AN APPLICATION OF FINITE GEOMETRY TO THE CHARACTERISTIC THEORY OF THE ODD AND EVEN THETA FUNCTIONS

BY

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Ordinary projective geometry has long been recognized as an important instrument of investigation in other and apparently quite distinct fields, such as the theory of equations and the theory of functions. One of the objects of this paper is to show that a similar purpose is served by the more recently formulated finite projective geometry. It may be said with some logical justification that this coördination between different subjects is due to their common use of a certain body of abstract theorems. But there are notions, such as that of projection and section, which are so essentially geometric in their origin and significance that their import can only be clouded by viewing them in another or more abstract light.

The characteristic theory of the odd and even theta functions has been the subject of numerous memoirs since the appearance of the original papers of Göpel and Rosenhain. An excellent account of the present state and a valuable history of the development of this theory is given by KRAZER in his Lehrbuch Der Thetafunktionen. As a matter of convenience KRAZER'S notation and formulæ are used in the following. For known theorems also reference is made to Krazer, since often they appear in the original in misleading or distorted form.

The following account is essentially geometrical, the principal notions used being those of linear and quadratic dependence, of the null system, and of projection and section. Practically all of the known theorems are reproved by short and direct methods, which in many cases suggest important generalizations. Two ideas, that of "projection and section of a null system" and

* Presented to the Society, February 24, 1912. Written under the auspices of The Carnegie Institution of Washington.
† VEBLER and BUSSEY, these Transactions, vol. 7 (1906), pp. 241–59. Much of DICKSON's Linear Groups can be interpreted as finite analytic geometry. Cf. also the author's article, A Configuration in Finite Geometry, etc., Johns Hopkins University Circulars, No. 7, 1908; and MITCHELL'S Determination of the Ordinary and Modular Ternary Linear Groups, these Transactions, vol. 12 (1911), pp. 207–42.
‡ Leipzig (Teubner), 1908; particularly, pp. 239–304; cited hereafter as K.

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that of "section of a null system," are constantly utilized. The grouping of characteristics due to the first of these has been employed to some extent by Frobenius; the grouping due to the second seems to have been noted only in a few particular cases.

§ 1 contains an elementary account of the finite geometry (modulo 2) in a linear space in which there may or may not be defined a null system. It is shown in § 2 that the period characteristics behave like the points of an $S_{2p-1}$ with reference to a given null system $C$. Quadrics in $S_{2p-1}$ are studied in § 3; in particular, those whose polar systems coincide with the null system $C$. In § 4 the theta characteristics are identified with the quadrics belonging to $C$. By mapping the quadrics belonging to $C$ upon a space $R_{2p}$, the period and theta characteristics are shown in § 5 to lie in a linear system. Numerous theorems concerning Steiner and Kummer groups are proved here. § 6 is devoted to the so-called systems of theta characteristics. In the earlier paragraphs translation schemes for the transition from the geometry to the characteristic theory are exhibited.

I think that even a hasty comparison of the presentation here given with the arithmetical method followed by Krazer and others will show that the geometrical point of view is very valuable, not only for suggesting novel ideas but also for giving precision to ideas* with consequent generality of statement.

§ 1. The Finite Geometry Modulo 2.

Let $x_0, x_1, \ldots, x_k$ be homogeneous coördinates in the linear space $S_k$ in which the coördinates of points and the coefficients of loci are restricted to the finite number field determined by the modulus 2. The coördinates can take either of the values 0 or 1 but cannot all be zero. If $P_k$ is the number of points in $S_k$, then

\[(1) \quad P_k = 2^{k+1} - 1, \quad P_k - P_l = 2^{l+1} P_{k-l-1} \quad (k > l).\]

An $S_{k-1}$ in $S_k$ is defined by the equation (congruence)

\[(2) \quad u_0 x_0 + u_1 x_1 + \cdots + u_k x_k = 0,\]

or equally well by the coördinates $u_0, u_1, \ldots, u_k$. Thus (2) is the condition that, in $S_k$, the point $x$ and the $S_{k-1}$ $u$ be incident. In this geometry the fundamental theorems of linear dependence and duality are true; and only those developments which differ essentially from the corresponding developments of ordinary geometry will be considered in detail.

\[(3) \text{Given } n \text{ linearly independent } S_{k-1}'s \text{ in } S_k, \text{ the number of points on } m \text{ of}\]

* See the closing remarks.
these $S_{k-1}$'s and not on the remaining $n - m$ $S_{k-1}$'s is $2^{k-n+1}$ if $m < n$, $2^{k-n+1} - 1$ if $m = n$. *

Let $z_1 = 0, \cdots, z_m = 0$ be the set of $m$ $S_{k-1}$'s; and $z_{m+1} = 0, \cdots, z_n = 0$ be the set of $n - m$ $S_{k-1}$'s. If $m < n$, the required points are those on the $n - 1$ $S_{k-1}$'s

$$z_1 = 0, \cdots, z_m = 0, z_{m+1} + z_n = 0, \cdots, z_{n-1} + z_n = 0,$$

which are not on $z_n = 0$ (since $z_{m+1} \equiv 1, \cdots, z_n \equiv 1$ modulo 2). The $n - 1$ spaces meet in an $S_{k-n+1}$ which cuts $z_n = 0$ in an $S_{k-n}$. Thus the required number is $P_{k-n+1} - P_{k-n} = 2^{k-n+1}$. But if $m = n$ the required points are in an $S_{k-n}$ and are $P_{k-n}$ in number.

The $k + 1$ points whose equations are $u_i = 0$ are linearly independent and constitute a point reference basis in $S_k$. Also the $k + 1$ $S_{k-1}$'s whose equations are $x_i = 0$ constitute an $S_{k-1}$ reference basis in $S_k$. Either basis determines the other and the two constitute a self-dual reference basis of $S_k$. A point reference basis in an $S_{k-1}$ and a point not in the $S_{k-1}$ determine a point reference basis in $S_k$ and each basis is thus determined in $k + 1$ ways. If $R_k$ is the number of reference bases in $S_k$ we have, since the number of the $S_{k-1}$ is $P_k$, the recursion formula

$$R_k = \frac{P_k (P_k - P_{k-1})}{k + 1} R_{k-1} = \frac{2^k}{k + 1} P_k R_{k-1}.$$  

(4) The number $R_k$ of reference bases in $S_k$ is

$$R_k = \frac{2^{k(k+1)/2}}{(k+1)!} P_k P_{k-1} \cdots P_1.$$  

An $S_h$ in $S_k$, where $h < k$, is fixed by choosing $h + 1$ linearly independent points in $S_k$; but the same $S_h$ can be thus fixed in $R_h$ ways. The $h + 1$ points, when ordered, can be chosen in

$$P_k (P_k - P_0) (P_k - P_1) \cdots (P_k - P_{h-1})$$

ways. This number divided by $R_h \cdot (h + 1)!$ is the number of $S_h$'s in $S_k$. Hence, from (1) and (4),

(5) For $h < k$, the number $P_h^{(k)}$ of $S_h$'s in $S_k$ is

$$P_h^{(k)} = \frac{P_k P_{k-1} \cdots P_{k-h}}{P_h P_{h-1} \cdots P_1}.$$  

We shall be concerned mainly with an odd space $S_{2p-1}$ and coordinates $x_1, x_2, \cdots, x_p, x_{p+1}, x_{p+2}, \cdots, x_{2p}$. A set of $2p + 1$ points must be linearly related. A set of $2p$ linearly independent points, e.g., $u_1 = 0, \cdots, u_{2p} = 0$,  

* K., p. 247, III.
determines uniquely a \((2p + 1)\)th point, \(u_1 + u_2 + \cdots + u_{2p} = 0\), such that any \(2p\) of the \(2p + 1\) points are linearly independent. Call such a set of \(2p + 1\) points a point basis in \(S_{2p-1}\). According to the dual of (3) there is a unique \(S_{2p-2}\) which is not on \(2p\) points of the basis and therefore must be on the \((2p + 1)\)th point. If

\[
\sigma = \sum_{i=1}^{2p} x_i,
\]

the set of \(2p + 1\) such \(S_{2p-2}\)'s derived from the point basis is

\[
\sigma - x_1 = 0, \quad \sigma - x_2 = 0, \quad \ldots, \quad \sigma - x_{2p} = 0, \quad \sigma = 0.
\]

Evidently any \(2p\) of these are linearly independent and the set constitutes an \(S_{2p-2}\) basis in \(S_{2p-1}\). The relation between the point basis and the \(S_{2p-2}\) basis is mutual. The basis can be defined similarly in an even dimension, the peculiarity of the odd dimension being that corresponding point and \(S_{2p-2}\) are incident. The two bases constitute a self dual basis in \(S_{2p-1}\).

(6) In a self dual basis of \(S_{2p-1}\) there are \(2p + 1\) incident elements (point, \(S_{2p-2}\)). Any set of \(2p\) points \([2p S_{2p-2}\)'s\] of the basis is a point \([S_{2p-2}\] reference basis.

A point basis is fixed by any one of the \(2p + 1\) point reference bases in it. Hence, from (4),

(7) The number of reference bases in \(S_{2p-1}\) is

\[
N_R = \frac{2p(2p-1)}{(2p)!} P_{2p-1} P_{2p-2} \cdots P_1.
\]

The number of bases is

\[
N_R = \frac{1}{2p + 1} N_S = \frac{2p(2p-1)}{(2p + 1)!} P_{2p-1} P_{2p-2} \cdots P_1.
\]

A collineation in \(S_{2p-1}\) is determined when two ordered point bases are made to correspond; a correlation is determined when an ordered point basis and an ordered \(S_{2p-2}\) basis are made to correspond. The totality of collineations or correlations is gotten by fixing one basis and allowing the other to vary, whence the number of each is the number of ordered bases.

(8) The order of the collineation group in \(S_{2p-1}\) is

\[
N = (2p + 1)! N_R = (2p)! N_R = \frac{2p(2p-1)}{(2p + 1)!} P_{2p-1} P_{2p-2} \cdots P_1.
\]

\(N\) is also the number of correlations in \(S_{2p-1}\) and \(2N\) is the order of the correlation group in \(S_{2p-1}\).

Of particular interest are those correlations for which corresponding point and \(S_{2p-2}\) are incident, the so-called null systems. If \(y_i = 0, i = 1, 2, \ldots, 2p + 1\), is an \(S_{2p-2}\) basis,
(9) $C = y_1 y'_1 + y_2 y'_2 + \cdots + y_{2p+1} y'_{2p+1} = 0, \quad \Sigma y = 0, \quad \Sigma y' = 0,$
is a proper correlation. Since $\Sigma y^2 = (\Sigma y)^2 \pmod 2,$ the correlation is a null system. For the $i$th point of the self dual basis, $y_i = 0,$ and $y_k = 1, \ k \neq i,$ whence the $i$th point of the basis corresponds to the $i$th $S_{2p-2}$ or the self dual basis $y$ is invariant under $C.$ Moreover, given any null system $C,$ there are self dual bases invariant under $C$ which are determined as follows. Let $z_1$ be any point of $S_{2p-1}$ ($P_{2p-1}$ choices), and let $w_1$ be its null $S_{2p-2}.$ Let $z_2$ be any point not on $w_1$ ($2^{2p-1}$ choices), and $w_2$ be its null $S_{2p-2}.$ On the $S_1$ $z_1 z_2,$ $w_1$ and $w_2$ cut out a reference basis and one point of $S_1$ lies on neither $w.$ Let then $z_3$ be any point not on $w_1$ or $w_2$ and not on $z_1 z_2$ ($P_{2p-3}$ choices) and $w_3$ be its null $S_{2p-2}.$ The $S_2$ $z_1 z_2 z_3$ is in $w_1 + w_2 + w_3$ since $w_i$ is on $z_i$ but not $z_j$ and $z_k.$ Thus $w_1, w_2, w_3$ cut $S_2$ in a pencil of $S_1$'s and every point of $S_2$ is on a $w.$ Let then $z_4$ be a point not on $w_1$ or $w_2$ and not on $z_1 z_2$ ($P_{2p-3}$ choices), and $w_4$ be its null $S_{2p-2}.$ The $S_3$ $z_1 z_2 z_3 z_4$ is not contained in an $S_{2p-2}$ on the $S_{2p-5}$ $w_1, w_2, w_3, w_4,$ whence the $w$'s cut the $S_3$ in $4$ linearly independent $S_2$'s containing all but one of the points of $S_3.$ Let then $z_5$ be a point not on $w_1, \ldots, w_5$ and not in the $S_3$ ($P_{2p-5}$ choices), and $w_6$ be its null $S_{2p-2}.$ The $S_4$ $z_1, z_2, \ldots, z_5$ is contained in $w_1 + \cdots + w_6,$ whence every point of $S_4$ is on a $w.$ Let then $z_6$ be a point not on $w_1, \ldots, w_6$ ($2^{2p-5}$ choices), and $w_7$ be its null $S_{2p-2}.$ Proceeding thus we find, in

$$P_{2p-1} \cdot 2^{2p-1} \cdot P_{2p-3} \cdot 2^{2p-3} \cdot \cdots \cdot P_1 \cdot 2 \cdot P_0$$

ways, a set of $2p + 1$ points, $z_1, z_2, \cdots, z_{2p+1},$ and a set of $2p + 1$ $S_{2p-2}$'s, $w_1, w_2, \cdots, w_{2p+1},$ subject only to the relation $w = 0, \Sigma w = 0,$ and such that $z_i$ and $w_i$ are incident while $z_i$ and $w_k, k \neq i,$ are not incident. Thus the two sets form a self dual basis which by its mode of formation is invariant under $C.$ Taking account of the order we see that

(10) The number of self dual bases invariant under the null system $C$ is

$$N_{BC} = \frac{2p^2}{(2p + 1)!} \cdot P_{2p-1} \cdot P_{2p-3} \cdots \cdot P_1.$$

Referred to a basis, $C$ takes the unique form (9). This form is unaltered by the group of order $(2p + 1)!$ determined by permutations of the basis, whence the order $N_C$ of the group of $C$ is $(2p + 1)! \cdot N_{BC}.$ Since all bases are conjugate under $G_N,$ all proper null systems $C$ are conjugate under $G_N$ and their number is $N / N_C.$

(11) All proper null systems are conjugate under $G_N.$ Each is unaltered by a group $G_{NC}$ of order

$$N_C = 2p^2 P_{2p-1} \cdot P_{2p-3} \cdots \cdot P_1.$$

*In general, in an $S_{2k+1}$ $2k + 1 S_{2k-1}$'s, whose sum is zero but subject to no other relation, contain all the points of $S_{2k};$ while in an $S_{2k+1}, 2k + 2$ similarly related $S_{2k}$'s contain all but one of the points of $S_{2k+1}.\text{ Cf. K., pp. 267-9.}$

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The number of proper null systems is
\[ \frac{N}{N_c} = 2^{p(p-1)} P_{2p-2} P_{2p-4} \ldots P_2. \]

If the null \( S_{2p-2} \) of \( x \) contains \( x' \) it contains the line \( xx' \), and the three null \( S_{2p-2} \)'s of points on this line contain the line. Such a line will be called a null line, any other line an ordinary line.

