets π_ξ in a line ξ corresponding to η. Hence

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* Bertini, *Geometria projettiva degli iperspazi*, p. 224.

† Bertini, l. c.

T is a $(5, 1)$ correspondence.

Comparing with theorem (d) what we have just said, we have the following:

The line l on π is a singular line of T; to l correspond lines of a conic in π_ξ; the line l is the only singular line of T in either plane.

Any space α on π_ξ marks a point y_α on π_η. All S_4's on α give lines η on y_α. To these lines correspond by T lines of the cubic curve $r_\alpha^3 \cdot \pi_\xi$. Hence

The correspondence T is cubic. To points y_α of π_η correspond lines of the curves $r_\alpha^3 \cdot \pi_\xi$.

There is a second correspondence, U, between π_ξ and π_η which we obtain by associating with a line, η, of π_η, the ten points of intersection of its corresponding lines ξ. The line η is unique when a general point x of the plane π_ξ is given, since there is on x a unique plane trisecant to R, and on this plane and π_ξ there is an S_4 which meets π_η in the line η. To lines on a point y_α of π_η correspond by U points of a curve $\Phi_\alpha\ \pi_\xi$. Hence

The relation U is an octavic 10-*to*-1 *line-point correspondence between the planes π_ξ and π_η. It has the six triple singular points $g_6\ \pi_\xi$.*

The latter statement follows from the fact that the curves $\Phi_\alpha\ \pi_\xi$ have $g_6\ \pi_\xi$ as triple points. To a cyclic point in π_ξ correspond by U, therefore, lines of a cubic curve in π_η. This cubic is necessarily rational.

§ 9. *T and U Considered Dually.*

We have now to study T and U from the point of view of the general theory of correspondences. It will be more convenient to discuss T as a point-point correspondence, and hence to consider dually the facts which we have obtained. We give briefly the dual statement of the essential facts, without reference to the arguments from hyperspace which we have been using hitherto.

Given a rational quintic curve, ρ^5, in a plane, π_x, its conjugate quintic, r^5, is determined to within projections. We consider r^5 as a curve of lines and regard it as situated in a plane π_y; the curves ρ^5 and r^5 then assume the forms (1) and (2) respectively (§ 1), and are in $(1, 1)$ correspondence through the parameter t.

Tangents to r^5 from a point y define on r^5 and hence on ρ^5 a set of the fundamental involution of ρ^5. The inflexional lines of mixed cubic osculants of pairs of parameters chosen from this set are ten lines of a 5-point, $x_1, \cdots, x_5$. To y correspond by the correspondence which we have called T the five points x. Given a point x, its y is uniquely determined. If y move on a line η in π_y, the corresponding points x_i move on a rational cubic, c_x, in π_x. All curves c_x are in a linear two-fold system: they have a common node O, which is a rational covariant point of ρ^5, and is characterized by the fact that the pencil of binary quintics cut out of ρ^5 by lines on O has a pencil of apolar quartics; in other words, *these quintics are the first polars of a binary sextic* (§ 3).

To proceed: A curve c_x is in natural $(1, 1)$ correspondence with ρ^5 and the groups x_i on c_x as a rational support are the pencil of the fundamental involution of ρ^5 which determines it. This pencil is cut out on any curve c_x by other curves of the system. In fact, the common parameter on all curves c_x is the parameter of a line in the pencil with vertex at O. Otherwise we would obtain from any two curves c_x two projective pencils with vertex at O and these two pencils would have five self-corresponding elements.

A rational plane quartic, ρ^4, has an important covariant conic defined in a manner similar to the curves c_x. It is the locus of vertices of triangles of stationary lines of cubic osculants of the quartic. The pencil of quartics in the fundamental involution of ρ^4 give a single infinity of 4-points of this conic. Stahl calls this conic the conic K of ρ^4.

If η is a line of r^5 with parameter t_0, $T\eta$ is the Stahl conic K of the osculant of ρ^5 at t_0, taken with a line through O. For the mixed cubic osculant $t_0 t_1$ is the osculant t_1 of the osculant t_0.* We have, then,

All conics K of first osculants of ρ^5 are on the point O.

