PROOF OF POINCARÉ'S GEOMETRIC THEOREM

BY

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In a paper recently published in the Rendiconti del Circolo Matematico di Palermo (vol. 33, 1912, pp. 375–407) and entitled *Sur un théorème de Géométrie,* Poincaré enunciated a theorem of great importance, in particular for the restricted problem of three bodies; but, having only succeeded in treating a variety of special cases after long-continued efforts, he gave out the theorem for the consideration of other mathematicians.

For some years I have been considering questions of a character similar to that presented by the theorem and it now turns out that methods which I have been using are here applicable. In the present paper I give the brief proof which I have obtained, but do not take up other results to which I have been led.†

1. Statement of the Theorem. Poincaré's theorem may be stated in a simple form as follows: Let us suppose that a continuous one-to-one transformation T takes the ring R, formed by concentric circles C_a and C_b of radii a and b respectively ($a > b > 0$), into itself in such a way as to advance the points of C_a in a positive sense, and the points of C_b in the negative sense, and at the same time to preserve areas. *Then there are at least two invariant points.*

In the proof of this theorem we shall use modified polar coördinates $y = r^2$, $x = \theta$ where r is the distance of the point (x, y) from the center of the circles, and θ is the angle which a line from the center to (x, y) makes with a fixed line through the center. The transformation T may be written then

$$x' = \varphi(x, y), \qquad y' = \psi(x, y).$$

The function $\psi(x, y)$ is a continuous function of (x, y), uniquely determined at each point of R, and so is periodic in x of period 2π. The function $\varphi(x, y)$ admits of an infinite number of determinations which differ from each

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† Some of my results are contained in a paper entitled *Quelques théorèmes sur les mouvements des systèmes dynamiques*, which is shortly to appear in the Bulletin de la Société Mathématique de France.

other by integral multiples of 2π, and these determinations can be grouped so as to form continuous branches. Since $(x+2\pi, y)$ and (x, y) represent the same point of R, the algebraic difference between the values of one of these determinations taken for $(x+2\pi, y)$ and (x, y) reduces to an integral multiple of 2π; and this difference must be one and the same multiple of 2π for all x and y because the difference is a continuous function. But if the point (x, y) makes a positive circuit of the circle C_a, the same is true of its image (x', y'); hence along this path $\varphi(x, y)$ increases by 2π when x increases by 2π. Thus the difference reduces identically to 2π; in other words, the function $\varphi(x, y)$ increases by 2π when x is increased by 2π.

In consequence of these properties of $\varphi(x, y)$ and $\psi(x, y)$, it is clear that $x'-x$ and $y'-y$ are both single-valued and continuous in R.

The precise meaning of the theorem is that if any determination of $\varphi(x, y)$ is made for which

$$x' > x \text{ along } C_a \quad \text{and} \quad x' < x \text{ along } C_b,$$

(the conditions on T make possible such a choice) then we shall have at least two points (x, y) of R for which

$$x' = x, \qquad y' = y.$$

2. On the Method of Proof. As Poincaré remarks (loc. cit., p. 377), the existence of one invariant point implies immediately the existence of a second invariant point. *Hence if the theorem is false we may assume that there is no point invariant for T.* In this case we shall have

$$(x'-x)^2 + (y'-y)^2 > d^2 > 0 \tag{1}$$

for all points of R since $x'-x$ and $y'-y$ are single-valued, continuous, and not simultaneously zero, over the ring R. We shall establish the theorem by proving that the hypothesis (1) admits a *reductio ad absurdum.*

3. The Auxiliary Transformation. If $0 < \epsilon < b^2$, the one-to-one continuous transformation T_ϵ given by

$$x' = x, \qquad y' = y - \epsilon$$

takes the circles C_a and C_b into the concentric circles $C'_a : y = a^2 - \epsilon$ and $C'_b : y = b^2 - \epsilon$ respectively, so that C'_a is within C_a at a distance $a - \sqrt{a^2 - \epsilon}$ from C_a, and C'_b is similarly within C_b at a distance $b - \sqrt{b^2 - \epsilon}$ from C_b. This transformation effects a shrinking of the plane toward the origin which leaves every point on its radius vector and which preserves areas, since dx and dy are unaltered by the transformation and the integral of areas is

$$\iint r\, dr\, d\theta = \tfrac{1}{2} \iint dx\, dy.$$

Now, as long as $\epsilon < d$, the auxiliary transformation TT_ϵ, formed by following T by T_ϵ, also has no invariant point. For *if we take x and y as the rectangular coördinates of a point in the strip S:*

