LEBESGUE INTEGRALS CONTAINING A PARAMETER, WITH APPLICATIONS*

BY

BURTON H. CAMP

1. **Introduction.**—In a recent memoir Lebesgue† has studied the following problem. Let $f(t)$ belong to one of a certain group of families of functions. What conditions should be imposed on another function $\phi(t, n)$ involving the parameter n, in order that the Lebesgue integral of the product of f and ϕ may exist and approach zero, as n becomes infinite, for all functions f of the family in question? I have studied elsewhere‡ the same problem when the functions involved are functions of several variables instead of one. The object of the present paper is to continue Lebesgue's work by considering other families of which most are narrower than his, and to make certain applications of the results. I have been led to do this because of an attempt to apply Lebesgue's theorems to certain series of orthogonal functions. It was found that the conditions were too severe, and the need was felt of simpler conditions, even at the expense of restricting the classes of functions for which the results would be valid. The conditions that should be imposed on ϕ in order that the integral of $f\phi$ shall remain limited (instead of approaching zero) are given in Theorem 4.

It is shown here how the theory may be applied to the series of Fourier and of Legendre to discover the degrees of their convergence and to find the rates at which their coefficients approach zero. In the case of the functions of Legendre certain inequalities are derived which may prove useful. The paper concludes with some theorems on the evaluation of definite integrals and on the integration of series.

2. The first theorem is taken from the paper of Lebesgue already mentioned. I copy it here, with certain slight changes, for which authority will be found in my earlier paper, because I shall need to refer to it frequently in the following pages.

* Presented to the Society April 27, 1912.

† Lebesgue, *Annales de la Faculté de Toulouse*, ser. 3, vol. 1 (1909), pp. 25–117.

‡ These *Transactions*, vol. 14 (1913), pp. 42–64.

THEOREM 1: *Necessary and sufficient conditions that the integral*

$$\int_k^l f(t)\,\phi(t,n)\,dt,$$

where f and ϕ are defined in the interval (k, l) and n is a parameter belonging to a sequence having plus infinity as its only limit point, shall exist for sufficiently large values of n as an absolutely convergent L-integral, and approach zero for all functions f belonging to one of the families specified below, may be stated in the several cases as follows:*

(F_1) *In the case of the family F_1 of absolutely L-integrable functions,* (1°) *M shall exist so that $|\phi|$ is less than M if $n > n_M$† at all points of (k, l), except perhaps at a null set, and* (2°) *the integral of ϕ over each interval (a, b) in (k, l) shall approach zero as n becomes infinite.*

(F_2) *In the case of the family F_2 of functions whose squares are L-integrable,* (1°) *M shall exist so that in (k, l) the integral of ϕ^2 is less than M if $n > n_M$, and* (2°) *condition* (2°) *for F_1 shall be satisfied.*

(F_3) *In the case of the family F_3 of limited L-integrable functions,* (1°) *the integral of ϕ shall be "equi-absolutely" continuous,‡ and* (2°) *condition* (2°) *for F_1 shall be satisfied.*

(F_4) *In the case of the family F_4 of simply discontinuous§ functions,* (1°) *M shall exist so that the integral of $|\phi|$ is less than M if $n > n_M$, and* (2°) *condition* (2°) *for F_1 shall be satisfied.*

(F_{4a}) *In the case of the family F_{4a} of continuous functions,* (1°) *condition* (1°) *for F_4 shall be satisfied,* (2°) *for each a in (k, l) the integral of $(t - a)\,\phi$ from a to l, and* (2a°) *the integral of ϕ from k to l shall approach zero as n becomes infinite.*

(F_5) *In the case of the family F_5 of functions having limited variation,* (1°) *ϕ shall be absolutely L-integrable,* (1a°) *M shall exist so that the integral of ϕ from k to t is less than M uniformly for all values of t if $n > n_M$, and* (2°) *condition* (2°) *for F_1 shall be satisfied.*

(F_{5a}) *In the case of the family F_{5a} of continuous functions having limited variation,* (1°, 1a°, 2°, 2a°) *conditions* (1°) *and* (1a°) *for F_5, and* (2°) *and* (2a°) *for F_{4a}, shall be satisfied.*

COROLLARY: *The conditions for F_1, F_2, and F_3 remain sufficient if, in place of* (2°), *it be required that $\phi(t, n)$ approach zero in (k, l), except perhaps at a null set of points.*

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* I. e., integral in the sense of Lebesgue.