If ξ is a line in π_x on O, $T\xi$ is a line of r^5, the conic TO factoring out of the transform of ξ. The conic TO and r^5 are in $(1, 1)$ correspondence with lines on O, directions around O corresponding to points of TO; we may therefore choose for each the parameter of a line of this pencil. The line t of r^5 is then on the point t of TO. Hence

The conic TO is the perspective conic of r^5.†

In the correspondence U a point y is made to correspond to the ten lines joining the five points x_i corresponding to y by I. $U\eta$ is a curve of class 8 having the cyclic lines of ρ^5 as triple tangents. These are triple singular lines of U. This correspondence has no singular points in π_y.

The locus $T\xi$ is a cubic curve C_y in π_y, necessarily rational, since it is in $(1, 1)$ correspondence with ξ. It follows that ξ contains a single pair of associated points, x_1, x_2, corresponding to the node y of $T\xi$. The transform $U\xi$ is the point y. The points x_1 and x_2 are the neutral pair of the $I_{2,1}$, of points in which ξ is met by curves of the net c_x.

Since the curve r^5 has six double tangents, it follows that there are six curves c_x which are made up of three lines. Call the double tangents of r^5, β_i. Then,

$$T\beta_i = \alpha_{i1}\alpha_{i2}\,\alpha_i;$$

where α_{i1}, α_{i2} are two lines on O and α_i is a cyclic line of ρ^5 : the latter statement follows from the fact that $U\alpha_i$ is indeterminate. Cubics c_x meet $\alpha_{i1}\alpha_{i2}\alpha_i$ in groups of 5 points x_i. Two of these, x_1 and x_2, say, are on α_{i1} and α_{i2}

* For the properties of K see papers by STAHL, Journal für die reine und angewandte Mathematik, vol. 101 (1887), p. 304, and vol. 104 (1889), p. 304.

† W. STAHL, Mathematische Annalen, vol. 3 (1891), p. 579.

respectively. The others then are on the line α_i and form an $I_{1,2}$. The points x_1, x_2, varying, determine on α_{i1} and α_{i2} two projective ranges and these are in turn projective to the corresponding points y of β_i. Lines joining corresponding points x_1, x_2, on α_i, α_{i2}, touch a conic q_i. This conic from its mode of generation touches α_{i1}, α_{i2}, and all lines α_j where $j \neq i$. Now Uq_i is β_i. But q_i is related projectively to β_i by U in such a way that to the line α_j corresponds the point where β_i meets β_j. This is a characteristic property of the singular lines of a quintic cremona line-line transformation with 6 double singular lines. Furthermore, given two sets of lines having this property, the transformation is uniquely determined. Then given a line ξ of π_x, and η, its transform in π_y, the points $\xi\alpha_i$ are projective to $\eta\beta_i$.* We may state this in the following theorem:

The cyclic lines of ρ^5 and the double lines of r^5 are double singular lines of a quintic cremona line-line transformation between the planes π_x and π_y.

A line ξ on O is transformed by T into a line η of r^5; since there is one variable point x on ξ, the points of ξ are related projectively to the points of η, and in such a way that $\xi\alpha_i$ are projective to $\eta\beta_i$. Hence,

This quintic transformation sends lines on O into lines of r^5.

The transforms of points y of π_y are a net of quintic curves with α_i as double tangents. To the 5 lines of r^5 on y correspond the five lines of a quintic of this net on O. This gives at once a determination of the fundamental involution of ρ^5 in the pencil of lines about O:

Given two lines t_1 and t_2 on O, there is a unique curve of class five on t_1 and t_2 and having the α_i as double lines. The other three lines of this curve, say t_3, t_4, t_5 on O form with t_1 and t_2 a set of the fundamental involution of ρ^5.

§10. *The Involution of Points x_i.*

Points $x_1, \cdots, x_5$ having the same correspondent y by T are groups of an involution, I, in π_x. The correspondence x_i, x_j is obviously $(4, 4)$. Since

$$T^2\,\xi = \xi I \xi,$$

and $T^2\,\xi$ must be a nonic curve, it follows that $I\xi$ is an octavic curve. Hence

The involution I is octavic.