From the canonical form (9) of \( C \) in terms of a self dual basis it is clear that the involution
\[ y'_i = y_2, \quad y'_3 = y_1, \quad y'_i = y_i \quad (i = 3, 4, \ldots, 2p + 1) \]
is contained in \( G_{NC} \). Every point of \( y_1 + y_2 = 0 \), including its null point \( 1, 0, 0, \ldots, 0 \), is a fixed point. Every point not on \( y_1 + y_2 = 0 \) with coordinates \( i, j, k, \ldots (i \neq j) \) is interchanged with \( j, i, k, \ldots \), the conjugate pair being on a line with the null point of \( y_1 + y_2 = 0 \). Since \( G_{NC} \) is transitive on the points of \( S_{2p-1} \), it contains a conjugate set of \( P_{2p-1} \) such involutions which generate \( G_{NC} \). For the transpositions of the \( y \)'s generate the subgroup of \( G_{NC} \) which leaves a basis unaltered. We have then only to show that one basis \( a_1, a_2, \ldots, a_{2p+1} \), self dual under \( C \) can be transformed by these involutions into any similar basis \( b_1, b_2, \ldots, b_{2p+1} \). Suppose that \( k - 1 \) points \( a \) already coincide with points \( b \). By means of the transpositions we adjust the case so that \( a_1 = b_1, \ldots, a_k-1 = b_k-1 \) while \( a_k \neq b_k \). If \( a_k, b_k \) is an ordinary line the null \( S_{2p-2} \)'s of the first \( k - 1 \) points all pass through \( a_k + b_k \) and the involution determined by \( a_k + b_k \) leaves the first \( k - 1 \) points fixed and transforms \( a_k \) into \( b_k \). If \( a_k b_k \) is a null line, let \( c_k \) be a point not on the null \( S_{2p-2} \)'s of \( a_1, a_2, \ldots, a_k, b_k \). Then the two points \( a_k + c_k, b_k + c_k \) are on the null \( S_{2p-2} \)'s of the first \( k - 1 \) points and the involution of the first followed by that of the second leaves \( a_1, a_2, \ldots, a_k-1 \) unaltered and transforms \( a_k \) into \( b_k \). The point \( c_k \) is subject to \( k + 1 \) conditions and can always be determined according to (3) until \( k = 2p \). But two self dual bases coincide if \( 2p - 1 \) of their points respectively coincide. Hence

(12) The group \( G_{NC} \) of the null system \( C \) is generated by a conjugate set of \( P_{2p-1} \) involutions. Each involution is associated with a point and its null \( S_{2p-2} \) in such a way that every point on the \( S_{2p-2} \) is fixed and every ordinary line on the point contains a conjugate pair of points.*

§ 2. Period Characteristics of the Thetas as Points in \( S_{2p-1} \) Modulo 2.

The theta function of \( p \) variables \( \vartheta (v) = \vartheta (v_1, v_2, \ldots, v_p) \) has \( 2p \) independent periods, each consisting of \( p \) quantities \( \omega_{1a}, \omega_{2a}, \ldots, \omega_{pa}, \) where

\( \alpha = 1, 2, \ldots, 2p. \) That is,
\[
\vartheta (v + \omega_\alpha) = \vartheta (v_1 + \omega_1, \ldots, v_p + \omega_p) = E \vartheta (v),
\]
\( E \) being an exponential factor. The value system
\[
\sum_{a=1}^{2p} \varepsilon_a \omega_1, \ldots, \sum_{a=1}^{2p} \varepsilon_a \omega_p
\]
is called a **half period**, whose **period characteristic or Per. Char.** is the set of \( 2p \) integers \( \varepsilon_1, \ldots, \varepsilon_{2p} \). Two characteristics whose half periods differ by a period are looked upon as not essentially distinct, whence the integers \( \varepsilon \) are reducible (modulo 2). Under integral linear transformation of the periods, the half periods are transformed so that the value of the expression
\[
\sum_{\mu=1}^{p} (\varepsilon_\mu \eta_{p+\mu} - \varepsilon_{p+\mu} \eta_\mu)
\]
is unaltered, \( \varepsilon \) and \( \eta \) being any two distinct characteristics.* Naturally the coefficients of the transformation are also reducible modulo 2. The zero characteristic \( \varepsilon_i = 0, i = 1, 2, \ldots, 2p \), differs from the remaining proper **Per. Char.** in that it is unaltered by every transformation. Thus our first fundamental theorem is apparent:

(13) **Under integral linear transformation of the periods of the theta function in** \( p \) **variables, the proper Per. Char. are transformed like the points (or their null \( S_{2p-1} \)'s) of a finite space \( S_{2p-1} \) **modulo 2** under the group \( G_{NC} \) of collineations which leaves unaltered the proper null system
\[
C = (x_1 x_{p+1}^' - x_{p+1} x_1^') + (x_2 x_{p+2}^' - x_{p+2} x_2^') + \cdots + (x_p x_{2p}^' - x_{2p} x_p^').
\]
Thus properties of sets of Per. Char. which are independent of integral period transformation—and these are the only properties of essential importance—can be inferred from the properties of sets of points in \( S_{2p-1} \) with reference to \( C \). The translation proceeds as follows:

<table>
<thead>
<tr>
<th>Point in ( S_{2p-1} )</th>
<th>Proper Per. Char.†</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two points on a null line.</td>
<td>Two syzygetic Per. Char.‡</td>
</tr>
<tr>
<td>Two points on an ordinary line.</td>
<td>Two azygetic Per. Char.†</td>
</tr>
<tr>
<td>(14) Sum of a number of points.</td>
<td>Sum of a number of Per. Char.†</td>
</tr>
<tr>
<td>Points of a self dual basis of ( C ).</td>
<td>Fundamental system (F. S.) of ( 2p + 1 ) Per. Char.; such that any two are azygetic.§</td>
</tr>
</tbody>
</table>

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* K., pp. 242-3.
† With Per. Char., the term proper will be understood hereafter.
‡ K., p. 244.
§ K., p. 267.
The duality in $S_{2p-1}$ established by $C$ permits of expressing an occurrence in several ways; thus two points are on a null line if either is on the null $S_{2p-2}$ of the other. Since the null lines on a point $x$ are in its null $S_{2p-2}$, we have

(15) A given point lies on a null line with $2P_{2p-3}$ other points; on an ordinary line with $2^{2p-1}$ other points.

A given Per. Char. is syzygetic with $2P_{2p-3}$ other Per. Char.; azygetic with $2^{2p-1}$ other Per. Char.*

The translation of (3) for $k = 2p - 1$ is

(16) The number of Per. Char. syzygetic with $m$ and azygetic with $n - m$ of $n$ given linearly independent Per. Char. is $2^{2p-n}$ if $m < n$, and $P_{2p-n-1}$ if $m = n$.†

From (10) the number‡ of F.S.'s is obtained. Some theorems, such as the first part of the following.§ are self-evident from the geometrical point of view:

(17) A F.S. of Per. Char. is transformed by integral linear transformation into a F.S. All F.S.'s are conjugate under such transformation.

Let $x^{(1)}, x^{(2)}, \ldots, x^{(r)}$ be $r$ points of $S_{2p-1}$ which are not linearly related. They lie in an $S_{r-1}$ and form a point reference basis of the $S_{r-1}$. Any space of dimension equal to or less than $2p - r - 1$ which has no point in common with $S_{r-1}$ will be called a skew space of $S_{r-1}$ in $S_{2p-1}$. The $r$ null $S_{2p-1}$'s of the points $x$ meet in an $S_{2p-r-1}$ called the null space of $S_{r-1}$. In general $S_{2p-r-1}$ meets $S_{r-1}$ in an $S_{m-1}$ called the null subspace of $S_{r-1}$. The line joining any point of $S_{r-1}$ to any point of its null space, $S_{2p-r-1}$, is a null line. If the null subspace of $S_{r-1}$ coincides with $S_{r-1}$, it is called a null $S_{r-1}$. The null space of largest dimension is an $S_{p-1}$ called a Göpel space. The translation to Per. Char. is made according to the following table.

\[
\begin{array}{l|l}
S_{r-1} & \text{Group } E_r \text{ of Per. Char. of rank } r . || \\
\text{Skew space of } S_{r-1} \text{ in } S_{2p-1} & \text{Group } H \text{ conjugate to the group } E_r. \| \\
\text{Null subspace of } S_{r-1} & \text{Syzygetic subgroup of } E_r. ** \\
(18) \text{Null space of } S_{r-1} \text{.} & \text{Group adjoint to } E_r. \uparrow \\
\text{Null } S_{r-1} \text{.} & \text{Syzygetic group } E. \uparrow \\
\text{Göpel space} \text{.} & \text{Göpel group}. \uparrow \\
\end{array}
\]

A null $S_k$ is determined by any point reference basis in it. The $k + 1$ points can be chosen as follows. Let $z_1$ be the first point ($P_{2p-1}$ choices), $w_1$ its null $S_{2p-2}$. Let $z_2$ be any point on $w_1$ other than $z_1$ ($P_{2p-2} - P_0$ choices), $w_2$ its null $S_{2p-2}$. Let $z_3$ be any point on the $S_{2p-3} w_1 w_2$ other than a point on the $S_1 z_1 z_2$ ($P_{2p-3} - P_1$ choices), $w_3$ its null $S_{2p-3}$; etc. The null $S_k$ is finally determined by means of an ordered point reference basis in it.

* K., p. 244. † K., p. 247. ‡ K., p. 268.
** K., p. 292.
Since the ordering and the particular basis in $S_k$ are not material we get

$$\frac{1}{(k+1)!} R_k \frac{1}{P_2} \left( P_{2p-2} - P_0 \right) \left( P_{2p-4} - P_1 \right) \cdots \left( P_{2p-k-1} - P_{k-1} \right)$$

null $S_k$'s. Or, by making use of (4),

$$P_{ca} = \frac{P_{2p-1} P_{2p-2} \cdots P_{2p-2k-1}}{P_1 P_2 \cdots P_k}$$

null $S_k$'s belonging to $C$ or syzygetic groups of rank $k + 1$.

Given a space $S_{m-1}$ with the null subspace $S_{m-1}$, let $S_{r-m-1}$ be a space skew to $S_{m-1}$ in $S_{r-1}$. Begin the construction of a self dual basis of $C$ by choosing $r - m$ points $y^{(m+1)}, y^{(m+2)}, \ldots, y^{(r)}$ in the $S_{m-1}$. The last point $y^{(r)}$, for example, must be on the $2p - (r - m)$ $S_{2p-2}$'s which meet in $S_{r-m-1}$ and outside the $r - m - 1$ null $S_{2p-2}$'s of $y^{(m+1)}, \ldots, y^{(r-1)}$. According to (3) such points can be found. Let $x^{(1)}, \ldots, x^{(m)}$ be any point reference basis in $S_{m-1}$. If the number of points $y^{(r)}$ is odd, the point $\Sigma y^{(r)}$, which is in $S_{r-m-1}$, is also on the null $S_{2p-2}$ of every point $y^{(r)}$ and of every point $x^{(1)}$ and is therefore in $S_{m-1}$. But $S_{m-1}$ is skew to $S_{r-m-1}$. Hence $r - m$ is even.

$(20)$ The difference $r - m$ of dimension of $S_{r-1}$ and its null subspace $S_{m-1}$ is even. A reference basis of $S_{r-1}$, $x^{(1)}, \ldots, x^{(m)}, y^{(m+1)}, \ldots, y^{(r)}$, can be selected so that every line $x^{(r)} x^{(k)}$ and $x^{(r)} y^{(k)}$ is a null line while every line $y^{(r)} y^{(k)}$ is an ordinary line. The difference $r - m$ of rank of $E_r$ and its syzygetic subgroup $E_m$ is even. $E_r$ has reference bases of the form $(\alpha_1), \ldots, (\alpha_m), (\beta_{m+1}), \ldots, (\beta_r)$, where the pairs $(\alpha_i) (\alpha_k)$ and $(\beta_i) (\beta_k)$ are syzygetic while the $(\alpha_i) (\beta_i)$ are asyzygetic.$^\dagger$

Such a reference basis of $S_{r-1}$ will be called a normal reference basis.

Further theorems in this paragraph will be stated in only one form, the translation being obvious.

Since $S_{m-1}$ is part or all of the space common to $r$ $S_{2p-2}$'s, $m + r < 2p$. Hence a space $S_{r-1}$ can be found skew to the null space $S_{2p-1-m}$ of $S_{m-1}$ and therefore skew to both $S_{m-1}$ and $S_{r-m-1}$. Then $S_{m-1}$ and $S_{r-m-1}$ lie in an $S_{r-1}$ in which part of a self dual basis of $C$ can be constructed beginning with $y^{(m+1)}, \ldots, y^{(r)}$, in $S_{r-m-1}$, and ending with $y^{(1)}, \ldots, y^{(m)}$. These $m$ points determine a space $S_{r-1}$ which may coincide with $S_{m-1}$, but which at all events is skew to $S_{2p-1-m}$, to $S_{m-1}$, and to $S_{r-m-1}$. The null $S_{2p-2}$'s of $y^{(1)}, \ldots, y^{(m)}$ cut $S_{m-1}$ in $S_{2p-2}$'s since these points are not found in $S_{2p-1-m}$. Thus an $S_{m-2}$ reference basis in $S_{m-1}$ is obtained which carries with it a point reference basis

$^*$ For $k = p - 1$, see K., p. 296.

$^\dagger$ K., p. 294.
$x^{(1)}, \ldots, x^{(m)}$ in $S_{m-1}$. The $m + r$ points

$y^{(m+1)}, \ldots, y^{(r)}, y^{(1)}, \ldots, y^{(m)}, z^{(1)} = y^{(1)} + x^{(1)}, \ldots, z^{(m)} = y^{(m)} + x^{(m)}$

form part of a self-dual basis of $C$. For $y^{(m+1)}$ is azygetic with $y^{(k)}$ by construction and, being syzygetic with $z^{(k)}$, is azygetic with $z^{(k)} = y^{(k)} + y^{(k)}$. Also $y^{(i)}$ is azygetic with $y^{(k)}$ and syzygetic with $x^{(k)}$ by construction hence azygetic with $z^{(k)}$. Also $y^{(i)}$ and $z^{(k)}$ are azygetic since $y^{(i)}$ and $y^{(k)}$ are azygetic while $y^{(i)}$ and $z^{(k)}$, $y^{(i)}$ and $x^{(i)}$, $x^{(i)}$ and $x^{(k)}$, are syzygetic. Since $x^{(i)} = y^{(i)} + z^{(i)}$, we have from (20),

(21) An $S_{r-1}$ with a null subspace $S_{m-1}$ has a reference basis of the form

$y^{(1)} + z^{(1)}, y^{(2)} + z^{(2)}, \ldots, y^{(m)} + z^{(m)}, y^{(m+1)}, \ldots, y^{(r)},$ such that the points $y, z$ form part of a self-dual basis of $C$.*

Two self-dual bases are conjugate in any order under $G_{NC}$, whence*

(22) Two spaces of the same dimension in $S_{2p-1}$ are conjugate under $G_{NC}$ if and only if their null subspaces have the same dimension.