$$-\infty < x < +\infty, \qquad b^2 \leqq y \leqq a^2,$$

corresponding to the ring R, we see that the point (x, y) is displaced at least d units in S by T and then is further displaced a distance ϵ by T_ϵ in the direction of the negative y-axis, so that if $\epsilon < d$, the point cannot come back to its initial position. We write this compound transformation TT_ϵ in the form

$$\bar{x}' = \varphi(x, y), \qquad \bar{y}' = \psi(x, y) - \epsilon.$$

Let us choose the positive quantity ϵ once for all and so small that

(2) $$\epsilon < b^2, \qquad \epsilon < d, \qquad \epsilon < a^2 - b^2.$$

Consider now the multiple-valued function

(3) $$\omega(x, y) = \operatorname{arc\,tan} \frac{y' - y}{x' - x},$$

or, more accurately, those branches of this function which give the angle that the vector drawn from the point (x, y) to the point (x', y') in the strip S makes with the positive direction of the x-axis (the other branches corresponding to the negative of this vector). In virtue of (1), this function is continuous at every point of S and accordingly falls into branches single-valued and continuous throughout S.

Moreover any such branch reduces to a fixed even and odd multiple of π along C_a and C_b respectively, since along C_a we have $x' > x$, $y' = y$ and along C_b, $x' < x$, $y' = y$.

The functions $y' - y$ and $x' - x$ have been seen to be periodic in x of period 2π, and so any such branch of $\omega(x, y)$ differs at $(x + 2\pi, y)$ and (x, y) by a multiple of 2π which may reduce to zero. Since the branches are continuous, this multiple is one and the same throughout S. But along C_a and C_b these branches have a constant value, as we have noted. Hence the multiple will in fact reduce to zero. Thus these branches of $\omega(x, y)$ are periodic in x of period 2π, i. e., are single-valued in R.

Likewise it is clear that the multiple-valued function

(4) $$\bar{\omega}(x, y) = \operatorname{arc\,tan} \frac{\bar{y}' - y}{\bar{x}' - x},$$

which gives the angle that the vector drawn from (x, y) to $(\bar{x}', \bar{y}')$ makes with the positive direction of the x-axis, falls into single-valued and continous branches. Moreover the functions $\bar{y}' - y = y' - y - \epsilon$ and $\bar{x}' - x = x' - x$ are periodic in x of period 2π; also each of the branches of $\bar{\omega}(x, y)$ is periodic in x along C_a and along C_b. We conclude thatthe branches of $\bar{\omega}(x, y)$ as well as of $\omega(x, y)$ are periodic in x of period 2π, i. e., are single-valued in R.

The branches of $\omega(x, y)$ and $\bar{\omega}(x, y)$ may be associated in pairs so that the maximum numerical difference of any pair for every (x, y) is regulated in accordance with the formula

$$|\omega(x, y) - \bar{\omega}(x, y)| < \frac{\pi}{2}. \tag{5}$$

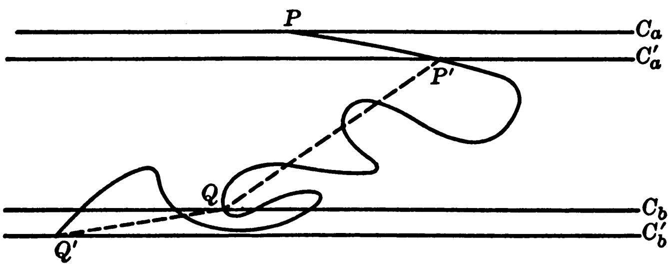
This is obvious geometrically, for the point (x', y') is at least d units distant from the point (x, y) in S, and $(\bar{x}', \bar{y}')$ is distant $\epsilon < d$ units from (x', y') so that the angle subtended at (x, y) by these last-named points can never become equal to $\frac{1}{2}\pi$. Thus if we associate the branches at a single point of S in accordance with (5), this inequality must continue to be true as the coordinates of the point vary continuously in S.