The only singular point of I is O. We have obviously

$$IO = Tc_0,$$

c_0 being the perspective conic of r^5 in π_y. The curve IO is therefore a sextic and it follows that $I\xi$ must have a 6-fold point at O. The curve $I\xi$ has besides the three nodes $a_3, a_4, a_5, a_1, \cdots, a_5$ being a group of I and a_1, a_2 being the unique pair of I on ξ.

* A. B. Coble, these Transactions, vol. 9 (1908), p. 398.

The curve I^2O is of order 48. It can be nothing but IO taken eight times. There is an involution of points x_2, x_3, x_4, x_5 on IO, x being indefinitely near to O. We should therefore expect IO to appear three times in $I^2 O$; to make up the necessary eight times IO must pass five times through O. Hence,

The curve IO is a Jonquières sextic with 5-fold point at O.

The Jacobian of the net of curves c_x is a Jonquières curve, J, with 5-fold point at O. Note also that J is the coincidence curve of the involution I. *The curve TJ is the curve r^5 taken in points, an octavic.*

Further, IJ is a curve of order 12, J factoring once and IO 5 times out of the transform of J by I.

Any question connected with the osculants of ρ^5 must be intimately associated with the transformations which we have been discussing. We give a few examples of this statement.

Let $x_1, \cdots, x_5$ be a set of the involution I; further let x_1 amd x_2 be indefinitely near to each other, and hence to J. Let

$$t_1 = t_2, \quad t_3, \quad t_4, \quad t_5$$

be the corresponding parameters on ρ^5—a set of the fundamental involution. The pure cubic osculant t_1 has a triangle of inflexional lines $x_3 x_4 x_5$. Hence

The curve IJ is the locus of vertices of triangles of inflexional lines of pure cubic osculants of ρ^5.

The curve $I^2 J$ is J taken six times; IJ twice; and IO six times to make up the necessary order of $I^2 J$, 96. Hence,

The transform IJ passes six times through O.

If a rational cubic have a cusp, two inflexional tangents have become the cusp tangent, the third is the single proper inflexional tangent of the curve. Consider the osculant t_4, t_5 with inflexional triangle x_1, x_2, x_3. The points x_1 and x_2 are indefinitely near; the lines $x_1 x_3$, $x_2 x_3$ are indefinitely near. This osculant must have a cusp; $x_1 x_2$ is the inflexional tangent and x_3 is the cusp. Hence

The curve IJ is also the locus of cusps of cuspidal mixed cubic osculants of ρ^5. Six such osculants have cusps at O.

The line $x_1 x_2$ is the inflexional tangent of the three cuspidal osculants t_4t_5, t_5t_3, t_3t_4. The locus of this line must be a rational curve, since it is in $(1, 1)$ correspondence with J. Call this locus Γ. Cubics c_x meet a line α_i in a pencil of binary cubics; there are four points on α_i at which a cubic of the pencil has a double root; hence,

The locus Γ, defined above, has the cyclic lines of ρ^5 as four-fold tangents.

Now $U\Gamma$ is ρ^5 taken in points, a curve of order eight. If m is the class of Γ we have

$$8m - 3 \cdot 4 \cdot 6 = 8,$$

whence $m = 10$. Hence

The locus of inflexional lines of cuspidal cubic osculants of ρ^5 *is a rational curve of class ten having the cyclic lines of* ρ^5 *as 4-fold lines. The transform of this curve by* U *is the curve* r^5, *taken in points.*

§ 11. *Perspective Curves of* r^5.

It will be seen at once that any Jonquières curve with multiple point at O in π_x is in one-one correspondence with lines on O and is transformed by T into a rational curve *perspective to* r^5. Thus lines of π_x give ∞^2 perspective cubics of r^5, that is,

The curves C_y *are perspective cubics of* r^5.

The transforms by T *of the* ∞^{2m} *curves* $j^{(m)}$ *of order* m *with* $(m-1)$*-fold point at* O, *are the* ∞^{2m} *perspective* $(m+2)$*-ics of* r^5.* *The points of contact of the perspective curves with* r^5 *are the transforms of the points of intersection of these curves* $j^{(m)}$ *with* J.†

Some interesting properties of systems of perspective curves are easily obtainable from this point of view. For instance, curves c_y on a point y break up into 5 systems, the transforms of pencils of lines on the 5 points x_i. The contacts of curves of each system are in a pencil. The locus of nodes of a system x_i is the curve Ux_i, a rational octavic. There are 25 perspective cubics of r^5 on two points.