In order to determine the number of $S_{r-1}$'s in a conjugate set it is convenient to introduce the notions of a "section" and of a "projection and section" of the null system $C$—notions that can be used later with advantage. According to (20) a space which has no null subspace is of odd dimension, $S_{2k-1}$. The null $S_{2p-2}$ of a point in $S_{2k-1}$ cannot contain $S_{2k-1}$ else the point is part of a null subspace. Then the null $S_{2p-2}$'s of the points in $S_{2k-1}$ cut $S_{2k-1}$ in $S_{2k-2}$’s and thus there is defined a null system $C_k$ in $S_{2k-1}$ which will be called the section of $C$ by $S_{2k-1}$. $2k$ points of a self-dual basis of $C_k$ in $S_{2k-1}$, being azygetic in pairs, are $2k$ points of a self-dual basis of $C$ in $S_{2p-1}$ and conversely. The number of such sets in $S_{2k-1}$ is $(2k + 1)! (N_{NC})_{p=k}$. On the other hand the number of such sets in $S_{2p-1}$ [see the enumeration before (10)] is

\[
\frac{1}{(2k)!} 2^{2p-1} \cdot 2^{2p-3} \cdots 2^{2p-2k+1} P_{2p-1} P_{2p-3} \cdots P_{2p-2k+1}.
\]

Dividing this number by the first we find that

(23) The number of $S_{2k-1}$'s without a null subspace is

\[
2^{2k(p-k)} P_{2p-1} P_{2p-3} \cdots P_{2p-2k+1} \over P_{2k-1} P_{2k-3} \cdots P_1.
\]

Each is unaltered by a subgroup of $G_{NC}$ of order

\[
2^{p^2-(p-k)+2k} P_{2p-2k-1} P_{2p-2k-3} \cdots P_1 \cdot P_{2k-1} \cdots P_3 P_1.
\]

Let $S_{r-1}$ have the null subspace $S_{m-1}$. The null space of $S_{m-1}$ is $S_{2p-m-1}$ which contains $S_{m-1}$. If $\pi = p - m$, there are within $S_{2p-m-1}$ and on $S_{m-1}$ precisely $P_{2\pi-1}$ spaces $S_m$. These we regard as "points $\Sigma_0$ in a space $\Sigma_{2\pi-1}$"  

* K., p. 295.
or $\Sigma$.' More generally then, spaces $S_{r-1}$ within $S_{2p-m-1}$ and on $S_{m-1}$ are "spaces $\Sigma_{r-m-1}$ in $\Sigma$." The null space of an $S_m$ within $S_{2p-m-1}$ and on $S_{m-1}$ is an $S_{2p-m-2}$ within $S_{2p-m-1}$ and on $S_m$ and therefore on $S_{m-1}$ also or "a point $\Sigma_0$ is on its null $\Sigma_{2p-2}$ with reference to the thus defined null system $\Gamma_\text{w}$ in $\Sigma$." This null system $\Gamma_\text{w}$ in the derived space $\Sigma_{2p-1}$ we call the projection and section of $C$ in $S_{2p-1}$ from the null space $S_{m-1}$ and by the null space $S_{2p-m-1}$ of $S_{m-1}$. Two points of $\Sigma$ are syzygetic or azygetic according as the null space of one corresponding $S_m$ does or does not contain the other corresponding $S_m$. A space $S_{r-1}$ within $S_{2p-m-1}$ and on $S_{m-1}$ has a null subspace $S_{m-1}$ which contains $S_{m-1}$. To $S_{r-1}$ there corresponds in $\Sigma$ a space $S_{r-m-1}$ which has with reference to $\Gamma_\text{w}$ a null subspace $\Sigma_{r-m-1}$ and conversely. In particular for $s = r$ and $m' = m$ we see that an $S_{r-1}$ with the null subspace $S_{m-1}$ corresponds to a $\Sigma_{r-m-1}$ in $\Sigma$ without a null subspace with reference to $\Gamma_\text{w}$. The number of these $\Sigma_{r-m-1}$'s has been determined in (23) where $2k$ is to be replaced by $r - m$ and $p$ by $p = p - m$. This is the number of $S_{r-1}$'s with a given null $S_{m-1}$ as null subspace. The number of null $S_{m-1}$'s is furnished by (19), whence

\begin{equation}
\frac{p^m}{2^{r-m} (r-m)!} \frac{P_{2p-1} P_{2p-3} \cdots P_{2p-m-1}}{P_{r-m-1} P_{r-m-3} \cdots P_1 \cdot P_{m-1} P_{m-3} \cdots P_1}.
\end{equation}

They all are conjugate and each is unaltered by a subgroup of $G_N$ of order

\begin{equation}
2^{p^m - (r-m) + 1} (r-m)! P_{2p-m-1} P_{2p-m-3} \cdots P_1 \cdot P_{r-m-1} P_{r-m-3} \cdots P_1 \cdot P_{m-1} P_{m-3} \cdots P_1.
\end{equation}

§ 3. Quadrics in $S_{2p-1}$ Modulo 2.

A quadric in $S_{2p-1}$ is defined by a congruence or equation of the form

\begin{equation}
f(xx) = \Sigma a_{ik} x_i x_k = 0 \quad (i, k = 1, \cdots, 2p; i \leq k).
\end{equation}

Points whose coordinates do or do not satisfy this equation will be called quadric or outside points respectively. The quadric will be called proper or degenerate according as it cannot or can be transformed by a collineation of $G_N$ into a form in less than $2p$ variables.

The point $x + y$ is on the quadric if

\begin{equation}
f(xx) + f'(xy) + f(yy) = 0,
\end{equation}

where

\begin{equation}
f'(xy) = \Sigma a_{ik} (x_i y_k + x_k y_i) \quad (i, k = 1, \cdots, 2p; i < k).
\end{equation}

Since

\begin{equation}
x_i y_k + x_k y_i = x_i y_k - x_k y_i = \pi_{ik},
\end{equation}
we see* that the polar system of \( f, f' (xy) = 0 \) is a null system whatever be the values of the coefficients \( a_{ij} \).

(27) The \( 2^{2p} \) quadrics \( f \), obtained by varying the coefficients \( a_{ij} \), have as polar systems the same null system. Conversely, any null system determines \( 2^{2p} \) quadrics whose polar systems coincide with the given null system.

An important relation between the quadric and its polar system is this:

(28) A quadric is proper or degenerate according as its polar system is a proper or degenerate null system.

For if the quadric is degenerate it can be transformed into a quadric in \( 2p - 1 \) variables at most, say \( y_2, \ldots, y_{2p} \), and the polar of 1, 0, \( \ldots, 0 \) is indeterminate, i.e., the null system is degenerate. Conversely, if the polar of 1, 0, \( \ldots, 0 \) is indeterminate the quadric either has the form \( g(y_2, \ldots, y_{2p}) \), and is degenerate or has the form \( y_1^2 + g(y_2, \ldots, y_{2p}) \). In the latter case, \( g(y_2, \ldots, y_{2p}) \) is a quadric in an even space whose polar system is a null system and therefore necessarily degenerate. Let \( y_2 = 1, y_{2+k} = 0 \), be a singular point. Then \( g \) has either the form \( y_1^2 + h(y_3, \ldots, y_{2p}) \) or the form \( h(y_3, \ldots, y_{2p}) \). The original quadric has either the form \( z^2 + h(y_3, \ldots, y_{2p}) \), where \( y_1^2 + y_2^2 = (y_1 + y_2)^2 = z^2 \) or the form \( y_1^2 + h(y_3, \ldots, y_{2p}) \), and in either case is degenerate.

We shall say that a quadric belongs to \( C \) if its polar system is the null system \( C \). A line will be called a skew line, tangent, secant, or generator of the quadric, according as it has 0, 1, 2, or 3 points in common with the quadric. It is easily verified from (26) that

(29) The \( 2^{2p} \) quadrics which belong to a given null system \( C \) have null lines for tangents and generators, ordinary lines for secants and skew lines.

Thus if \( x \) is a point of the quadric, \( u \) its null, or polar, or tangent \( S_{2p-2} \), all lines on \( x \) in \( u \) are generators or tangents of the quadric, and all lines on \( x \) and not in \( u \) are secants of the quadric; if however \( x \) is an outside point, all lines on \( x \) in \( u \) are tangents of the quadric, and all lines on \( x \) and not in \( u \) are secants or skew lines of the quadric.

(30) Quadrics in \( S_k \) have real points if \( k > 1 \).

For, if (25) contains no reference point, every \( a_{ii} \neq 0 \); if no point like 1, 1, 0, \( \ldots, 0 \), every \( a_{ik} \neq 0 \), but then it must contain points like 1, 1, 1, 0, \( \ldots, 0 \). 

Let \( C(x, x') = 0 \) be the equation of the null system \( C \), \( f(x, x) = 0 \) be a quadric belonging to \( C \); \( y \) be a point on \( f \), and \( I_y \) be the involution determined by \( y \) [see (12)]. \( C(x, y) = 0 \) is the tangent space of \( f \) at \( y \) and contains the tangents and generators of \( f \) which pass through \( y \). \( I_y \) transforms \( f \) into a quadric \( \phi(x, x) = 0 \), which has in common with \( f \) the point \( y \) and its tangent space. Any line through \( y \) and not in the tangent space has a further point.

* Cf., Dickson, Linear Groups, p. 201, footnote.
on $f$ and a further distinct point on $\varphi$. Thus $f$, $\varphi$, and $[C(x, y)]^2$ constitute a pencil, or since 

$$[C(x, y)]^2 = C(x^2, y^2) = C(x^2, y) \pmod 2,$$

we have

$$f(x, x) + C(x^2, y) = \varphi(x, x).$$

If $z$ is another point on $f$,

$$f(x, x) + C(x^2, z) = \psi(x, x),$$

whence

$$\varphi(x, x) + \psi(x, x) = C(x^2, y + z).$$

If $yz$ is a null line, the point $y + z$ is on $f$, $\varphi$, $\psi$; if $yz$ is an ordinary line, $y + z$ is not on $f$ nor $C(x^2, y) = C(x, y)$ and therefore is on $\varphi$ and similarly is on $\psi$. Then $\varphi$ and $\psi$ also are in a pencil with the square of their common tangent space, $C(x^2, y + z)$. If $f$ contains $n$ points, the $n + 1$ quadrics, $f$, $\varphi$, $\psi$, being conjugate with $f$ each contain $n$ points and each determines the set by means of its tangent spaces. If $r$ is a point outside $f$, $I_r$ leaves unaltered the contacts of tangents from $r$ to $f$ but interchanges the points on a secant line through $r$, i.e., leaves $f$ unaltered. Hence the set of $n + 1$ quadrics is a complete conjugate set under $G_{NC}$.

Let $y$ and $z$ be two points not on $f$, i.e., let $f(y, y) = 1$ and $f(z, z) = 1$. Let

$$f(x, x) + C(x^2, y) = \varphi(x, x), \quad f(x, x) + C(x^2, z) = \psi(x, x).$$

Then

$$\varphi(x, x) + \psi(x, x) = C(x^2, y + z).$$

But $y + z$ is a point of $\varphi$ and $\psi$, since

$$\varphi(y + z, y + z) = f(yy) + C(y, z) + f(zz) + C(z, y) = 0.$$ 

Hence the set of quadrics obtained by adding to $f$ the squares of its secant spaces is such that any one of the set differs from the others by the squares of its own tangent spaces. Again this set is a complete conjugate set. But all the quadrics which belong to $C$ differ from any one by the square of a tangent or secant space of the one. Hence there are only two distinct types of quadrics which belong to $C$. Since any proper quadric belongs to a proper null system and all proper null systems are conjugate under $G_N$ there are only two distinct types of proper quadrics in $S_{2p-1}$. *

$G_{NC}$ is doubly transitive on either conjugate set of quadrics. For two quadrics in the set conjugate to $f$ are associated with points $y$ and $z$ on $f$. If $yz$ is an ordinary line, $I_{r+1}$ leaves $f$ unaltered and interchanges $y$ and $z$. If $yz$ * Cf. Dickson, Linear Groups, p. 197.
is a null line there is a point \( t \) on \( f \) and not on the tangent spaces of \( y \) and \( z \); e. g., an ordinary line on \( y \) and a point outside the tangent space of \( z \) meets the quadric again at a point \( t \). The product \( I_{y+t} \cdot I_{y+t} \) leaves \( f \) unaltered and sends \( y \) into \( z \).

\( G_{NC} \) is simultaneously simply transitive on both sets of quadrics. For if \( y \) and \( z \) are two points not on \( f \) and if \( yz \) is an ordinary line, then \( y + z \) is not on \( f \) and \( I_{y+z} \) leaves \( f \) unaltered and interchanges \( y \) and \( z \). If \( yz \) is a null line touching \( f \) at \( y + z \), let \( t \) be a point on the tangent space of \( y + z \) but not on \( f \) nor \( C(x, y) \) and therefore not on \( C(x, z) \). Then \( yt \) and \( zt \) are skew to \( f \) and the product \( I_{y+t} \cdot I_{y+t} \) leaves \( f \) unaltered and sends \( y \) into \( z \). But \( y \) and \( z \) determine any two quadrics of the set which does not contain \( f \).

(31) The \( 2^{2p} \) proper quadrics which belong to \( C \) divide into two sets conjugate under \( G_{NC} \). Quadrics from the same set have contact along an \( S_{2p-2} \) tangent to both; quadrics from different sets have contact along a space secant to both. Every proper quadric in \( S_{2p-1} \) is conjugate under \( G_X \) with one or the other type. \( G_{NC} \) is doubly transitive on the quadrics in either set; and simultaneously simply transitive on the quadrics of both sets. The group of any quadric is simply transitive on the points of the quadric and also on the outside points of the quadric.