4. Construction of a Curve Invariant for the Auxiliary Transformation. The transformation TT_ϵ of R takes C_a to the circle C'_a of radius $\sqrt{a^2 - \epsilon}$, *between* C_a and C_b, since we chose $\epsilon < a^2 - b^2$. The repetition of TT_ϵ takes C'_a into C''_a, a simple closed curve. Now TT_ϵ is a one-to-one and continuous transformation of the ring R into the ring R', limited by C'_a, C'_b and obtained by letting R shrink under the transformation T_ϵ. Since C'_a lies within R and encloses C_b, its image C''_a must lie within R' (and accordingly within C'_a) and enclose C'_b. Furthermore the transformation TT_ϵ takes the ring formed by $C_a C'_a$ in R into a second ring $C'_a C''_a$ in R', which abuts on the inner boundary C'_a of the first ring. If C''_a lies wholly in R it will enclose C_b as well as C'_b, and the image C'''_a of C''_a by TT_ϵ will be a simple curve within C''_a and enclosing C'_b. Thus $C''_a C'''_a$ will form the boundary of a third ring abutting on the inner boundary of the second ring $C'_a C''_a$. This process may be continued to form simple curves C'_a, C''_a, $\cdots$ lying one within the other, and corresponding rings $C_a C'_a$, $C'_a C''_a$, $\cdots$, so long as the curves continue to lie wholly in R and not in part or wholly within C_b.

The area of the ring $C_a C'_a$ is $\pi\epsilon$ and will, *since the transformations T and T_ϵ each preserve areas,* be the same as that of its image rings $C'_a C''_a$, $C''_a C'''_a$, $\cdots$. This series of rings can only terminate when a curve $C^{(n)}_a$ is reached $(n \gtreqless 2)$ part, at least, of which lies within the circle C_b. But for every l, the area included between C_a and $C^{(l)}_a$ is $\pi l\epsilon$, which if l is sufficiently large exceeds the area of R. It follows that such a curve $C^{(n)}_a$ exists. Hence we can find a particular point P of C_a whose nth image lies inside of the circle C_b.

Now let us turn to the strip S and to one of the representations of P on the upper side C_a of the strip. Here C'_a is represented by a straight line at distance ϵ below C_a, and C''_a, C'''_a, $\cdots$ are each represented by a simple open curve congruent by sections $2k\pi \gtreqless x \gtreqless 2(k+1)\pi$ and extending indefinitely to right and left. The rings $C_a C_a'$, $C_a' C_a''$, $\cdots$ clearly are represented by the successive strata between successive curves of this set in S. Let P' be the first image of P and join PP' by a straight line which will lie entirely within the strip $C_a C_a'$. Let the arc $P' P''$ be the image of PP' under TT_ϵ, and let $P'' P'''$ be the image of $P' P''$ under the same transformation, and so on. In this way we construct successively PP', $P' P''$, $\cdots$, $P^{(n-1)} P^{(n)}$ lying respectively in the strata $C_a C_a'$, $C_a' C_a''$, $\cdots$, $C_a^{(n-1)} C_a^{(n)}$ in S, the end point $P^{(n)}$ of the last of these arcs falling below the lower side C_b of S.

Let Q be the first intersection of this succession of arcs with the lower side



C_b of the strip (see figure). It is obvious that the curve PQ formed by this succession of arcs is a simple curve, for PP', $P' P''$, $\cdots$ are successive simple arcs which lie in the successive strata $C_a C_a'$, $C_a' C_a''$, $\cdots$ in S. Furthermore PQ lies wholly between C_a and C_b.

The image $P' Q'$ of PQ under the transformation TT_ϵ is made up of $P' Q$ and an extended simple arc QQ', the image of the arc $Q^{-1} Q$ where Q^{-1} is the point that goes into Q by TT_ϵ. The arc QQ' being the image of points of R by TT_ϵ lies wholly below the straight line C_a'; the end-point Q' of this arc lies of course C_b'. Furthermore QQ' has no point but Q in common with PQ. For if such a point exists it must lie on $P' Q$, and by performing the transformation inverse to TT_ϵ we see that $Q^{-1} Q$ has a point other than Q^{-1} in common with PQ^{-1}, which is not possible since PQ has no multiple points. Thus PQ' forms a simple curve.

The transformation TT_ϵ takes the arc PQ of PQ' into the arc $P' Q'$ of PQ', advancing each point of PQ along PQ'. In this sense the curve PQ is *invariant* under the transformation.

The properties of T and T_ϵ ensure that P' has an x greater than that of P, and that Q' has an x less than that of Q, as indicated in the figure.

5. The Rotation of the Auxiliary Point-Image Vector on the Invariant Curve. If a point B moves along PQ' from P to Q, it is clear that its image $\bar{B}'$ by TT_ϵ moves from P' to Q' along the same curve, never coinciding with B. We shall now establish the fact (intuitively almost self-evident) that the corresponding rotation of the vector $B\bar{B}'$ thus obtained is $-\pi$ plus the sum of the two acute angles which the straight lines PP' and QQ' make with the x-axis.