It is known that, if a rational sextic plane curve is given, and a quadratic involution on it, the joins of corresponding pairs of this involution touch a curve of class 5. A curve related in this way to r^5 is TC, where C is any conic of π_x. Now a rational sextic with a given involution on it has 19 constants, the number involved in our scheme if we include the conic C. Presumably *all* rational sextics related to r^5 in this way are obtainable as transforms of conics of π_x. The contacts of these curves with r^5 are the transforms of the meets of conics of π_x with J. These groups of 12 points are in an $I_{5,7}$.

A word as to the possibility of extension of this method seems advisable. Curves of a net of Jonquières curves $c_x^{(m)}$, of order m, in a plane π_x, and with fixed multiple point O, may be put into homographic correspondence with the lines of a second plane π_y. It is easily seen that an m-ic point-point $(2m-1, 1)$ correspondence, $T^{(m)}$, is thus determined between π_x and π_y.

To degenerate curves of the net $c_x^{(m)}$ *correspond by* $T^{(m)}$ *lines of a curve* r^{2m-1} *of class* $2m-1$ *in* π_y.

The constants are exactly right for r^{2m-1} to be a general rational curve of class $2m-1$; in fact there is an argument from hyperspace which develops the apparatus and proves that this curve is general.

* W. Stahl, loc. cit.

† A. B. Coble, *Symmetric Binary Forms and Involutions*, American Journal of Mathematics, vol. 32 (1910), p. 352.

The jacobian of the net $c_x^{(m)}$ is a Jonquières $(3m-3)$-ic with $(3m-4)$-fold point at O. Call this $J^{(3m-3)}$. Then $T^{(m)}\,J^{(3m-3)}$ is r^{2m-1} taken in points.

The curve $T^{(m)}O$ is the unique perspective $(m-1)$-ic of r^{2m-1}. The curves $T^{(m)}\,\xi$, ξ being a line of π_x, are the ∞^2 perspective m-ics of r^{2m-1}. We see, then, that the properties of systems of perspective curves which we have stated for the rational quintic are extensible to all rational curves of odd order.

Somewhat similar theorems are true for curves of even order; for, by making the curves $c_x^{(m)}$ pass through a second point O', the class of $r^{(2m-1)}$ is reduced by one.

§ 12. *Determination of ρ^5 and r^5 from the Curves c_x.*

Given the net of curves c_x with a common node O, the transformation T is, at once

$$y_i = (c_i x)^3,$$

where $(c_i x)^3$ are three linearly independent curves of the net, and r^5 is the locus of lines η whose transforms are degenerate cubics.

Stahl* obtains the general rational space quintic curve from a norm-curve R^3 and a binary octavic $(at)^8$ on it, in the following manner: A cubic polar $(a\tau)^5\,(at)^3$ considered as a binary cubic on R^3 defines a point x to which is associated the parameter τ. The locus of x is the rational quintic. This is the point of view of § 4 of this paper, the curve r_a^3 being the Stahl cubic of $D_2\,\alpha$. The involution $I_{1,4}$ on R^3 is the involution of quintics apolar to $(at)^8$.

By projection from a point of space the curve R^3 becomes a cubic c_x of the quintic ρ^5 which we obtain; hence ρ^5 is determined from *any* of the curves c_x (non-degenerate), in the following manner. On c_x is a pencil of binary quintics, an $I_{1,4}$, its intersection-groups with the other cubics of the system. This pencil has a unique apolar octavic $(at)^8$. Given a cubic polar $(a\tau)^5\,(at)^3$, of a, a point x is determined by requiring line-sections of c_x on the point x to be apolar to this cubic in t. Then x is the point τ of ρ^5.

The net c_x is determined uniquely by the lines α_i and the point O. On all lines α except α_i there is a conic q_i. The line α_i, taken with the tangents, α_{i1}, α_{i2}, from O to q_i is a cubic. The six cubics thus obtainable are in a net, and the net c_x and hence the quintic ρ^5 are uniquely determined. This leads to the following theorem:

The invariant theory of a plane rational quintic is identical with the covariant theory of six lines in a plane.

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December, 1910.

* W. STAHL, loc. cit.