† I use n_M in place of the usual n_0 to denote a fixed number.

‡ In the sense of Vitali, Rendiconti del Circolo Matematico di Palermo, vol. 23 (1907), p. 139. See also my other paper, loc. cit., p. 44.

§ I. e., having at most discontinuities of the first kind.

For I have shown in another paper* (by the use of a theorem of Vitali's) that termwise integration of the sequence $\{ \phi (t, n) \}$ is permissible, if condition (1°) for the family in question is satisfied. Therefore, if ϕ approaches zero, so does its integral.

3. The following theorem is of interest because of its bearing on the development of an arbitrary function in series of orthogonal functions. There follow quite readily from it, for example, certain known facts concerning the convergence of Fourier's and Legendre's developments at the point x.

THEOREM 2: *In order that the integral of Theorem* 1 *shall exist for sufficiently large values of* n, *and approach zero for all functions* f *belonging to the family* F_6 *of functions which are continuous in* (k, l), *and have at a previously chosen point* x *a derivative, it is necessary and sufficient that* (1°) *the integral of* ϕ *from* k *to* l *approach zero as* n *becomes infinite, and that* (2°) *the conditions of Theorem* 1 *relative to* F_{4a} *be satisfied when* ϕ *is replaced by* $(t - x)\phi$.

Proof. Except for $t = x$, we have

$$f(t) = f(x) + (t - x)\frac{f(t) - f(x)}{t - x}$$

identically, and therefore

$$(1) \quad \int_k^l f\phi\, dt = f(x)\int_k^l \phi\, dt + \int_k^l \frac{f(t) - f(x)}{t - x}[(t - x)\phi]\, dt = \mathrm{I} + \mathrm{II}.$$

It is necessary and sufficient that (1) approach zero. I approaches zero if (1°) is fulfilled. Since, however, (1°) is clearly necessary (for if it is not satisfied the function $f = 1$ contradicts the theorem), it follows that it is necessary as well as sufficient that I and II separately approach zero. In II $[f(t) - f(x)]/(t - x)$ is, if properly defined at x, continuous, since $f(t)$ is continuous, and at x its derivative, that is, the limit of this fraction, exists. By (2°) we may therefore apply to II the conditions of Theorem 1 relative to F_{4a}, ϕ being replaced by $(t - x)\phi$, and it follows that II approaches zero, and the conditions stated are sufficient. Moreover the conditions in 2° are necessary, for, if any one of them is not satisfied, there exists by Theorem 1 a continuous function $q(t)$ such that

$$\int_k^l q(t)(t - x)\phi\, dt$$

does not approach zero. In that case the function $f = q(t)(t - x)$ contradicts the theorem, for it is continuous in (k, l) and has at x a derivative, viz:

$$\lim_{t=x}\frac{q(t)(t - x) - q(x)(x - x)}{t - x} = q(x).$$

* These Transactions, loc. cit., pp. 63, 64.

COROLLARY 1: *If condition* (1°) *be omitted, the theorem applies to the family of functions which have the properties of* F_6 *and, in addition, vanish at* x.

COROLLARY 2: *If, in the theorem, it is desired that the integral approach* $f(x)$ *instead of zero, it is necessary and sufficient that* (2°) *hold, and that in* (1°) *the integral be required to approach unity instead of zero.*

4. We now consider the same problem relative to the families $G_1', \cdots, G_6'; G_1'', \cdots, G_6''; \cdots$, defined as follows:

DEFINITION 1: The function f belongs to $G_j^{(i)}$ if

$$f(t) = C_i + \int_k^t f'\,dt,$$

where f' is a function belonging to $G_j^{(i-1)}$, and C_i is a constant, and $G_j^{(0)} = F_j$ (i. e., $f^{(i)}$ is in F_j). We do not here assert that f' is necessarily the derivative.

DEFINITION* 2: The function f belongs to $G_j^{(i)}$ if its $(i-1)$th derivative exists,† and is an indefinite integral, and its ith derivative (which will then exist except perhaps at a null set) may be defined, or redefined, in a null set so that it becomes a function of F_j.