Let there be \( \tau \) points on, and \( (2p-1 - \tau) \) points outside, the proper quadric \( f \) which belongs to \( C \). According to (31) the quadric points are each of the same type; the same is true of the outside points. Let there be on an outside point, \( \rho_0, \rho_1, \rho_2 \) lines skew, tangent, secant, respectively to \( f \); on a quadric point, \( \sigma_1, \sigma_2, \sigma_3 \) lines respectively tangent to, secant to, on, \( f \). By joining an outside point to the other outside points and to the quadric points we get the equations

\[
2\rho_0 + \rho_1 = 2p+1 - \tau - 1, \quad \rho_1 + 2\rho_2 = \tau.
\]

In the same way from a quadric point we get

\[
2\sigma_1 \rho_2 = 2p+1 - \tau - 1, \quad \sigma_2 + 2\sigma_3 = \tau - 1.
\]

But \( \rho_1 = \sigma_1 + \sigma_2 \) is the number of null lines on a point, i. e., \( p_{2p-3} \). Hence we have, in terms of \( \tau \),

\[
2\rho_0 = 2p-1 - 2p-3 - \tau - 1, \quad \rho_1 = 2p-3, \quad 2\rho_2 = \tau - 2p-3,
\]

\[
2\sigma_1 = 2p-1 - \tau - 2^{2p-2}, \quad \sigma_2 = 2^{2p-2}, \quad 2\sigma_3 = \tau - 1 - 2^{2p-2}.
\]

The total number of tangents of \( f \) is either \( \sigma_1 \tau \) or \( \frac{1}{2} \rho_1 \) (\( 2p-1 - \tau \)). Equating these values we get

\[
\tau^2 - (2^{2p} - 2) \tau + (2^{2p} - 1) (2^{2p-2} - 1) = 0,
\]

\[
\tau = 2^{p-1} [2^{p+1}] - 1, \quad \tau = 2^{p-1} [2^{p+1}] - 1.
\]

Set

(32) \( E_p = 2p-1, \quad O_p = 2p-1 \).
For a quadric $E$ with $E_p - 1$ quadric points,
\[ \begin{align*}
\rho_0 &= 2^{p-2} P_{p-2} = O_p - 1, \\
\rho_1 &= P_{2p-3}, \\
\rho_2 &= E_{p-1},
\end{align*} \tag{33} \]
\[ \begin{align*}
\sigma_1 &= 2^{p-2}, \\
\sigma_2 &= 2^{2p-2}, \\
\sigma_3 &= E_{p-1} - 1.
\end{align*} \]

For a quadric $O$ with $O_p - 1$ quadric points,
\[ \begin{align*}
\rho_0 &= E_{p-1}, \\
\rho_1 &= P_{2p-3}, \\
\rho_2 &= O_{p-1},
\end{align*} \tag{34} \]
\[ \begin{align*}
\sigma_1 &= E_{p-1}, \\
\sigma_2 &= 2^{2p-2}, \\
\sigma_3 &= O_{p-1} - 1.
\end{align*} \]

If $\pi_0$, $\pi_1$, $\pi_2$, $\pi_3$, denote respectively the total number of skew lines, tangents, secants, and generators, of a quadric, then

For a quadric $E$,
\[ \begin{align*}
\pi_0 &= \frac{1}{2} \rho_0 [P_{2p-1} - \tau_X] = \frac{1}{2} O_p O_{p-1}, \\
\pi_1 &= \frac{1}{2} \rho_1 [P_{2p-1} - \tau_X] = \sigma_1 \tau_X = \frac{1}{2} P_{2p-4} O_p = O_{p-1} [E_p - 1], \\
\pi_2 &= \rho_2 [P_{2p-1} - \tau_X] = \frac{1}{2} \sigma_2 \tau_X = O_p E_{p-1} = 2^{2p-3} (E_p - 1), \\
\pi_3 &= = \frac{1}{2} \sigma_3 \tau_X = \frac{1}{3} (E_p - 1) (E_{p-1} - 1). \\
\end{align*} \tag{35} \]

For a quadric $O$,
\[ \begin{align*}
\pi_0 &= \frac{1}{2} \rho_0 [P_{2p-1} - \tau_0] = \frac{1}{3} E_p E_{p-1}, \\
\pi_1 &= \frac{1}{2} \rho_1 [P_{2p-1} - \tau_0] = \sigma_1 \tau_0 = \frac{1}{2} P_{2p-4} E_p = E_{p-1} (O_p - 1), \\
\pi_2 &= \rho_2 [P_{2p-1} - \tau_0] = \frac{1}{2} \sigma_2 \tau_0 = E_p O_{p-1} = 2^{2p-3} (O_p - 1), \\
\pi_3 &= = \frac{1}{3} \sigma_3 \tau_0 = [O_p - 1] [O_{p-1} - 1].
\end{align*} \tag{36} \]

If $\phi = f + C (x^2, y)$, then $\varphi$ contains the points of both or neither of $f$ and $C (x^2, y)$. If $f$ is an $E$ quadric and $y$ a point not on it, $\varphi$ contains
\[ P_{2p-3} + [O_p - (P_{2p-2} - P_{2p-3})] = O_p - 1 \]
points, and $\varphi$ is an $O$ quadric. If $f$ is an $O$ quadric and $y$ a point not on it, $\phi$ contains
\[ P_{2p-3} + [E_p - (P_{2p-2} - P_{2p-3})] = E_p - 1 \]
points, and is an $E$ quadric. Hence

(37) The $2^{2p}$ quadrics which belong to $C$ divide into a set of $E_p = 2^{p-1} (2^p + 1)$ quadrics $E$, each containing $E_p - 1$ points; and a set of $O_p = 2^{p-1} (2^p - 1)$ quadrics $O$ each containing $O_p - 1$ points. The number of skew lines, tangents, secants, and generators, of these quadrics, and the number of similar lines on a quadric point or an outside point are furnished by the formulæ (33), · · ·, (36).
It is convenient to use $Q$, $\bar{Q}$ to denote at the same time $E$, $O$ and $O$, $E$. The following relations among the numbers defined above are sometimes useful:

$$2Q_{p-1} (\bar{Q}_p - 1) = P_{2p-1} Q_p, \quad Q_p \bar{Q}_p = 2^{2p-3} P_{2p-1},$$

$$Q_p (Q_p - 1) = 2P_{2p-1} Q_{p-1}, \quad Q_{p-1} \bar{Q}_p = 2^{2p-3} (Q_p - 1).$$

A quadric associated with the null system $C$ in (13) is

$$q_p = x_1 x_{p+1} + x_2 x_{p+2} + \cdots + x_p x_{2p} = 0.$$  

Of the $p$ terms in $q_p$ each takes the value 0 in 3 ways, namely, 00, 01, 10; the value 1 in one way, 11. But $q_p = 1$ if $2k + 1$ terms take the value 1 and $p - 2k - 1$ terms take the value 0 which occurs in $(2^p + 1) 2^{p-2k}$ ways. Hence the number of points not on $q_p$ is

$$\sum_k (2^p + 1) 2^{p-2k} = \frac{1}{2} [(3 + 1)^p - (3 - 1)^p] = \frac{1}{2} [2^{2p} - 2^p] = O_p.$$  

Hence $q_p$ is an $E$ quadric. The quadric

$$q_p + \sum_{i=1}^p a_i x_i^2 = q_p + \left[ \sum_{i=1}^p a_i x_i \right]^2$$

is of the same type as $q_p$ provided $\sum a_i x_i$ is the tangent space of a point on $q_p$, i.e., if $\sum a_i a_{i+p} = 0$. Hence

$$(40) \quad The 2^{2p} quadrics obtained by varying the a's in$$

$$\sum_{j=1}^p x_j x_{j+p} + \sum_{i=1}^p a_i x_i^2 = 0,$$

which belong to the null system

$$C = \sum_{j=1}^p (x_j x_{j+p} - x_{j+p} x_j) = 0,$$

are of the type $E$ or $O$ according as $\sum a_j a_{j+p} = 0$ or $= 1$.

Other canonical forms of a quadric are useful. Let

$$y_1 y'_1 + y_2 y'_2 + \cdots + y_{2p+1} y'_{2p+1} = 0$$

be the equation of $C$ referred to one of its self dual bases. The quadric

$$\Sigma y_i y_k = 0 \quad (i, k = 1, \cdots, 2p + 1; i < k),$$

has the polar system $C$. Any two points of the basis are on a line with a third point all of whose coordinates except two are zero; call these the residual...
points of the basis. Evidently no residual point is on the quadric. Conversely if a quadric contains no residual point it is the above quadric. For, using \( y_2, \ldots, y_{2p+1} \) as a reference basis, the residual points become the reference points and the points with only two coordinates not zero. Since the reference points are not on the quadric every square appears. Also every product term appears, else the corresponding residual point is on the quadric. But

\[
(y_1^2 + \cdots + y_{2p+1}^2) = (y_1 + \cdots + y_{2p+1})^2 = y_1 (y_2 + \cdots + y_{2p+1}),
\]

and the quadric has the given form.

To find the number of points on the quadric and therefore its kind, let \( 2A \) be the number of coordinates of a point which are not zero. The point is on the quadric if \( 2A (2A - 1) \equiv 2 \pmod{4} \), i.e., if \( 2A - 1 \) is even. Then the total number of points on the quadric is \( \sum_{i=1}^{g} \binom{2p+1}{i} \), where \( g \) is the greatest integer for which \( 4g < 2p + 1 \). Since

\[
4 \sum_{i=0}^{g} \binom{2p+1}{i} = (1 + 1)^{2p+1} + (1 + i)^{2p+1} + (1 - 1)^{2p+1} + (1 - i)^{2p+1},
\]

we find that

\[
\sum_{i=0}^{g} \binom{2p+1}{i} = \frac{1}{4} \left( 2^{2p+1} + (1 + i)^{2p+1} + (1 - i)^{2p+1} \right) - 1.
\]

If \( p = 4n + m \) this reduces to

\[
2^{2p-1} + 2^{p-1} \left[ \frac{(1 + i)(2i)^m + (1 - i)(-2i)^m}{2^{m+1}} \right] - 1,
\]

i.e., to \( E_p - 1 \) if \( p = 0, 3 \pmod{4} \); and to \( O_p - 1 \) if \( p = 1, 2 \pmod{4} \). If \( p \) is even the basis points are on the quadric.

(41) A basis, \( y_1, \ldots, y_{2p+1} \), self dual under \( C \), determines uniquely a quadric

\[
Q = \sum_{i} y_i y_k = 0 \quad (i, k = 1, \ldots, 2p + 1; i < k),
\]

belonging to \( C \), which contains none of the residual points of the basis. If \( p = 0 \pmod{2} \), \( Q \) contains the basis points; if \( p = 1 \pmod{2} \), the lines of the basis are skew to \( Q \). If \( p = 0, 3 \pmod{4} \), \( Q \) is an \( E \) quadric; if \( p = 1, 2 \pmod{4} \), an \( O \) quadric.

§4. The Theta Characteristics as Quadrics in \( S_{2p-1} \) Modulo 2 Belonging to \( C \).

If \( \omega_* \) is the half period whose Per. Char. is \( \epsilon \), and \( E \) is a proper exponential factor, then \( E\theta(v + \omega_*) \), considered as a function of \( v \), is a theta function

\[\text{K.}, \text{p. 240. Formulae (a), (b), (c) are given by K., p. 242; formula (d), p. 247.}\]
θ [ε]_2 (v), whose \textit{theta characteristic}, or Th. Char., is ε. In particular the zero Per. Char. gives rise to the original theta function whose Th. Char. therefore is ε_0 = 0. Two functions whose Th. Char. are congruent modulo 2 arise from two similar Per. Char. and are not essentially distinct. There are then \(2^{2p}\) Th. Char. including the zero Th. Char. The function with Th. Char. ε is \textit{even} or \textit{odd} according as

\[ \sum_{μ=1}^{p} ε_μ ε_{p+μ} = 0 \text{ or } \equiv 1 \pmod{2}. \]

Under integral linear transformation of the periods,

(a) \[ \bar{ω}_{μα} = \sum_{β=1}^{2p} c_{αβ} ω_{μβ} \quad (μ = 1, 2, \cdots, p; α = 1, 2, \cdots, 2p), \]

the Per. Char. are transformed as follows:

(b) \[ ε_β = \sum_{α=1}^{2p} c_{αβ} \bar{ε}_α \quad (β = 1, 2, \cdots, 2p). \]

The coefficients \(c_{αβ}\) are such that \(\sum_{μ=1}^{p} (ε_μ \eta_{p+μ} - ε_{p+μ} \eta_β)\) is invariant, i.e.,

(c) \[ \sum_{μ=1}^{p} (c_{μβ} \epsilon_{p+μ, γ} - c_{μγ} \epsilon_{p+μ, β}) = \begin{cases} 1 & \text{if } γ = p + β, \\ 0 & \text{if } γ ≠ p + β. \end{cases} \]

The \(2^{2p}\) functions \(θ [ε]_2 (v)\) are transformed, to within exponential factors, into a similar system \(\mathfrak{F} [\bar{ε}]_2 (v)\), the Th. Char. of the two systems being connected by the equations

(d) \[ \bar{ε}_γ = \sum_{μ=1}^{p} (c_{μv} ε_μ - c_{ν, p+μ} ε_{p+μ} + c_{νμ} c_{ν, p+μ}), \quad (v = 1, 2, \cdots, p), \]

\[ \bar{ε}_{p+v} = \sum_{μ=1}^{p} (-c_{p+v, v} ε_μ + c_{p+v, p+μ} ε_{p+μ} + c_{p+v, μ} c_{p+v, p+μ}) \]

Since these equations hold as congruences modulo 2 and since \(c_{αβ}^2 = c_{αβ}\) and \(-1 ≡ 1 \pmod{2}\), we can modify them so as to read

\[ \bar{ε}_β = \sum_{μ=1}^{p} (c_{βμ} c_{β, p+μ} + c_{βμ}^2 ε_μ + c_{β, p+μ} ε_{p+μ}) \quad (β = 1, 2, \cdots, 2p). \]

In § 2 we have identified the points, \(x_1, x_2, \cdots, x_{2p}\), of \(S_{2p-1} \pmod{2}\) with the Per. Char. ε; and the collineations \(x_β = \sum_{α=1}^{2p} c_{αβ} \bar{x}_α\) of \(S_{2p-1}\) which leave the null system, \(C = \sum_{μ=1}^{p} (x_μ x_{p+μ} - x_{p+μ} x_μ)\), unaltered with the transformations

\[ * K., p. 240. \] Formulae (a), (b), (c) are given by K., p. 242; formula (d), p. 247.
Consider the effect of such a collineation upon the quadric
\[ \sum_{\mu=1}^{P} (x_{\mu} x_{p+\mu} + \epsilon_{\mu} x_{\mu}^2 + \epsilon_{p+\mu} x_{p+\mu}^2), \]
which belongs to \( C \). It must be transformed into another of the same sort, say with coefficients \( \tilde{\epsilon} \). By effecting the collineation upon the quadric and making use of the relations (c) the coefficients \( \tilde{\epsilon} \) of the transformed quadric turn out to be those defined by (42). Hence by making use of (40), we obtain the second fundamental theorem:

\begin{equation}
(44) \text{Under integral linear transformation of the periods the } 2^{2p} \text{ Th. Char. are permuted just as the } 2^{2p} \text{ quadrics in } S_{2p-1} \text{ modulo 2 which belong to } C \text{ are permuted under the collineation group } G_{N_{c}} \text{ of } C. \text{ The theta function with given Th. Char. is odd or even according as the corresponding quadric is an } O \text{ or an } E \text{ quadric. Thus the parity of the characteristic is invariant under such transformation.}\end{equation}