At the outset it is obvious that the rotation can differ from this value only by a multiple of 2π. That the rotation has precisely the value stated depends entirely on simple considerations of *analysis situs*. The fact on which this conclusion rests is that a continuous deformation of the curve $PP'QQ'$ through a series of simple curves containing P, P', Q, Q' brings the curve to the broken line position $PP'QQ'$ (see figure).

Let t be any monotonic parameter for the curve PQ taking on the increasing values t_0, t_0', t_1, t_1', at P, P', Q, Q' respectively. Let $\tau(t)$ be the value of this parameter for $\bar{B}'$ where t is its value for the corresponding point B. Clearly $\tau(t)$ is a continuous increasing function of t $(t_0 \overline{\gtrless} t \overline{\gtrless} t_1)$ which has the property $\tau(t) > t$. Consider now any varied simple curve through the same four points P, P', Q, Q' on which a monotonic parameter t is so chosen that as before t_0, t_0', t_1, t_1' correspond to P, P', Q, Q' respectively.* If the distance between any point of the varied curve and the point of PQ with the same parameter value is uniformly small, the corresponding vector $B\bar{B}'$ along the varied curve will undergo precisely the same total rotation as along PQ, since the initial and final positions are the same in either case and the angular differences of the vectors in all intermediate positions are uniformly small also.

As a consequence of this reasoning it follows that we may deform the curve $PP'QQ'$ continuously through any series of simple curves containing the same four points P, P', Q, Q' provided that for the varied curves the parameter is properly chosen, and the total rotation of $B\bar{B}'$ will not thereby be altered.

The curve PQ lies wholly in S, so that the straight line QQ', being outside of S, does not intersect the curve PQ. It is therefore apparent that the arc QQ' may be continuously deformed to this straight line position QQ' without taking the curve QQ' outside of the strip formed by C_a' and C_b'; for the continuum formed by this strip is simply-connected after a cut in it by the curve $P'Q$ is made, and hence any simple curve joining the points Q and Q' of this continuum, and lying in it, can be continuously deformed into any other such simple curve through a series of simple arcs not having any points but Q and Q' in common with the boundary of the cut strip.

Next, the arc $P'Q$ may be continously deformed, through a series of simple arcs joining P' to Q in the strip $C_a'C_b$ in which $P'Q$ wholly lies, into the straight line PQ, inasmuch as the straight lines QQ' and PP' have no points within this strip.

* I include in the term "varied simple curve" any simple curves with the same geometric locus but with a different parameter.

Hence the rotation of $B\bar{B}'$ during its first series of positions is precisely the same as it is along the broken line formed by the three straight segments PP', $P'Q$, QQ'. The rotation along the broken line from the initial to the final position is clearly $-\pi$ plus the sum of the acute angles which the vectors PP' and QQ' make with the x-axis. Thus our statement is proved.

In the first series of positions, $\bar{B}'$ is obtained from B by the transformation TT_ϵ. If the coördinates of B are (x, y), those of $\bar{B}'$ are accordingly $(\bar{x}', \bar{y}')$, and *the rotation of* $B\bar{B}'$ *is measured by the change in the function* $\bar{\omega}(x, y)$ *as* (x, y) *moves from* P *to* Q *along the invariant curve* PQ.

Let us fix upon that continuous branch $\bar{\omega}_1(x, y)$ of this function which is measured at the point P by the negative of the acute angle which the vector PP' makes with the x-axis. At the point Q this determination will have the value $-\pi$ plus the acute angle which the vector QQ' makes with the x-axis.

6. The Rotation of the Point-Image Vector for T. Now fix upon that continuous branch $\omega_1(x, y)$ of the function $\omega(x, y)$ which, along the upper side C_a of S, takes on the value zero.

These two functions $\bar{\omega}_1(x, y)$ and $\omega_1(x, y)$ differ by less than $\frac{1}{2}\pi$ at the point P and hence are branches of $\bar{\omega}(x, y)$ and $\omega(x, y)$ associated by the inequality (5), which holds throughout S. Moreover the terminal value of the function $\bar{\omega}_1(x, y)$ has been shown to differ from $-\pi$ by less than $\frac{1}{2}\pi$ at Q, so that $\omega_1(x, y)$ differs from $-\pi$ by less than π at Q. Any branch of the function $\omega(x, y)$ however has been seen to be precisely equal to a fixed odd multiple of π along C_b. Hence this function $\omega_1(x, y)$ must have the value $-\pi$ at Q and also at all points of C_b.

The variation of $\omega_1(x, y)$ is therefore $-\pi$ when the point (x, y) moves in any manner whatever from a point of C_a to a point of C_b. In other words *if we let the point* B *move in any manner from* C_a *to* C_b *in* S, *and let* B' *denote the image of* B *by the transformation* T, *the total rotation in the vector* BB' *will be precisely* $-\pi$.