Of these definitions the first will be the more convenient to use in our theoretical work, the second the easier to apply in practice. Before proceeding to discuss the problem, I will show that the two definitions are equivalent. The following theorems‡ will be used.

(*a*) *If* ϕ *is continuous in* (k, l), *it equals the derivative of its indefinite integral.*

(*b*) *The same is true, except perhaps at a null set, if* ϕ *is absolutely L-integrable in* (k, l).

(*c*) *If* f *is an indefinite integral, its derivative exists, except perhaps at a null set, and* f *is the indefinite integral of its derivative.*

(*d*) If df/dt *exists and is absolutely L-integrable in* (k, l), *then*

$$f(t) - f(k) = \int_k^t \frac{df}{dt}\,dt.$$

Assuming, first, Definition 1, let

$$f(t) = C_1 + \int_k^t f'\,dt, \tag{1}$$

* In particular, f satisfies this definition for $G_{4a}^{(i)}$, $G_{5a}^{(i)}$, or $G_6^{(i)}$, if its ith derivative is a member of F_{4a}, F_{5a}, or F_6, respectively. But a similar statement does not hold for $G_j^{(i)}$ when $j = 1, 2, 3$, or 4. In fact, we cannot even say in these cases that, if df/dt exists, except perhaps at a null set, and after being defined, or redefined, in a null set becomes a member of F_j, then f belongs to $G_j^{(i)}$. For example, if $f = 0$ in $(0 \leqq t \leqq 1)$ and 1 in $(1 < t \leqq 2)$, df/dt exists in $(0, 2)$, except at 1, and if it be defined there in any manner it becomes a member of F_4, but f is not an integral.

† If a derivative is infinite at a point, I say it does not exist at that point.

‡ (*a*) is well known. For (*b*) and (*c*) see de la Vallée Poussin, *Cours d'Analyse Infinitésimale*, 2d ed., vol. 1 (1909), p. 267; for (*d*) see page 263 of the same volume.

$f^{(i)}$ being in F_j. By (a), since by the definition f' is continuous when $i > 1$,

$$\frac{df}{dt} = f'(t) = C_2 + \int_k^t f''\,dt,$$

(2)
$$\frac{d^2 f}{dt^2} = f''(t) = C_3 + \int_k^t f'''\,dt,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{d^{i-1} f}{dt^{i-1}} = f^{(i-1)}(t) = C_i + \int_k^t f^{(i)}\,dt.$$

Then, since $f^{(i)}$ is given absolutely L-integrable, by (b), except perhaps at a null set,

(3)
$$\frac{d^i f}{dt^i} = f^{(i)}(t).$$

We may define, or redefine, $d^i f / dt^i$ in this null set so that (3) is true at every point of (k, l). Then this new function belongs to the same family as $f^{(i)}$, namely to F_j.

Now let us suppose Definition 2 to be satisfied. Since $d^{i-1} f / dt^{i-1}$ is an indefinite integral, $d^i f / dt^i$ exists by (c), except perhaps at a null set, and, no matter how it is defined in this or in any other null set,

$$\frac{d^{i-1} f}{dt^{i-1}} - C_1 = \int_k^t \frac{d^i f}{dt^i}\,dt.$$

By successive applications of (d), it follows that

$$\frac{d^{i-2} f}{dt^{i-2}} - C_2 = \int_k^t \frac{d^{i-1} f}{dt^{i-1}}\,dt,$$

etc. Finally,

$$f(t) - C_i = \int_k^t \frac{df}{dt}\,dt,$$

and df / dt belongs to $G_j^{(i-1)}$, since $d^i f / dt^i$ may be defined, or redefined, in a null set so that it belongs to F_j. This statement means that f satisfies the requirements of Definition 1.

We have now the following general theorem, which holds for all positive integral values of i.

THEOREM 3: *In order that the integral of Theorem* 1 *shall exist, for sufficiently large values of n, and approach zero as n becomes infinite, for all functions belonging to the family* * $G_j^{(i)}$, *it is necessary and sufficient that* (1°) *the conditions of the corresponding theorem for the family $G_j^{(i-1)}$ be satisfied* ($G_j^{(0)}$ *being* F_j) *when*

* Defined at the beginning of this section.