According to the formula
\[ \theta [\epsilon]_{2}(v + \omega_{n}) = E\theta [\epsilon + \eta]_{2}(v), \]
the function \( \theta [\epsilon]_{2}(v) \) vanishes when \( v \) is \( \omega_{n} \) if \( \theta [\epsilon + \eta]_{2}(v) \) is an odd function. Regarding \( \theta [\epsilon]_{2}(v) \) as a quadric \( Q \) and \( \omega_{n} \), as a point \( P \), then \( \theta [\epsilon + \eta]_{2}(v) \) is the quadric \( Q' \) obtained by adding to \( Q \) the square of the null \( S_{2p-2} \) of \( P \). If \( Q' \) is an \( O \) quadric then, either \( Q \) is an \( E \) quadric and \( P \) is not on \( Q \) or \( Q \) is an \( O \) quadric and \( P \) is on \( Q \). We have therefore a further translation scheme:

\begin{align*}
\text{An } E \text{ (or } O \text{) quadric which belongs to } C. & \quad \text{A theta function with an even (or odd) Th. Char.} \quad \text{An even theta function does not (or does) vanish for a half period with given Per. Char.} \\
\text{An } E \text{ quadric does (or does not) contain a given point.} & \quad \text{An odd theta function does (or does not) vanish for a half period with given Per. Char.} \\
\text{An } O \text{ quadric does (or does not) contain a given point.} & \quad \text{A theta function with an even (or odd) Th. Char.} \quad \text{An even theta function does not (or does) vanish for a half period with given Per. Char.} \\
\end{align*}

The number \( \sharp \) of odd and of even thetas is gotten from (37), while the enumerations contained in (32), \( \cdots \), (36) characterize very fully the behavior of a particular theta with regard to the sets of three syzygetic or three azygetic Per. Char.

* Cf. the proof of the invariance of parity given by K., pp. 247–50.
† K., p. 240 (VII).
‡ K., p. 252.
§ 5. The Period and Theta Characteristics as a Linear System in $R_{2p}$ with Reference to a Quadric. Projection and Section Applied to Quadrics.

Steiner and Kummer Sets.

We have already remarked that the sum of two quadrics belonging to $C$ is the square of an $S_{2p-2}$. More generally, a sum of a number of quadrics and a number of squared $S_{2p-2}$’s is a quadric or a squared $S_{2p-2}$ according as the number of quadrics in the sum is odd or even.* Thus the $2^{2p}$ quadrics and $2^{2p} - 1$ squared $S_{2p-2}$’s lie in a linear system containing $2^{2p+1} - 1$ elements, i.e., in a linear space $R_{2p}$. A concrete representation of this linear system is obtained by mapping the points of $S_{2p-1}$ on the points of a quadric $M$ in $R_{2p}$ by means of $2p + 1$ independent quadrics in $S_{2p-1}$ belonging to $C$. Taking the convenient canonical form of (40), let

$$z_0 = \sum_{j=1}^{p} x_j \cdot x_{p+j}, \quad z_i = x_i^2 \quad (i = 1, 2, \ldots, 2p).$$

Then the points $x$ of $S_{2p-1}$ are mapped on the points $z$ of

$$M = z_0^2 + \sum_{j=1}^{p} z_j z_{p+j} = 0.$$

$M$ is the general type of non-degenerate quadric in $R_{2p}$. The collineation group which leaves it unaltered is simply isomorphic with $G_{2p}$.† The null system of $M$ is necessarily degenerate whence there is one point, $z_0 = 1$, $z_i = 0$, whose polar $R_{2p-1}$ as to $M$ is evanescent. This point we shall call the vertex $V$ of $M$. Since $C$ connects a point of $S_{2p-1}$ with its null $S_{2p-2}$ or also with the square of its null $S_{2p-2}$ we can identify the period characteristics with the squared $S_{2p-2}$’s and thus show that

(48) The Per. Char. and Th. Char. can be represented as the linear system of $2^{2p+1} - 1$ $R_{2p-1}$’s in a space $R_{2p}$ with reference to a proper quadric $M$. The $R_{2p-1}$’s on the vertex $V$ of $M$ correspond to the Per. Char.; those not on $V$ correspond to the Th. Char., which are odd or even according as the $R_{2p-1}$ cuts $M$ in an $O$ or an $E$ quadric.

An $R_{2p-1}$ on $V$ cuts $M$ in a quadric section with a double point $z$ on $M$. The point $z$ is the map of a point $x$ in $S_{2p-1}$ whose null $S_{2p-2}$ corresponds to the $R_{2p-1}$. This is the trace in $R_{2p}$ of the null system $C$.

The above representation with reference to $M$ in $R_{2p}$ will be retained only in the background for purposes of suggestion, the important feature being the linearity of the entire system of characteristics. As an instance of the usefulness of this feature let $x$ be a given point in $S_{2p-1}$. It is a linear condition that a quadric or $S_{2p-2}$ be on $x$, whence there are $2^{2p} - 1$ quadrics and $S_{2p-2}$’s

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* K, p. 254.
† Dickson, Linear Groups, p. 197, §§ 199, 200.
and $2^{2p-1} - 1 S_{2p-2}$'s on $x$. There are then $2^{2p-1}$ quadrics on $x$. These quadrics are paired by the involution, $I_x$, the members of a pair having the same section by the null $S_{2p-2}$ of $x$. Such a set of $2 \cdot 2^{(p-1)}$ quadrics will be called a first Steiner set or the Steiner set of $x$. Again if $x^{(1)}$ and $x^{(2)}$ are two points of a null line a quadric or $S_{2p-2}$ on both must contain the third point of the line. This imposes two linear conditions on the quadrics and $S_{2p-2}$'s whence there are $2^{2p-1} - 1$ quadrics and $S_{2p-2}$'s and $2^{2p-2} - 1 S_{2p-2}$'s on the line. Hence there are $4 \cdot 2^{(p-2)}$ quadrics on the line which divide into $2^{2(p-2)}$ sets of 4, each set of 4 having the same section by the null $S_{2p-3}$ of the null line $x^{(1)} x^{(2)}$. The members of a set of 4 are permuted transitively by the Abelian $G_4$ generated by the involutions $I_{x^{(1)}}$ and $I_{x^{(2)}}$. Such a set of $2^2 \cdot 2^{(p-2)}$ quadrics will be called a second Steiner set or the Steiner set of the null line $x^{(1)} x^{(2)}$. Evidently the argument can be carried on to the limit set by the null $S_{p-1}$ and we can say generally that

(49) A null $S_{m-1}$ ($m = 1, \ldots, p$) determines an $m$th Steiner set of $2^m \cdot 2^{(p-m)}$ quadrics which contain $S_{m-1}$. They divide into $2^{2(p-m)}$ sets of $2^m$ quadrics. Each set of $2^m$ quadrics has the same section by the null $S_{2p-m-1}$ of $S_{m-1}$ and its members are permuted regularly by the Abelian $G_m$ generated by the involutions $I_x$ of the points $x$ of a reference basis of $S_{m-1}$.†

Further properties of the quadrics of an $m$th Steiner set can be derived from the projection and section of $C$ from $S_{m-1}$ and within $S_{2p-m-1}$ as explained after (23). We have then a derived space $\Sigma_{2p-1}$, $\pi = p - m$, whose points correspond to $S_{m}$'s on $S_{m-1}$ and within $S_{2p-m-1}$. Evidently a set of $2^m$ quadrics on $S_{m-1}$ which have the same section by $S_{2p-m-1}$ determine in $\Sigma_{2p-1}$ a single quadric. Hence the above $2^{2r}$ sets of $2^m$ quadrics determine in $\Sigma_{2p-1}$ the $2^{2r}$ quadric belonging to the null system $\Gamma_x$. If a point of $\Sigma_{2p-1}$ lies on a quadric, the corresponding $S_m$ on $S_{m-1}$ lies on each of the corresponding set of $2^m$ quadrics in $S_{2p-1}$. For $m = 1$, we see from the values of $s_3$ in (33) and (34) that the section of a quadric of type $Q$ is a quadric of the same type. Since the general projection and section can be effected by projections and sections from successive points it is clear that the type of quadric is unaltered in the process. Thus from (37) for $p = \pi$ we find that

(50) The $2^{(p-m)}$ sets of $2^m$ quadrics in the $m$th Steiner set of a null $S_{m-1}$ divide into $Q_{p-m}$ sets of $2^m$ $O$ quadrics and $E_{p-m}$ sets of $2^m$ $E$ quadrics.

For the important particular case, $m = 1$, the enumerations contained in (32) to (36) lead to the following results in $S_{2p-1}$:

(51) Given a $Q$ quadric and a line $l$ tangent to $Q$ at $x$; there are $Q_{p-2}$ planes on $l$ containing two further tangents at $x$, $P_{2p-5}$ planes on $l$ containing a further tangent and generator on $x$, and $Q_{p-2}$ null planes on $l$ containing two further

* When $p = 3$, the odd quadrics of the set correspond to the well known Steiner complex of 12 double tangents of a plane quartic curve.

† For $m = 1, 2, 3$, cf. K., pp. 255-65.
generators on \( x \); given a generator \( m \) of \( Q \) on \( x \), there are \( Q_{p-2} \) planes on \( m \) containing two further tangents at \( x \), \( 2^{2p-4} \) planes on \( m \) containing a further tangent and generator on \( x \), and \( Q_{p-2} - 1 \) null planes on \( m \) containing two further generators on \( x \). The number of planes on \( x \) containing respectively three tangents, two tangents and a generator, a tangent and two generators, and three generators, of \( Q \) on \( x \) is \( \frac{1}{3} Q_{p-2} Q_{p-1}, Q_{p-2} [Q_{p-1} - 1], Q_{p-2} Q_{p-1}, \) and \( \frac{1}{3} (Q_{p-2} - 1) \).

Call a null \( S_{m-1} \) contained in \( Q \) a generator \( S_{m-1} \) of \( Q \); and a null \( S_m \) which meets \( Q \) in a null \( S_{m-1} \) a tangent \( S_m \) of \( Q \). Then after projection and section from \( S_{m-1} \) on \( Q \), the earlier enumerations lead to the following results:

(52) Given a generator \( S_{m-1} \) of \( Q \), and a tangent \( S_m \) containing it; there are on \( S_m \), \( Q_{p-m-1} S_{m+1} \)'s containing two other \( S_m \)'s tangent along \( S_{m-1} \), \( P_2(p-m-2) \) \( S_{m+1} \)'s containing another \( S_m \) tangent along \( S_{m-1} \) and a generator \( S_m \), and \( Q_{p-m-1} \) null \( S_{m+1} \)'s containing two generator \( S_m \)'s; given a generator \( S_m \) on \( S_{m-1} \), there are on \( S_m \), \( Q_{p-m-1} S_{m+1} \)'s containing two \( S_m \)'s tangent along \( S_{m-1} \), \( 2^{2(p-m-1)} \) \( S_{m+1} \)'s containing a generator \( S_m \) and a tangent \( S_m \), and \( Q_{p-m-1} - 1 \) null \( S_{m+1} \)'s containing two other generator \( S_m \)'s. On \( S_{m-1} \), the number of \( S_{m+1} \)'s containing respectively three tangent \( S_m \)'s, two tangents \( S_m \)'s and a generator \( S_m \), a tangent \( S_m \) and two generator \( S_m \)'s, and there generator \( S_m \)'s is \( \frac{1}{3} Q_{p-m} Q_{p-m-1}, Q_{p-m} Q_{p-m-1}, \) and \( \frac{1}{3} [Q_{p-m} - 1][Q_{p-m-1} - 1] \).

Some fairly obvious deductions from the above general theorems can now be drawn; e. g., from (50), for \( m = 1 \), it is clear that

(53) An \( S_{2p-2} \) can be expressed in \( Q_{p-1} \) ways as a sum of two \( O \) quadrics; in \( E_{p-1} \) ways as a sum of two \( E \) quadrics; and in \( 2^{2p-2} \) ways as a sum of an \( O \) and an \( E \) quadric. Or, if a squared \( S_{2p-2} \) be added to the \( 2^{2p} \) quadrics, then of the \( Q \) \( Q \) quadrics, \( 2Q_{p-1} \) become \( Q \) quadrics while the remaining \( 2^{2p-2} \) become \( Q \) quadrics.*

Two points determine a third on their join. If the join is a null line every quadric contains only one or all of the three points; those containing all constituting the second Steiner set of the line. If the join is an ordinary line a quadric on one point must contain a second point but cannot contain all three. In particular a pair on one of the points do not meet in another of the three. Calling the first Steiner sets of the points syzygetic or azygetic according as the points are syzygetic or azygetic [see (14)], we have

(54) A first Steiner set is determined uniquely by any one of its \( Q_{p-1} \) pairs of \( Q \) quadrics. Two first Steiner sets determine another first Steiner set, the three being symmetrical. If the three are syzygetic, they together contain all the quadrics and have in common a second Steiner set; i. e., the sets have \( 4Q_{p-1} \) \( Q \) quadrics in common, while \( 3 \cdot 2^{p-2} \) \( Q \) quadrics occur in only one set. If the three are azygetic, they have no common quadrics, any two have \( Q_{p-1} \) \( Q \) quadrics in com-

* K., p. 258 (VI) and (VII).
mon, but have no pairs in common. Together the three contain $3Q_{p-1}$ $Q$ quadrics, and $Q_{p-1}$ $Q$ quadrics are found in none.*

Applying (54) to the projection from $S_{m-1}$ and translating the result from $\Sigma_{a-1}$ back to $S_{a-1}$, we obtain the theorem:

(55) If three null $S_m$'s on a null $S_{m-1}$ are in an ordinary $S_{m+1}$, no quadric contains all three but any two are contained in $2^m Q_{p-m-1}$ $Q$ quadrics; if the three are in a null $S_{m+1}$ every quadric on $S_{m-1}$ contains at least one, while $3 \cdot 2^m 2^{2(p-m)-3}$ $Q$ quadrics contain only one and $4 \cdot 2^m Q_{p-m-2}$ $Q$ quadrics contain all three.

