CONCERNING VAN VLECK'S NON-MEASURABLE SET*

BY

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The construction of a point-set which is not measurable in the sense of LEBESGUE has been described by VAN VLECK.† In view of the great interest attaching to the question of the existence and structure of such sets, it is thought desirable to formulate the construction with explicit reference to a definite list of postulates. The general procedure will be that of Van Vleck; with minor departures however, to which attention will be called by notes as they occur. The postulates employed are selected from those given by ZERMELO.‡

Postulate I. *If $E(x)$ is a significant proposition for every element x of a set M, then there is a set M_1 which has as elements all those elements x of M for which $E(x)$ is true and no other element.*

Postulate II. *For every set T there is a set UT which has as elements all subsets of T and no other element.*

Postulate III. *For every set T, whose elements are sets, there is a set ST which has as elements all the elements of the elements of T and no other element.*

Postulate IV. *If T is any set whose elements are all sets, each different from the zero set, and such that no two have an element in common, then the set ST contains a subset which contains one and only one element of each element of T (Prinzip der Auswahl).*

Definition. A set of points T is *homogeneous* on an interval as 01 if there exists a set R, everywhere dense on 01, such that if a and b are points of R

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† E. B. VAN VLECK: *Non-measurable sets of points*. These Transactions, vol. 9 (1908), p. 237.

‡ E. ZERMELO: *Untersuchungen über die Grundlagen der Mengenlehre*, Mathematische Annalen, vol. 65 (1908), p. 261.

then the subset of T which lies on ab is similar to the set T on 01. We also say that the set T is homogeneous with respect to the set R.*

The exterior Lebesgue measure $Me(T)$ of a set T is the greatest lower bound of the sum of the lengths of the segments of a non-overlapping set of segments covering the set T. If the set T is measurable the measure $M(T)$ of the set is equal to its exterior measure. Let C be the complementary set of T on 01. Then if T is measurable, C is measurable and we have $M(T) + M(C) = 1$.

The theorem of Van Vleck. *If a set T is measurable and homogeneous in the interval* 01 *then its measure is either zero or one.*

Proof. Suppose that $M(T) = 1 - \Sigma$ where Σ is a definite number such that $0 < \Sigma < 1$. Let T be homogeneous with respect to a set R. Then there exists a set of non-overlapping segments $[\sigma]$ covering T such that all end-points of segments of $[\sigma]$ are points of R, and further such that $M[\sigma] = 1 - \Sigma + \epsilon$, where ϵ is an arbitrary positive number such that $1 - \Sigma + \epsilon \overline{<} 1$. Let ab be any segment of $[\sigma]$. Then on ab there is a set of segments $[\sigma]_{ab}$ similar to $[\sigma]$ and covering all points of T on ab. Consequently $1/(1 - \Sigma + \epsilon) = ab/M[\sigma]_{ab}$. A similar set of segments exists on each segment of $[\sigma]$. Denote the set of all these segments by $[\sigma]_l$. Then T is covered by a set of non-overlapping segments $[\sigma]_l$ such that

$$\frac{1}{1 - \Sigma + \epsilon} = \frac{1 - \Sigma + \epsilon}{M[\sigma]_l},$$

or

$$M[\sigma]_l = (1 - \Sigma + \epsilon)^2,$$

Since ϵ is arbitrary, it may be so chosen that $(1 - \Sigma + \epsilon)^2 < 1 - \Sigma$, contrary to the assumption that $M(T) = 1 - \Sigma$.

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* VAN VLECK's definition of homogeneity is as follows: If in any subinterval whatsoever ab of the interval 01 there can be found an interval as nearly equal in length to ab as we please such that the subset of T included in it shall be similar to the set T on 01 then T is homogeneous (loc. cit., p. 238; see also certain modifications). A set which is homogeneous according to the definition of this paper is obviously homogeneous according to Van Vleck's definition. It would seem however that the converse need not be true, though I have not found an example to show this. It should be observed however that the degree of generality of these sets is of no moment for our present purpose. Here it is sufficient so to define homogeneity that the theorem of Van Vleck shall hold of all homogeneous sets, and then to construct two such sets with a peculiar property as regards measures.

We shall now reject, as in Van Vleck's construction, the rational points of the interval 01, and shall separate the remaining points (whose measure is 1) into two homogeneous sets T and C which are superposable and have therefore the same measure either 0 or 1. Hence $M(T) + M(C) = 0$ or 2, and as this is absurd the sets are non-measurable.*

Denote by Q the set of all irrational points of 01. Then for x_1, any point of Q, the sets $[x_1 r_1 + s_1]$ and $[1 - (x_1 r_2 + s_2)]$, (r_1, r_2, s_1, s_2, having as their range all rational numbers with the restriction that $r_1 + r_2 \neq 0$, $s_1 + s_2 \neq 1$), have no element in common. We say that a set of points has the property J if it has as elements all points on 01 of a pair of sets of the type $[x_1 r + s]$, $[1 - (x_1 r + s)]$ and no other element. The point x_1 may be regarded as a distinctive element of a set of having property J. Clearly if any other point of the set is taken as the distinctive element the same set will be generated. If two sets having property J have an element in common, they are identical. The sets $[x_1 r + s]$ and $[1 - (x_1 r + s)]$ are superposable.

By Postulate II there is a set UQ which has as elements all subsets of Q and no other element. For every element of UQ it is a significant proposition whether or not it has the property J. Hence by Postulate I there is a definite subset M of UQ which has as elements all those subsets of Q which have property J, and no other element. Each element of M is a set different from the zero set, and no two of these sets have an element in common. Hence by Postulate IV there is a set N which contains as elements one and only one element of each set of M. But each element x of N may now be regarded as the distinctive element of a set having the property J.

We now consider that subset K of UQ whose elements are of the form $xr + s$, where x is an element of N. By Postulate III there is a set $SK = T$ whose elements are the elements of the elements of K. The remaining irrational points of 01, $Q - T$, we call a set C. The sets T and C are superposable, since one set is a reflection of the other with respect to the point $1/2$. Further, T is homogeneous on 01 with respect to the set of all rational points on this

* Van Vleck effects the required separation by describing a set of pairs of sets of the type

$$\left[\frac{x}{2^{\pm p}} \pm \frac{m}{2^n}\right], \quad \left[1 - \frac{x}{2^{\pm p}} \pm \frac{m}{2^n}\right]$$

and making an *arbitrary choice* of one set of each of these pairs to be in T and the other in C. These sets T and C are superposable and homogeneous. We shall effect the separation by citing a postulate of Zermelo, thus basing it upon an explicitly recognized fundamental proposition.

interval, since the points of T on 01 may be projected into the points of T on any interval ab whose end-points are rational points.* This completes the proof outlined above.

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