ϕ is replaced by its integral over (k, t), and that* (2°) *the integral of ϕ over (k, l) approach zero as n becomes infinite.*

Proof. Using Definition 1 and integrating by parts,† we have

$$\int_k^l f\phi\, dt = f(l) \int_k^l \phi\, dt - \int_k^l \left(\int_k^t \phi\, dt\right) f'\, dt = \mathrm{I} + \mathrm{II}. \tag{1}$$

By (2°), I approaches zero. By (1°) and the corresponding theorem for $G_j^{(i-1)}$, II approaches zero. Hence the conditions are *sufficient.* To show that they are necessary we note first that the function $f = 1$ proves that (2°) is necessary, and that therefore, in (1), I must approach zero. It follows that II must approach zero. Now, if (1°) is not satisfied, there exists, by the theorem for $G_j^{(i-1)}$, a function $q(t)$ belonging to $G_j^{(i-1)}$ such that

$$\int_k^l \left(\int_k^t \phi\, dt\right) q\, dt \tag{2}$$

does not approach zero. Let $f = \int_k^t q\, dt$. Then f belongs to $G_j^{(i)}$ and (1) holds, q replacing f'; but by (2) II does not approach zero, a condition which we have just found to be necessary.

Corollary 1: *If condition* (2°) *be omitted, the theorem applies to that family of functions which have the properties of $G_j^{(i)}$ $(i > 0)$ and, in addition, vanish at l.*

Sufficiency.—In (1), II approaches zero as before, and I vanishes since $f(l) = 0$.

Necessity.—Suppose (1°) to be not satisfied, and consider the function $q(t)$ given in (2). Let

$$f(t) = \int_k^t q(t)\, dt - \int_k^l q(t)\, dt.$$

For this function we may write, analogous to (1),

$$\int_k^l f\phi\, dt = \left[f(t) \int_k^t \phi\, dt\right]_k^l - \int_k^l \left(\int_k^t \phi\, dt\right) q\, dt = -\int_k^l \left(\int_k^t \phi\, dt\right) q\, dt,$$

and, by (2), this does not approach zero.

Corollary 2: *In order that the integral from k to k' of $f\phi$ shall approach $f(l)$ for all functions f which belong to $G_j^{(i)}$ $(i > 0,\ j < 6)$ in the interval $(k \gtreqqless l \gtreqqless k')$, it is sufficient that the conditions of Corollary* 1 *be satisfied for each of the intervals (k, l), (k', l), and that the integral from k to k' of ϕ approach unity.*

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* This supposes that ϕ is absolutely L-integrable for sufficiently large values of n, a condition which is necessary, for if it is not satisfied the function $f = 1$ contradicts the theorem.

† Lebesgue, loc. cit., p. 46.

For, by the last condition the expressions

$$f(l) - \int_k^{k'} f\phi dt, \qquad \int_k^{k'} [f(l) - f(t)]\phi dt$$

have the same limit, and the function $f(l) - f(t)$ has the properties of $G_f^{(i)}$ in each of the intervals stated, and in addition vanishes at l. By the preceding corollary, therefore, the first expression approaches zero.

5. Sometimes it is desirable to know whether the integral of Theorem 1 remains limited,* not whether it approaches zero, as n becomes infinite. The following statement will enable one to determine from the previous theorems the necessary and sufficient conditions that this shall be true.

THEOREM 4: *If n is as in the preceding theorems and I_n is a number depending on it, then a necessary and sufficient condition that two numbers, M and n_M, exist so that I_n shall be numerically less than M when $n > n_M$ is that, for every positive divergent sequence $\{q_n\}$, $\lim_{n=\infty} | I_n | / q_n = 0$.*

APPLICATIONS.