In (54) we have considered the Steiner sets of three points of a line. Let us suppose the three points form a triangle. To contain the points is three independent linear conditions on a quadric or $S_{2p-2}$ whence there are $2^{2p-2} - 1$ quadrics and $S_{2p-2}$'s and $2^{2p-3} - 1 S_{2p-2}$'s on the three points or

(56) The Steiner sets of three points which form a triangle have $2^{2p-3}$ quadrics in common.

Four cases are possible according as 0, 1, 2, or 3, of the three sides of the triangle are null lines. Drawing null lines full, we see from the figure of Case I

that a quadric on $y, z, t$ cannot contain $y + z + t$, else the three null lines are generators and therefore also the sides contrary to the hypothesis that they are ordinary lines. The null space $S$ of $y + z + t$ contains the triangle and is a secant space of all quadrics on the triangle. If $S$ be added to all these quadrics each $Q$ quadric on the triangle becomes a $Q$ quadric on the triangle whence the number of $Q$ and $Q$ quadrics is the same. Thus from (56) we find

(57) The Steiner sets of three points which form a triangle with ordinary sides have in common $2^{2p-4}$ $Q$ quadrics unpaired in each set. Each $Q$ quadric meets a definite $Q$ quadric on the null $S_{2p-2}$ of the null subspace of the plane of the triangle.†

In case IV the plane of the triangle is a null plane and every quadric on the triangle contains the plane whence

(58) The first Steiner sets of three points which form a triangle with null sides have in common the third Steiner set of the plane of the triangle.‡

†Cf. K., p. 263 (X).
‡Cf. K., p. 265 (XIII).
From the figure in case II we see that a quadric on \( y, z, t \) contains two null lines on \( z + t \), but not the third, and conversely. Hence theorem (55), for \( m = 1 \), can be applied. In case III a quadric on \( y, z, t \) contains two null lines on \( t \) but not the third, and the same theorem applies. Hence

(59) The Steiner sets of three points which form a triangle with one or with two null sides have in common 2 \( Q_{p-2} \) Q quadrics. In the first case the quadrics are not paired in any set though they are paired in the Steiner set of the third point of the null side. In the second case the quadrics are paired in the Steiner set of the point on the two null sides.*

Some of these theorems are easily generalized. Thus (58) and the first case of (54) are special cases of the following:

(60) If any number of null spaces lie in a null \( S_{m-1} \) their Steiner sets have in common the \( m \)th Steiner set of the null \( S_{m-1} \).

Following out (57), let \( x^{(1)}, \ldots, x^{(2k+1)} \) be an odd number of points of a self dual basis of \( C \). They lie in an \( S_{2k} \) which has an \( S_0, x^{(1)} + \cdots + x^{(2k+1)} \), as a null subspace. As in (56), there are \( 2^{2p-2k-1} \) quadrics on the points which are interchanged in type by adding the null \( S_{2p-2} \) of \( S_0 \) which is secant to all. Hence

(61) The Steiner sets of an odd number, \( 2k + 1 \), of points of a self dual basis of \( C \) have in common \( 2^{2p-2k-2} \) Q quadrics which are unpaired in each set.

If however we have an even number \( 2p \) of points of a self dual basis of \( C \) as in the second case of (54), their \( S_{2p-1} \) has no null subspace and the above argument does not apply. \( S_{2p-1} \) determines its null space \( S_{2(p-p)-1} \), which also has no null subspace; in fact, each is the null space of the other and the two are skew. Let \( \Gamma_\rho \) be the section of \( C \) by \( S_{2p-1} \) [as defined after (22)], \( \Gamma_{p-\rho} \) the section of \( C \) by \( S_{2(p-p)-1} \). These two sections define \( C \). For if \( x \) is a point of \( S_{2p-1} \), its null \( S_{2p-2} \) under \( \Gamma_\rho \) together with \( S_{2(p-p)-1} \) are contained in an \( S_{2p-2} \), the null space of \( x \) under \( C \); similarly, if \( y \) is a point of \( S_{2(p-p)-1} \). If \( z \) is a point of neither, the \( S_{2p} \) joining \( z \) to \( S_{2p-1} \) meets \( S_{2(p-p)-1} \) in a point \( y \), the line \( yz \) meets \( S_{2p-1} \) in a point \( x \) and the null \( S_{2p-2} \) of \( z \) is determined from those of \( x \) and \( y \); the line \( yzx \) is of course a null line. If a reference basis of \( S_{2p-1} \) be chosen by taking a reference basis \( x \) of \( S_{2p-1} \) and a reference basis \( y \) of \( S_{2(p-p)-1} \), a quadric belonging to \( C \) can have only product terms in \( x \) and product terms in \( y \). Since the squared terms also are separable the quadric is determined by its two sections. Hence

(62) If \( S_{2p-1} \) and \( S_{2(p-p)-1} \) are skew null spaces of each other under \( C \), and if \( \Gamma_\rho \) and \( \Gamma_{p-\rho} \) are their sections of \( C \), then \( C \) is determined by the two sections. If \( Q' \) is a quadric in \( S_{2p-1} \) belonging to \( \Gamma_\rho \), \( Q'' \) a quadric in \( S_{2(p-p)-1} \) belonging to \( \Gamma_{p-\rho} \) there is a single quadric \( Q \) belonging to \( C \) which contains the sections \( Q' \) and \( Q'' \). \( Q \) is an \( E \) quadric if \( Q' \) and \( Q'' \) are of the same type, an \( O \) quadric if \( Q' \) and \( Q'' \) are of different types.

The last statement is proved as follows: Any line $xy$ is a null line and therefore a tangent or generator of $Q$. If $x$ is on $Q'$, $y$ on $Q''$ it is a generator and contains a point $z$ of $Q$. If $x$ is on $Q'$ and $y$ not on $Q''$ it must be a tangent to $Q$ at $x$. If $x$ is not on $Q'$ and $y$ is on $Q''$ it must be a tangent to $Q''$ at $y$. If $x$ is not on $Q'$ and $y$ not on $Q''$ it must be a tangent to $Q$ at $z$. Since every point $z$ of $Q$ is on one such line, the number of points on $Q$ if $Q'$ and $Q''$ are of the same sort is

$$(Q_p - 1) (Q_{p-p} - 1) + (Q_{p-p} - 1) + (Q_p - 1) + Q_p Q_{p-p} = E_p - 1$$

and $Q$ is an $E$ quadric. If $Q'$ and $Q''$ are unlike the number of points on $Q$ is

$$(Q_p - 1) (Q_{p-p} - 1) + (Q_p - 1) + (Q_{p-p} - 1) + Q_p Q_{p-p} = O_p - 1$$

and $Q$ is an $O$ quadric.

The above suggests for $p = 1$ an obvious construction for the null system $C$ and its quadrics in $S_{2p-1}$ when a null system and its quadrics in $S_{2p-3}$ are given.

According to (62), the $2^{2p}$ quadrics $Q$ are determined by pairing the $2^{2p}$ quadrics $Q'$ with the $2^{2(p-p)}$ quadrics $Q''$. If $Q$ has the section $Q'$ every quadric $Q + S^2$ where $S$ is a null $S_{2p-2}$ on $S_{2p-1}$ has the same section. $S$ then must be the null $S_{2p-2}$ of a point $y$ on $S_{2(p-p)-1}$. If $y$ lies on $Q''$, $Q + S^2$ is of the same type as $Q$, otherwise of a different type. Thus

(63) If $S_{2p-1}$ has no null subspace and if $E (p)$ is an $E$ quadric belonging to the section of $C$ by $S_{2p-1}$, there are $E_{p-p} E$ quadrics and $O_{p-p} O$ quadrics containing $E_p$; if $O (p)$ is an $O$ quadric belonging to the section of $C$ by $S_{2p-1}$ there are $O_{p-p} E$ quadrics and $E_{p-p} O$ quadrics containing $O (p)$.

For $p = 1$ we have the second case in (54), since then a quadric $E (p)$ is a pair of points and a quadric $O (p)$ has no real points.

We have therefore a method for determining the number and kind of quadrics on a group of points which lie in a space which has no null subspace. Taking up again the case of $2p$ points, azygetic in pairs, which lie in $S_{2p-1}$ we first find how many quadrics $Q (p)$ are on these points. Taking the points as a reference basis in $S_{2p-1}$ the quadric on them must be

$$\Sigma x_i x_k = 0 \quad (i, k = 1, \ldots, 2p; \ i < k).$$

To find the type of this quadric we note that a point lies on it if $4r$ or $4r + 1$ of its coordinates are not zero. The number of its points is $(\frac{1}{2}) + (\frac{1}{2}) + (\frac{1}{2}) + \cdots$. Adding and subtracting $(\frac{1}{2}) = 1$, this number is obtained from the expansions of $(1 + i)^{2p}$, $s = 0, 1, 2, 3$, and is

$$\frac{1}{2} \{2 (1 + 1)^{2p} + (1 - i) (1 + i)^{2p} + (1 + i) (1 - i)^{2p}\} - 1;$$

i. e., $E_p - 1$ if $p = 0, 1 \ (mod \ 4)$ and $O_p - 1$ if $p = 2, 3 \ (mod \ 4)$. Hence from (63),
The Steiner sets of $2p$ points of a self dual basis of $C$ have in common $Q_{p-p}$ quadrics if $p \equiv 0, 1 \pmod{4}$; and $\bar{Q}_{p-p}$ quadrics if $p \equiv 2, 3 \pmod{4}$. When $p = 1$ we obtain again the second case of (54). Theorems (61) and (64) exhaust the cases arising from points of a self dual basis of $C$.

The above canonical form of a quadric is of especial interest for $p = p$. It is determined by $2p$ points of a basis and must coincide with the quadric of (41) when $p$ is even.

If the $2p$ points of the reference basis $x$ of $S_{2p-1}$ belong to a self dual basis of $C$, the unique quadric belonging to $C$ and on the $2p$ basis points

$$\sum x_i x_k = 0$$

coincides with the quadric (41) when $p$ is even. When $p$ is odd the quadric does not contain the $(2p + 1)$th basis point nor the residual points of the reference basis, but does contain the remaining residual points of the basis. It is an $E$ or an $O$ quadric according as $p \equiv 0, 1$ or $p \equiv 2, 3 \pmod{4}$.

We define a Kummer set of quadrics to be all the quadrics which do not pass through a point. The Kummer and Steiner sets of a point exhaust the $2^{2p}$ quadrics. A Caporali set of quadrics consists of all the quadrics which are skew to an ordinary line.* By analogy with the Steiner sets, a Caporali set might be called a second Kummer set. But the analogy could be carried no further, since any $S_k$, $k > 1$, meets every quadric in real points.

Quadrics and squared $S_{2p-2}$s constitute a linear system, $R_{2p}$; squared $S_{2p-2}$s constitute a linear system $S_{2p-1}$. The quadrics common to the Kummer sets of $r$ linearly independent points satisfy $r$ inequalities and by the use of (3) we see that

The Kummer sets of $r$ linearly independent points have $2^{2p-r}$ quadrics in common.

An obvious argument from (54) shows that

The Kummer set of a point contains $2^{2p-2}$ quadrics. The Kummer sets of two syzygetic points have $2^{2p-4}$ quadrics in common, of three points on a null line no quadrics in common. The Kummer sets of two or three points on an ordinary line have $\bar{Q}_{p-1}$ quadrics in common which constitute a Caporali set.

Let us consider the Kummer sets of three points which form a triangle taking up the Cases I, •••, IV defined above. In case I, a quadric common to the three sets must contain $y + z + t$ and that point only. By projection from the point we ask for the quadrics not on the three points of an ordinary line, i.e., a Caporali set of the projected space. We originally had therefore $2 \cdot \bar{Q}_{p-2}$ quadrics. In case IV, the common quadrics contain the null line $y + z$,
z + t, t + y, and are projected from this line into the Kummer set of a point whence originally there were

\[ 4 \cdot 2^{2(p-2)-2} = 2^{2p-4} \]

\( Q \) quadrics. In case II, the common quadrics contain \( z + t \) and are projected from this point into a Caporali set so that originally there were \( 2 \cdot \tilde{Q}_{p-2} \) \( Q \) quadrics. In case III, a quadric not on \( y, z, t \) must be on \( y + t, z + t, y + z + t \) and we thus get again the case of (57) with the same result as our present case IV. Hence

\[ \text{(68)} \] The Kummer sets of the vertices of a triangle have in common \( 2^{2p-3} \) quadrics. These consist of \( 2 \tilde{Q}_{p-2} \) \( Q \) quadrics if none or one of the sides of the triangle are null lines, of \( 2^{2p-4} \) \( Q \) quadrics if two or three sides of the triangle are null lines.

Case IV is easily generalized. Let \( x(1), \ldots, x(m) \) be a reference basis of a null \( S_{m-1} \). A quadric common to their Kummer sets must contain a null \( S_{m-2} \) in \( S_{m-1} \) without containing \( S_{m-1} \). Projected from \( S_{m-2} \), \( S_{m-1} \) becomes a point and the quadric belongs to the Kummer set of the point; whence

\[ \text{(69)} \] The Kummer sets of the points of a reference basis of a null \( S_{m-1} \) have in common \( 2^{2p-m-1} \) \( Q \) quadrics which divide into sets of \( 2^{m-1} \).

To determine the quadrics common to the Kummer sets of a number of basis points we can utilize a new canonical form of the quadric. Take for reference point basis \( 2p \) points of a basis of \( C \) and let \( x_1, \ldots, x_{2p} \) be the coordinates. Any quadric not on the \( 2p \) points must contain every squared term and, the pairs of points being azygetic, must also contain every product term and therefore is

\[ q = \sum_{i=1}^{2p} x_i^2 + q', \]

where

\[ q' = \sum_{i=1} x_i x_k \quad (i, k = 1, \ldots, 2p; i < k). \]

Since \( \sum x_i \) is the polar \( S_{2p-2} \) of the unit point as to \( q' \) and this point lies on \( q' \) if \( p \) is even, \( q \) is of the same type as \( q' \) if \( p \) is even. Or, from (65), \( q \) is an \( E \) or an \( O \) quadric according as \( p = 0, 3 \) or \( 1, 2 \) (mod 4).

\[ \text{(70)} \] If the \( 2p \) points of the reference basis \( x \) of \( S_{2p-1} \) belong to a self dual basis of \( C \), the unique quadric not on the \( 2p \) basis points

\[ \sum_{i=1}^{2p} \sum x_i^2 + \sum x_i x_k = 0 \quad (i, k = 1, \ldots, 2p; i < k) \]

is the quadric (41) when \( p \) is odd. When \( p \) is even, the quadric does not contain the residual points of the reference basis but does contain the \( (2p + 1) \)th basis point and the remaining residual points. It is an \( E \) or an \( O \) quadric according as \( p = 0, 3 \) or \( 1, 2 \) (mod 4).