7. Completion of the Proof. Consider now the transformation T^{-1} inverse to T, which is in every respect similar to T except that points on C_a and C_b, and hence on the two sides of the strip S, are moved in the reverse direction. By symmetry, the vector $B'B$ which joins B' to its image B under T^{-1} must now rotate through an angle $+\pi$ as B and B' move from C_a to C_b.

Here the vector $B'B$ is of sense opposite to that of BB'. But the actual rotation of the vectors BB' and $B'B$ is of course one and the same quantity, so that a contradiction has been reached. Hence the theorem is completely proved.

8. Generalizations of the Theorem. In phrasing his geometric theorem Poincaré makes the hypothesis that we have *some* integral invariant $\iint P(x, y)\,dx\,dy$ $(P(x, y) > 0)$, not necessarily the invariant of areas. However, we can, by a suitable change of coördinates from (x, y) to (ξ, η), change the integral invariant to the simpler area invariant.

Let the lines $\eta =$ const. be the circles concentric with C_a and C_b, and let the number $\eta(y)$ be so chosen for each circle that as a point moves from C_b to C_a the double integral taken over the ring between C_b and the concentric circle through the moving point is equal to η:

$$\eta(y) = \int_{b^2}^{y} \left\{ \int_0^{2\pi} P(x, y)\, dx \right\} dy.$$

This function $\eta(y)$ is clearly eontinuous and increasing and has a positive continuous derivative, namely

$$\frac{d\eta(y)}{dy} = \int_0^{2\pi} P(x, y)\, dx.$$

Speaking somewhat inexactly, the η-curves are all so placed as to measure off equal increments of $\int\int P(x, y)\, dx\, dy$ for equal increments of η. Now (speaking in the same inexact sense) choose the ξ-curves so as to measure off equal increments of the same double integral between these successive η-curves and the line $x = 0$.

The possibility of making such a choice of curves $\xi =$ const. may be seen as follows: Let $x = f(y)$ be any curve lying in R, joining C_a to C_b, and such that the part of the area between the initial line $x = 0$, this curve, the circle C_b and the circle $y =$ const. is constantly proportional to $\eta(y)$:

$$\int_{b^2}^{y} \left\{ \int_0^{f(y)} P(x, y)\, dx \right\} dy = \frac{k}{2\pi} \eta(y).$$

In view of the above mentioned properties of $\eta(y)$, this equation is equivalent to

$$\int_0^{f(y)} P(x, y)\, dx = \frac{k}{2\pi} \frac{d\eta(y)}{dy}. \tag{6}$$

For a given k, the quantity $f(y)$ is clearly a single-valued continuous function of y since $P(x, y)$ and $d\eta(y)/dy$ are positive and continuous. Moreover an increase in k continuously increases $f(y)$. Thus, we get a set of non-intersecting curves $x = f(y)$ which, like the circles $\eta =$ const., fill up the ring R. It is necessary to notice that $x = 0$ corresponds to $k = 0$, and that $x = 2\pi$ is the curve for $k = 2\pi$ by definition of $\eta(y)$ We take k as determined by the above method to be the coördinate ξ of any point on the curve $x = f(y)$.

Thus we have coördinates (ξ, η) for which the integral invariant is the invariant $\int\int d\xi d\eta$. Taking (ξ, η) as the modified polar coördinates of a point in a new plane, we find that, expressed in these coördinates, T has all the properties specified for T in the theorem. Thus we infer that there are at least two invariant points as before.

A further generalization is that the curves C_a and C_b may be allowed to be any simply closed curves, one within the other, bounding a ring R; we may again state a similar theorem, for we can make a preliminary transformation, for instance a conformal one, to take these curves into concentric circles, when of course the integral invariant merely changes form. Here it may be necessary to consider with care the nature of the integral invariant near the boundaries after the transformation of coördinates.

Finally we may permit the function $P(x, y)$ to vanish at some or all points of the curves C_a and C_b. Under certain restrictions it is certain that in this limiting case there will be invariant points also.

9. Poincaré's Method. It is interesting to notice that Poincaré uses only a single property consequent on the existence of an invariant integral, namely that no continuum on R can be transformed into part of itself by the transformation T (loc. cit., p. 377). It seems improbable that this condition is equivalent to the condition that there exists an invariant integral. The existence of such an invariant integral is a fact which enters more intimately into the proof which I have given above. I do not know whether the modification of Poincaré's theorem which results when the condition that an integral invariant exists is replaced by this weaker condition is true.

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