6. **Fourier's Coefficients.**—In this section I discuss the rates at which the coefficients of Fourier approach zero.† Let f be defined and absolutely L-integrable in the interval $(c, c + 2\pi)$. As is customary, we write

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(t) \sin nt\, dt, \qquad b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(t) \cos nt\, dt.$$

Let $\{ q_n \}$ be a positive sequence which diverges at an arbitrarily small, positive rate. From the preceding theorems the following may be easily established:

(*a*) *The coefficients a_n and b_n approach zero.*‡

(*b*) *There exist continuous, odd* § *functions for which $q_n a_n$ does not approach zero, and continuous, even functions for which $q_n b_n$ does not approach zero.*

(*c*) *If f has limited variation, na_n and nb_n remain finite.*||

(*d*) *If f is continuous and has limited variation, but not necessarily if f has limited variation and is not continuous, na_n approches zero when c is an odd multiple of $\pi / 2$ and n is odd, and nb_n approaches zero when c is an even multiple of $\pi / 2$.*

(*e*)¶ *Let f belong to $G_1^{(i)}$, let $f^{(0)} = f$, $f^{(p)}$ be the pth derivative of f, and let $p=0$,*

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* Cf., e. g., § 6, c, f, and § 7, c, f.

† Cf. also a footnote to an article by Bôcher, Annals of Mathematics, ser. 2, vol. 7 (1906), p. 109.

‡ This follows from Theorem 1 for F_1, and is well known.

§ This shows that (a) is the most that may be said even of families of functions as restricted as those in (b). The function $f(t)$ is odd in $(c, c + 2\pi)$, if $f(c + \pi - t) = -f(c + \pi + t)$; even, if $f(c + \pi - t) = f(c + \pi + t)$.

|| This is known.

¶ Cf. Young, Proceedings of the London Mathematical Society, ser. 2, vol. 10 (1911), pp. 256–7.

$1, \cdots, i-1$. *Then* $n^i a_n$ *approaches zero if* $c = 0$ *or a multiple of* π, *and* $f^{(p)}(c+2\pi) = f^{(p)}(c)$ *when* p *is even;* $n^i b_n$ *approaches zero if* $c = 0$ *or a multiple of* π, *and* $f^{(p)}(c+2\pi) = f^{(p)}(c)$ *when* p *is odd; both approach zero if* $f^{(p)}(c+2\pi) = f^{(p)}(c)$ *for all the values of* p.

(*f*) *If* f *belongs to* $G_5^{(i)}$, *then with the same provisions, respectively, as in* (*e*), $n^{i+1} a_n$ *and* $n^{i+1} b_n$ *remain finite.*

Proof of (*b*).—Consider condition 1° of Theorem 1 for F_{4a}.

$$\int_c^{c+\pi} q_n \left| \begin{matrix} \sin nt \\ \cos nt \end{matrix} \right| dt \gtreqless q_n \int_c^{c+\pi} \begin{matrix} \sin^2 nt \\ \cos^2 nt \end{matrix} dt = \frac{\pi}{2} q_n,$$

which is not limited.

Proof of (*c*).—Consider Theorem 1 for F_5.

$$\frac{n}{q_n} \left| \int_c^{c+t} \begin{matrix} \sin nt \\ \cos nt \end{matrix} dt \right| = \frac{1}{q_n} \left| \begin{matrix} \cos nc - \cos n(c+t) \\ \sin n(c+t) - \sin nc \end{matrix} \right| \overline{\lessgtr} \frac{2}{q_n},$$

which approaches zero.

Proof of (*d*).—Consider Theorem 1 for F_{5a}.

$$n \left| \int_c^{c+t} \begin{matrix} \sin nt \\ \cos nt \end{matrix} dt \right| \overline{\lessgtr} 2.$$

$$n \left| \int_a^{c+2\pi} (t-a) \sin nt \, dt \right| = \frac{1}{n} \left| \sin nc - \sin na + n(a-c-2\pi) \cos nc \right|,$$

which approaches zero if nc is an odd multiple of $\pi/2$. Similarly,

$$n \left| \int_a^{c+2\pi} (t-a) \cos nt \, dt \right| = \frac{1}{n} \left| \cos nc - \cos na + n(c+2\pi-a) \sin nc \right|,$$

which approaches zero if c is an even multiple of $\pi/2$. The conditions of the theorems for F_{5a} are thus satisfied, but condition (2°) for F_5 is not satisfied. For example, if $c = 0$ and $t = \pi/2$,

$$n \left| \int_0^{\frac{\pi}{2}} \cos nt \, dt \right| = \left| \sin \frac{n\pi}{2} \right|,$$

which does not approach zero.