Given then an even number \( 2p \) of basis points. They lie in an \( S_{2p-1} \) without
a null subspace. A quadric not on these points is cut by $S_{2p-1}$ in a quadric $Q(\rho)$ not on these points. According to (70) there is a unique quadric $Q(\rho)$ of this type. From $Q(\rho)$ we pass to the original quadric as in (63) and find that

(71) The Kummer sets of $2p$ points of a self dual basis of $C$ have in common $Q_{p-1}$ quadrics if $\rho \equiv 0, 3 \pmod{4}$, and $Q_{p-1}$ quadrics if $\rho \equiv 1, 2 \pmod{4}$.

$2p + 1$ points of a self dual basis lie in an $S_{2p}$ which has a null subspace $S_0$ and any quadric not on the $2p + 1$ points is on $S_0$. By projection and section from $S_0$ we obtain the case of (71) and have therefore shown that

(72) The Kummer sets of $2p + 1$ points of a self dual basis of $C$ have in common $2Q_{p-1}$ quadrics if $\rho \equiv 0, 3 \pmod{4}$, and $2Q_{p-1}$ quadrics if $\rho \equiv 1, 2 \pmod{4}$.

The Steiner and Kummer sets have received considerable attention in the particular case, $p = 3$. Numerous other sets are suggested by the smaller values of $p$ and can be readily generalized and discussed by the foregoing methods. In the next paragraph a somewhat different point of view is emphasized and again the geometrical method of treatment seems most effective.


A linear system $F_r$ of quadrics is determined by $r + 1$ linearly independent quadrics $Q, Q_1, \ldots, Q_r$. The sum of any even number of the quadrics is the square of an $S_{2p-2}$ which has a null point $y$. Let $Q + Q_i$ have the null point $y^{(i)}$. Then $F_r$ determines an $S_{r-1}$ with the reference basis $y^{(1)}, \ldots, y^{(r)}$. Moreover any reference basis of $S_{r-1}$ together with any quadric of $F_r$ determines $F_r$. Let us call $S_{r-1}$ the allied space of $F_r$. Two systems, $F_r$ and $F'_r$, with the same allied space and a common quadric coincide. Thus the $2^{2p}$ quadrics can be divided in a single way into $2^{2p-r}$ systems $F_r$ with a given common allied space $S_{r-1}$. These $2^{2p-r}$ systems $F_r$ are called a complex allied with $S_{r-1}$. Two systems, $F_r$ and $F_s$, are skew systems or null systems of each other if their allied spaces are respectively skew spaces or null spaces of each other. Two skew systems may or may not have one common quadric. The two cases can be distinguished by the respective terms partially skew or completely skew. A Göpel system $F_r$ has for allied space a null $S_{r-1}$; a Göpel system is allied with a Göpel space $S_{p-1}$.

There are two types of system $F_0$, namely, an $E$ quadric and an $O$ quadric. There are three types of system $F_1$, a pair of $E$ quadrics, a pair of $O$ quadrics, and an $E$ and $O$ quadric. The allied space $S_0$ is on the quadrics in the first two types but not in the last.

Let a system $F_2$ be determined by the quadric $Q$ and the allied space $S_1$

* Cf., for these definitions, K., p. 296, § 9.
with points $y, z, y + z$. The system contains four quadrics,

$$Q, Q + C(x^2, y), Q + C(x^2, z), Q + C(x^3, y + z),$$

whose sum is identically zero. Any three of the four determine the system. According as the line $S_1$ is skew to, tangent to, secant to, or on, $Q$, three, two, one, or none, of the quadrics are $\bar{Q}$ quadrics. That is, the set of four contains an even number of each kind if $S_1$ is a null line, an odd number of each kind if $S_1$ is an ordinary line. In the first case we say that any three of the four are syzygetic;\* in the second case any three of the four are azygetic.\* There are then five types of system $F_2$: three syzygetic types with 4, 2, 0, $E$ quadrics respectively; and two azygetic types with 3, 1, $E$ quadrics respectively. In the first and third types $S_1$ is a generator of the four quadrics; in the second type $S_1$ touches one pair at one point, the other pair at another point; in the fourth and fifth types $S_1$ is skew to the one quadric and cuts the other three in two out of three of its points.

Given two quadrics of the same type, $Q, Q' = Q + C(x^2, y)$, where $y$ is on $Q$, the number of pairs which can be added to the given pair to form a syzygetic or azygetic system is determined by the numbers $\sigma_1, \sigma_2, \sigma_3, [33]$ and $[34]$. If the given pair are of opposite type, $Q, \bar{Q}$, the numbers $\rho_0, \rho_1, \rho_2$, serve. Hence

(73) Given two quadrics $Q, Q'$, there are $\bar{Q}_{p-1}$ pairs $\bar{Q}, \bar{Q}'$, and $Q_{p-1} - 1$ pairs $Q''$, $Q'''$, each syzygetic with the two and $2^{p-2}$ pairs $Q''', \bar{Q}$, each azygetic with the two; given two quadrics $Q, \bar{Q}$, there are $P_{2p-3}$ pairs, $Q', \bar{Q}'$, each syzygetic with the two, and $\bar{Q}_{p-1}$ pairs $\bar{Q'}, \bar{Q}'$, and $Q_{p-1}$ pairs $Q', Q''$, each azygetic with the two.

Similarly, the numbers $\tau_0, \tau_1, \tau_2, \tau_3, [35]$ and $[36]$ serve to determine the number of sets of three quadrics syzygetic or azygetic with one given quadric.

(74) A $Q$ quadric is syzygetic with $\bar{Q}_{p-1}$ ($Q_p - 1$) triads $Q', \bar{Q}, \bar{Q}'$ and with $\frac{1}{2} (Q_p - 1) (Q_{p-1} - 1)$ triads $Q', Q'', Q'''$, azygetic with $\frac{3}{2} Q_p Q_{p-1}$ triads $\bar{Q}, Q', Q''$, and with $2^{p-3} (Q_p - 1)$ triads $Q', Q'', \bar{Q}$.

An obvious enumeration and the use of (38) leads to the following result:

(75) There are $\frac{1}{2} Q_p (Q_p - 1) (Q_{p-1} - 1) = \frac{1}{3} P_{2p-1} P_{2p-3}$ syzygetic tetrads $Q, Q', \bar{Q}, \bar{Q}'$; $\frac{1}{2} Q_p \bar{Q}_{p-1} (Q_p - 1) = 2^{2p-4} P_{2p-1} P_{2p-3}$ syzygetic tetrads $Q, Q', \bar{Q}, \bar{Q}$; and $\frac{1}{2} 2^{p-3} P_{2p-1} Q_p$ azygetic tetrads $Q, Q', Q''$. Let $s^{(1)}, \ldots, s^{(2p+1)}$, be a self dual basis of $C$ and let $Q^{(0)}$ be any quadric. $Q^{(0)}$ and the $2p + 1$ quadrics $Q^{(i)} = Q^{(0)} + s^{(i)}$ are subject to the single relation $\sum_{i=0}^{2p+1} Q^{(i)} = 0$. Any three of the quadrics are azygetic if $S^{(0)} + S^{(x)}$, $S^{(x)} + S^{(y)}$, $S^{(x)} + S^{(z)}$ do not form a null pencil. $S^{(0)} + S^{(y)}$ and $S^{(x)} + S^{(z)}$ are

\*K., p. 253.
are in an ordinary pencil if

\[(S^{(i)} + S^{(k)}, S^{(k)} + S^{(i)}) = (S^{(i)}, S^{(k)}) + (S^{(k)}, S^{(i)}) \neq 0.\]

But by the definition of the basis, \((S^{(i)}, S^{(k)}) \neq 0\). Such a set of \(2p + 2\) quadrics any three of which are azygetic is called a fundamental set and is denoted by F.S. If the set be given and one be added to the others the squares of \(2p + 1\) \(S_{2p-2}\)'s are obtained which must form a basis because of the azygetic property of the quadrics. Thus a basis determines \(2^{2p}\) F.S.'s while an F.S. is determined from \(2p + 2\) bases. The number of sets then is \(N_{BC}2^{2p} / (2p + 2)\).

(76) A fundamental set, F.S., of \(2p + 2\) quadrics (a set such that all are connected by one linear relation and any three are azygetic) is obtained by adding one quadric to the squared \(S_{2p-2}\)'s of a basis of C. If any quadric of a F.S. be added to the others a self dual basis of C is obtained. The number \(N_F\) of F.S.'s is

\[N_F = \frac{2^{2p} + p^2(2^{2p} - 1)(2^{2p-2} - 1) \cdots (2^2 - 1)}{(2p + 2)!} = \frac{2^{p(p+2)}}{(2p + 2)!} P_{2p-1} P_{2p-3} \cdots P_1.\]

In (41) we showed that with a self dual basis \(y_1, \ldots, y_{2p+1}\), there is associated a unique quadric

\[R_y = \Sigma y_i y_k \quad (i, k = 1, \ldots, 2p + 1, i < k).\]

From the basis and \(R_y\) we can construct a definite F.S. which has certain special properties and which will be called a normal F.S. Since \(R_y\) contains the basis points if \(p\) is even, the quadrics of the normal F.S. are then all of the same type. When \(p\) is odd \(R_y\) is the only one of its type.

(77) The normal F.S. determined by the basis \(y\) of C and its quadric \(R_y\) contains only E quadrics if \(p = 0 \pmod{4}\); contains only O quadrics if \(p = 2 \pmod{4}\); contains the O quadric \(R_y\) and \(2p + 1\) E quadrics if \(p = 1 \pmod{4}\); and contains the E quadric \(R_y\) and \(2p + 1\) O quadrics if \(p = 3 \pmod{4}\). Moreover a F.S. which contains \(2p + 1\) quadrics of the same type is a normal F.S. for which the remaining quadric is \(R_y\).†

The last statement is proven as follows: Let \(Q, Q + y_i, i = 1, 2, \ldots, 2p + 1\), be a F.S. such that the quadrics \(Q + y_i\) are all of the same type. The null points of \(y_i\) and \(y_2\) are both on or both off \(Q\) according to the type of \(Q + y_i\). Their line is an ordinary line since any three of the quadrics are azygetic and is either a secant or skew line of \(Q\). In either case the residual point, the null point of \(y_1 + y_2\) is not on \(Q\), and \(Q\) is the quadric \(R_y\) associated with the basis.

Any F.S., \(Q, Q + y_i\), determines not only the basis \(y\) but also the \(2p + 1\) bases gotten by varying \(k\) in \(y_k, y_k + y_i (i \neq k)\). This set of \(2p + 2\) bases

* Cf. K., pp. 283–5.
† Cf. K., pp. 274–6; also p. 288.
is a symmetrical set consisting of any one basis and its residual points. The set is transformed into itself by a $G_{(2p+2)}$, the symmetric group on the $2p+2$ bases, which is generated by the $(p+1)(2p+1)$ involutions $I_x$ on the point $x$ belonging to the bases. Sample involutions are $y_i = y_i$, $y'_k = y_k + y_i (i \neq k)$ and $y'_i = y_k$, $y'_k = y_i$, $y'_i = y_i (l \neq i, l \neq k)$. The quadric $R_y$ attached to one basis is also attached to the others if $p$ is odd since according to (41) it contains neither the basis points nor the residual points. But if $p$ is even, the basis points are on $R_y$ and the residual points are not, whence there are $2p + 2$ quadrics $R_y$.

(78) The points and residual points of a basis form a basis configuration, i.e., a set of $(p+1)(2p+1)$ points which can be divided into a basis and its residual points in $2p+2$ ways, each point lying in two bases. The configuration is unaltered by a $G_{(2p+2)}$ generated by the involutions on its points, which is symmetric on its bases. Any F.S. determines a basis configuration.

If $p$ is even, the basis configuration determined by a normal F.S. contains bases whose quadrics $R_y$ make up the normal F.S. But if $p$ is odd the basis configuration of the normal F.S. determines the unique quadric $R_y$, isolated in the normal F.S. $R_y$ and the basis configuration determine $2p+2$ normal F.S.'s containing $(p+1)(2p+1)$ quadrics (each quadric in two F.S.'s) apart from $R_y$ which occurs in each F.S.

(79) If $p$ is even a normal F.S. contains the $2p+2$ quadrics attached to the bases of the configuration determined by the F.S. and is invariant under the configuration $G_{(2p+4)}$. If $p$ is odd, a normal F.S. is one of a set of $2p+2$ normal F.S.'s each having the same isolated quadric and basis configuration. The configuration $G_{(2p+2)}$ permutes the $2p+2$ F.S.'s symmetrically, each F.S. being invariant under the $G_{(2p+1)}$ attached to its particular basis.

For example, when $p = 2$ the 15 points of $S_3$ form a single basis configuration containing the 6 bases in $S_3$. There is but one normal F.S., consisting of the 6 $O$ quadrics attached to the 6 bases. When $p = 3$, a normal F.S. containing one $E$ quadric and 7 $O$ quadrics determines a basis configuration. This configuration contains 8 bases and arises from 8 normal F.S.'s each having the same $E$ quadric. That is, the Aronhold sets of seven can be grouped in 36 ways, corresponding to the even characteristics, into 8 sets, each odd characteristic occurring in two sets.

The following enumeration is immediate:

(80) If $p = 0$ or $3 \pmod{4}$, for each $E$ quadric there are $N_{BC}/E_p$ self dual bases of $C$ whose lines are secant or skew lines respectively of $E$. If $p = 2$ or $1 \pmod{4}$ there are $N_{BC}/O_p$ self dual bases of $C$ whose lines are secant or skew lines respectively of $O$. If $p$ is odd the bases can be grouped into basis configurations.

The normal F.S. affords a convenient method for studying the general F.S.
Let \( Q, Q_1, \cdots, Q_{2p+1} \), where \( Q + Q_i = y_i \), be any F.S. Since \( \sum_{i=1}^{2p+1} y_i = 0 \), any \( S_{2p-2} \) can be expressed as \( \sum_{i=1}^{2k} y_i \), where \( k = 1, \cdots, p \). Hence \( Q \) itself can be written as, \( Q = R_v + \sum_{i=1}^{2k} y_i \), and the F.S. takes the form

\[
Q = (R_v + \sum y_i), \quad Q_1 = (R_v + \sum y_i) + y_1, \quad \cdots, \quad Q_h = (R_v + \sum y_i) + y_h, \quad \cdots.
\]

\( Q \) and \( R_v \) are of the same or different type according as \( k \) is even or odd. The quadrics \( Q_h \) divide into two sets, according as \( y_1 \) does or does not occur in \( \sum y_i \). Hence we have

1. 1 quadric \( Q \) of the form \( R_v + \sum_{i=1}^{2k} y_i \),
2. \( 2k \) quadrics of the form \( R_v + \sum_{i=1}^{2k-1} y_i = R_v + \sum_{i=1}^{2p-2k+2} y_i \),
3. \( 2p + 1 - 2k \) quadrics of the form \( R_v + \sum_{i=1}^{2p+1} y_i = R_v + \sum_{i=1}^{2p+2} y_i \).