Proof of (*e*) and (*f*).—I use here a well-known method.* Integration by parts is allowable† because f is an indefinite integral, and so we have

$$\int_c^{c+2\pi} f \sin nt \, dt = -\frac{\cos nc}{n} [f(c+2\pi) - f(c)] + \frac{1}{n} \int_c^{c+2\pi} f' \cos nt \, dt.$$

* E. g., Hobson, *Theory of Functions of a Real Variable* (1907), p. 718.

† Lebesgue, loc. cit., p. 46.

Therefore, if (1) either $\cos nc$ or $f(c+2\pi)-f(c)$ equals zero,

$$a_n = \frac{b_n'}{n}, \qquad \text{where} \qquad b_n' = \frac{1}{\pi}\int_c^{c+2\pi} f' \cos nt\, dt.$$

Likewise

$$\int_c^{c+2\pi} f \cos nt\, dt = \frac{\sin nc}{n}\left[f(c+2\pi)-f(c)\right] - \frac{1}{n}\int_c^{c+2\pi} f' \sin nt\, dt,$$

and, if (2) either $\sin nc$ or $f(c+2\pi)-f(c)$ equals zero,

$$b_n = \frac{a_n'}{n}, \qquad \text{where} \qquad a_n' = \frac{1}{\pi}\int_c^{c+2\pi} f' \sin nt\, dt.$$

We can obviously satisfy (2) by choosing $c = 0$ or a multiple of π. We can satisfy both by making $f(c+2\pi) = f(c)$.

Repeating these processes and applying (a) we have (e); applying (c) we have (f).

From the foregoing propositions follow many corollaries concerning the convergence of Fourier's series. I will state two of them:

Corollary 1: *If f has limited variation, the series whose general term is* $|a_n^p \sin^p nx| + |b_n^p \cos^p nx|$ *converges uniformly when* $p > 1$. For, by (c), this may be compared with the series whose general term is $1/n^p$.

Corollary 2: *If f is the integral of a function of limited variation and if* $f(c) = f(c+2\pi)$, *the series of Corollary* 1 *converges uniformly when* $p > 1/2$. For, by (f) for G_5', it may be compared with the series whose general term is $1/n^{2p}$.

7. **Fourier's Development.**—In this section n and q_n are used as in the preceding section, and R_n represents the difference between $f(x)$ and the sum of the first terms of its Fourier's series up to and including the terms whose argument is nx. The function $f(t)$ is supposed defined everywhere, to be periodic of period 2π, and absolutely L-integrable in the interval $(-\pi, \pi)$. At x the limits $f(x+0)$ and $f(x-0)$ are supposed to exist, and $f(x)$ to be equal to one half their sum. As is well known,*

$$\text{(1)} \qquad \pi R_n = \int_{x-\pi}^{x+\pi} \left[f(t)-f(x)\right] \frac{\sin \dfrac{2n+1}{2}(t-x)}{2\sin \dfrac{t-x}{2}}\, dt$$

$$\text{(2)} \qquad = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[f(x+2v)-f(x)\right] \frac{\sin \nu v}{\sin v}\, dv \qquad (\nu = 2n+1).$$

Set

$$\psi(v) = \frac{f(x+2v)-f(x)}{\sin v} \quad \text{in} \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

* E. g., Pierpont, *Theory of Functions of Real Variables*, vol. 2, pp. 421, 2.

Reasoning analogous to that of the preceding article gives us the following propositions:

(*a*) *For some function* ψ *which is absolutely L-integrable,* $q_n R_n$ *does not approach zero, though it is known that* R_n *does approach zero.*

(*b*) *For some function* $f(t)$ *which is continuous in* $(-\pi, \pi)$ *and has a first derivative at* x, *the same is true.*

(*c*) *If* ψ *has limited variation,* nR_n *remains finite.*

(*d*) *The same is true if* $1/v[f(x+2v) - f(x)]$ *has limited variation in* $(-\pi/2, \pi/2)$.