Quadrics (1), (2), (3) are of the type \( R \) if respectively \( k \) is even, \( p - k \) is odd, \( p - k \) is even. Thus we have the table:

<table>
<thead>
<tr>
<th>Residue of ( p ) modulo 4</th>
<th>( R ) of type</th>
<th>Number of ( O ) quadrics</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( E )</td>
<td>( 2k )</td>
</tr>
<tr>
<td></td>
<td>( O )</td>
<td>( 2k + 1 )</td>
</tr>
<tr>
<td></td>
<td>( O )</td>
<td>( 2 (p - k) + 2 )</td>
</tr>
<tr>
<td></td>
<td>( E )</td>
<td>( 2 (p - k) + 1 )</td>
</tr>
</tbody>
</table>

Hence if \( s \) is the number of \( O \) quadrics, \( s = p \) (mod 4). Moreover, for any such number \( s \), a F.S. containing \( s \) \( O \) quadrics can be constructed by determining \( k \) in the table from the given \( s \). The required F.S. can be written down in terms of the basis and \( 2k \) arbitrarily selected \( S_{2p-2} \)'s of it. Since any two bases are conjugate and within a basis any two sets of \( 2k \) spaces are conjugate, then any two F.S.'s with the same number \( s \) are conjugate, provided that the same F.S. can be determined from \( k \) even or \( k \) odd—a double possibility for the same \( s \). Denote by \( y' \) the basis

\[
y'_1 = y_1, \quad y'_i = y_1 + y_i \quad (i = 2; \cdots, 2p + 1).
\]

If \( p \) is odd let \( y_1 \) be not contained in \( \sum y_i \). Then \( R_v = R_v', \sum_{i=1}^{2k} y_i = \sum_{i=1}^{2k} y_i' \), and the above F.S. is

\[
Q_1 = (R_v + \sum_{i=1}^{2k} y_i'^2), \quad Q = (R_v + \sum_{i=1}^{2k+1} y_i'^2) + y_1'^2, \quad Q_2 = (R_v + \sum_{i=1}^{2k+1} y_i'^2) + y_2'^2, \quad \cdots.
\]
But
\[ \sum_{i=1}^{2^k+1} y_i^2 = \sum_{j=1}^{2(\frac{p-k}{2})} y_j^2. \]

Hence, if \( p \) is odd, \( k \) can be replaced by \( p - k \). If \( p \) is even, let \( y_1 \) be in \( \sum y_i^4 \). Then
\[ R_v = R_v + y_1^2, \quad \sum y_i^2 = \sum y_i^2, \]
and the original F.S. is
\[ Q_1 = (R_v + \sum y_i^2), \quad Q = (R_v + \sum y_i^2) + y_1^2, \quad Q_2 = (R_v + \sum y_i^2) + y_2^2, \quad \ldots. \]

Since \( \sum y_i^2 = \sum y_j^2 \), if \( p \) is even, \( k \) can be replaced by \( p - k + 1 \). Thus the same F.S. can be gotten from an even or an odd \( k \). If \( Q \) and \( Q_i \) are of the same type, the involution, \( I_s \), determined by \( Q + Q_i \), interchanges \( Q \) and \( Q_i \) and leaves all the other quadrics unaltered. No collineation other than the identity can leave every quadric unaltered.

(81) A F.S. contains \( s \) O quadrics, where \( s \equiv p \pmod{4} \). If \( s \equiv p \pmod{4} \), there are \( N/\{s!(2p+2-s)\!\} \) F.S.'s containing \( s \) O quadrics, all conjugate under \( G_N \) and each invariant under a subgroup \( G_{(2p+2-s)!} \) of \( G_N \). This subgroup is the product of the interchangeable groups \( G_s \) and \( G_{(2p+2-s)!} \); the symmetric groups on the O and E quadrics respectively of the F.S. The subgroup has an invariant quadric or an invariant space (the sum of the O or E quadrics in F.S.), according as \( p \) is odd or even. It has also two invariant skew spaces \( S_{p-2} \) and \( S_{2p-s} \), the allied spaces of the O and of the E quadrics. The F.S. can be obtained from a normal F.S. by adding a squared \( S_{2p-2} \).

Let \( S_{p-1} \) be a Göpel space, \( F_p \) a Göpel system. If \( F_p \) contains two quadrics, \( Q \) and \( Q \), of different types, any third quadric \( Q' \) of \( F_p \) is syzygetic with the two and therefore paired with a fourth \( Q' \) of different type, so that \( Q + \bar{Q} + Q' + \bar{Q}' = 0 \). Hence \( F_p \) contains \( 2^{p-1} E \) quadrics and \( 2^{p-1} O \) quadrics. If however the quadrics of \( F_p \) are all of the same type, they all must contain \( S_{p-1} \). Since there are, on a null \( S_{k} \), \( 2^{k+1} Q_{p-k-1} \) quadrics [60], and \( E_0 = 1 \) while \( O_0 = 0 \), \( S_{p-1} \) is on \( 2^p E \) quadrics and those only. Hence

(82) In every complex of \( 2^p \) Göpel systems there is one Göpel system which contains only \( E \) quadrics; each of the \( 2^p - 1 \) other Göpel systems contains \( 2^{p-1} Q \) quadrics. Each set of \( 2^{p-1} \) \( Q \) quadrics has an allied null space \( S_{p-2} \) which lies on every quadric of the set. Any three quadrics in a system are syzygetic.†

A precisely similar argument applies to the case where the \( S_{m-1} \) allied to the system \( F_m \) is a null \( S_{m-1} \). If \( F_m \) contains a quadric of each type half of the \( 2^m \) quadrics in \( F_m \) are \( Q \) quadrics. If all the quadrics are of one type they

* Cf. K., pp. 286–9; XXVIII–XXXI.
† K., p. 300, XXXIV.
all contain \( S_{m-1} \). There are \( 2^m Q_{p-m} \) \( Q \) quadrics containing \( S_{m-1} \) and these divide in a unique way into sets of \( 2^m \) with the allied space \( S_{m-1} \).

(83) If \( S_{m-1} \) is a null space, the complex of \( 2^{2p-m} \) \( F_m \)'s with the allied space \( S_{m-1} \) contains \( Q_{p-m} \) systems \( F_m \) containing only \( Q \) quadrics. The members of these systems contain \( S_{m-1} \). The remaining \( 2^{2(p-m)} P_{m-1} \) systems \( F_m \) contain \( 2^{m-1} \) quadrics of each type.*

Denote by \( \phi_{0,m} \) the system \( F_m \) containing quadrics of the type \( Q \) only. Such a system becomes, after projection and section from \( S_{m-1} \), a single quadric of type \( Q \) in the derived space \( \Sigma_{2^m-1}, \pi = p - m \). From the definition of syzygetic and azygetic quadrics in \( \Sigma_{2^m-1} \) we have

(84) Three systems \( \phi_{0,m} \) allied with the null \( S_{m-1} \) are contained in a system \( F_{m+2} \) allied with an \( S_{m+1} \) on \( S_{m-1} \). The three systems are syzygetic or azygetic according as \( S_{m+1} \) is a null or an ordinary space. The entire theory here developed of the quadrics \( Q \) belonging to \( C \) in \( S_{2p-1} \), when applied to the quadrics belonging to \( C_\pi \) in \( \Sigma_{2^m-1} \) yields an analogous theory of the systems \( \phi_{0,m} \) in \( S_{2p-1} \).

This theorem is due essentially to Frobenius† though its remarkable utility, exemplified in many of the preceding theorems, is clearly apparent only when it is viewed as a result of the general process—projection and section from a null space.

Let us now investigate the “section \( C_\rho \) of \( C \) by the space \( S_{2p-1} \) which has no null subspace”; i.e., by the Rosenhain space \( S_{2p-1} \). Let us call a system of quadrics, \( F_{2p} \), with the allied Rosenhain space, \( S_{2p-1} \), a Rosenhain system.‡ A Rosenhain space, \( S_{2p-1} \), determines its complementary skew Rosenhain space, \( S_{2(p-p)-1} \), the two being null spaces of each other. According to (62) and (63),

(85) The \( 2^{2p} \) quadrics of a Rosenhain system \( F_{2p} \) allied with an \( S_{2p-1} \) have the same section \( Q \) \( (p - \rho) \) by the complementary Rosenhain space \( S_{2^{(p-p)-1}} \). They divide into \( E_\rho Q \) quadrics and \( O_\rho \) \( \bar{Q} \) quadrics. The complex of \( 2^{2(p-p)} \) Rosenhain systems with the allied \( S_{2p-1} \) contains \( E_\rho O \) systems with \( E_\rho Q \) quadrics and \( O_\rho O \) quadrics and \( O_{p-p} \) systems with \( E_\rho O \) quadrics and \( O_\rho E \) quadrics.

Denoting by \( \psi_{\epsilon,\rho} \) and \( \psi_{0,\rho} \) these respective systems we obtain the theorem analogous to (84):

(86) Three Rosenhain systems \( \psi_{0,\rho} \) allied with the Rosenhain space \( S_{2p-1} \) are contained in a system \( F_{2p+2} \) allied with an \( S_{2p+1} \) which cuts the complementary Rosenhain space \( S_{2^{(p-p)-1}} \) in \( S_1 \). The three systems are syzygetic or azygetic according as \( S_{2p+1} \) is not or is a Rosenhain space; or also according as \( S_1 \) is not or is an ordinary line, i.e., a Rosenhain \( S_1 \). The theory developed above of the quadrics \( Q \) belonging to \( C \) in \( S_{2p-1} \), applied to the quadrics belonging to \( C_{p-p} \) in \( S_{2^{(p-p)-1}} \), yields an analogous theory of the systems \( \psi_{0,\rho} \) in \( S_{2p-1} \).

* K., p. 303, XXXVI.
† Cf. K., pp. 302-5, where references are given.
‡ For \( p = 2, \rho = 1 \), cf. K., pp. 337-8.
Let us determine finally the number of $E$ and $O$ quadrics in the general system $F_r$ with allied space $S_{r-1}$. Let $S_{r-1}$ have the null subspace $S_{m-1}$ with reference basis $x^{(1)}, \ldots, x^{(m)}$. Let $S_{r-m-1}$ be a space skew to $S_{m-1}$ in $S_{r-1}$ with reference basis $y^{(m+1)}, \ldots, y^{(r)}$ which is part of a self-dual basis of $C$. According to (20), $r - m$ is even and $S_{r-m-1}$ is a Rosenhain space. $F_r$ is fixed by means of $S_{r-1}$ and any one of its quadrics $Q$. If $Q$ contains $S_{m-1}$ so also does every quadric of $F_r$. Then either all or none of the quadrics of $F_r$ contain $S_{m-1}$. In the latter case $Q$ and $S_{m-1}$ determine a system $F_m$ and by adding to $F_m$ the null spaces of the $2^m - 1$ points in $S_{r-m-1}$, $2^m$ systems $F_m$ are obtained each consisting [see (83)] of $2^{m-1} Q$ quadrics. The systems $F_m$ exhaust the system $F_r$, whence $F_r$ contains $2^{r-1} Q$ quadrics. In case, however, $Q$ contains $S_{m-1}$, the system $F_r$ becomes by projection and section from $S_{m-1}$, a system $F_{r-m}$ with an allied Rosenhain space derived from the projection of $S_{r-m-1}$. In $S_{2p-1}$ the complementary Rosenhain space of $S_{r-m-1}$ is an $S_{2p-(r-m)-1}$ which has in common with the null $S_{2p-m-1}$ of $S_{m-1}$ an $S_{2p-m-(r-m)-1}$ which contains $S_{m-1}$ and a skew space $S_{2(p-m)-(r-m)-1}$. This last space and $S_{r-m-1}$ itself project from $S_{m-1}$ into complementary Rosenhain spaces in $S_{2p-1}$. Then theorem (85) can be applied to the system $F_{r-m}$ in $S_{2p-1}$ and the result translated to $S_{2p-1}$. Hence

(87) A space $S_{r-1}$ with null subspace $S_{m-1}$ is determined by the Göpel space $S_{m-1}$ and $q$ skew Rosenhain space $S_{r-m-1}$. The complementary Rosenhain space cuts the null space of the Göpel space in an $S_{2p-m-(r-m)-1}$ which can be determined by the Göpel space and a skew $S_{2(p-m)-(r-m)-1}$. By projection and section from the Göpel space, $S_{r-m-1}$ and $S_{2(p-m)-(r-m)-1}$ become complementary Rosenhain spaces.

A system $F_r$ determined by $Q'$ and $S_{r-1}$ contains $2^{r-1} Q$ quadrics if $Q'$ does not contain $S_{m-1}$. If $Q'$ contains $S_{m-1}$ and meets $S_{2(p-m)-(r-m)-1}$ in a quadric of type $Q''$, then $F_r$ contains $2^m E_{(r-m)/2}$ quadrics of type $Q''$ and $2^m O_{(r-m)/2}$ quadrics of type $Q''$.*

Krazer remarks (p. 266) that the Per. and Th. Char. have been confused by various writers. He notes (pp. 253-4) some points of difference between the two but does not call express attention to the fact—fundamental in the exposition here given—that the coefficients of the transformation occur linearly in the transformation of the Per. Char. and quadratically in the transformation of the Th. Char. Though we find (p. 254) that "die Summe einer geraden Anzahl von Th. Char. sich wie eine Per. Char., die Summe einer ungeraden Anzahl von Th. Char. aber wie eine Th. Char. transformiert," yet it is stated (p. 284) that "Die Summe der $2p + 2$ Th. Char. eines F.S. is [0]," i. e., is the zero Th. Char.; and (p. 305) that "Durch . . . der Addition einer beliebigen Th. Char. zu den sämtlichen Th. Char. eines Systems geht ein System von Th. Char. immer wieder in ein System von Th. Char. über.”

* Cf. K., p. 301, XXXV.
Consider also the statement (p. 270): “Man fasse nun die (Per.) charakteristiken des F.S. als Th. Char. . . . auf.” A Per. Char. can be regarded as a Th. Char. only through the intervention of some given Th. Char. which Krazer implicitly takes to be the zero Th. Char. However if entire accuracy is sought in the use of such a process one should state whether the resulting theorems are or are not independent of the given Th. Char. employed for the transition, i. e., whether the results are covariant under $G_N$ or covariant only under the subgroup of $G_N$ defined by the given Th. Char. Such distinctions or limitations are almost self evident from the geometrical point of view.

Baltimore,

June 1, 1912.