(*e*) *If* ψ *is continuous and has limited variation,* nR_n *approaches zero, but this is not true of all functions having limited variation.*

(*f*) *If* ψ *is the integral of a function having limited variation* (*i. e., if* ψ *belongs to* G_5'), $n^2 R_n$ *remains finite.*

(*g*) *There exists a function* ψ *belonging to all the other G families, for which* $n^2 R_n$ *does not approach zero.*

Proof of (*a*).—We use (2) and condition 1° of Theorem 1 for F_1, putting $\phi = q_n \sin \nu v$. This function is not uniformly limited in $(-\pi/2, \pi/2)$. Therefore there exists an absolutely *L*-integrable function $g(v)$, such that the integral from $-\pi/2$ to $\pi/2$ of $g \sin \nu v$ does not approach zero. It can now be shown that a function $f(t)$ may be defined so that $g(v)$ has the form $\psi(v)$ above.

Proof of (*b*).—We use (1). The function $f(t) - f(x)$ satisfies the conditions on f in Corollary 1 of Theorem 2 for F_{6a}. We have, then, corresponding to ϕ,

$$\frac{q_n \sin \dfrac{\nu}{2}(t-x)}{2 \sin \dfrac{1}{2}(t-x)}.$$

Since, in the interval $(x-\pi, x+\pi)$, except at most at the three points, $x-\pi, x, x+\pi$,

$$\left| \frac{t-x}{2 \sin \dfrac{1}{2}(t-x)} \right| \overline{\gtrless} 1,$$

$$\int_{x-\pi}^{x+\pi} |(t-x)\phi| \, dt \overline{\gtrless} q_n \int_{x-\pi}^{x+\pi} \left| \sin \frac{\nu}{2}(t-x) \right| dt$$

$$\overline{\gtrless} q_n \int_{x-\pi}^{x+\pi} \sin^2 \frac{\nu}{2}(t-x)\, dt = q_n \pi,$$

and is not uniformly limited.

Proof of (c).—Using (2) and Theorem 1 for F_5, we have

$$\left|\frac{n}{q_n}\int_{-\frac{\pi}{2}}^{t}\sin\nu v\,dv\right| = \frac{n}{q_n\,\nu}\left|-\cos\nu t\right| \overline{\gtrless} \frac{n}{q_n\,\nu},$$

and this approaches zero.

Proof of (d).—We use the same theorem, but now corresponding to ϕ we have $v\,\sin\nu v/\sin v$. By an inequality,* due to Lebesgue, for monotone increasing functions, and since in $(0,\pi/2)$ $v/\sin v$ is monotone increasing from 1 to $\pi/2$, we have,† if $0 \overline{\gtrless} t \overline{\gtrless} \pi/2$,

$$\left|\int_0^t \frac{v\sin\nu v}{\sin v}\,dv\right| < \frac{3\pi}{2}\max_{\lambda \text{ in } (0,t)}\left|\int_0^\lambda \sin\nu v\,dv\right| \overline{\gtrless} \frac{3\pi}{2}\,\frac{2}{\nu}.$$

The same is true if $-\pi/2 \overline{\gtrless} t \overline{\gtrless} 0$, since the integrand is an odd function. Therefore

$$\frac{n}{q_n}\left|\int_{-\frac{\pi}{2}}^{t}\frac{v\sin\nu v}{\sin v}\,dv\right| < \frac{n}{q_n}\,\frac{6\pi}{\nu},$$

which approaches zero as n becomes infinite.

Proof of (e).—Using (2) and Theorem 1 for F_{5a}, we have

$$n\left|\int_{-\frac{\pi}{2}}^{t}\sin\nu v\,dv\right| \overline{\gtrless} \frac{2n}{\nu} \overline{\gtrless} 1,$$

$$n\left|\int_a^{\frac{\pi}{2}}(v-a)\sin\nu v\,dv\right| = \frac{n}{\nu^2}\left|\sin\frac{\nu\pi}{2} - \sin\nu a\right| \overline{\gtrless} \frac{2n}{\nu^2},$$

$$n\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin\nu v\,dv = 0.$$

These show that the conditions in question are satisfied, and establish the first part of (e). As for the second part, we note that

$$n\left|\int_{-\frac{\pi}{2}}^{t}\sin\nu v\,dv\right| = \frac{n}{\nu}\left|\cos\nu t\right|.$$

This equals n/ν when $t=0$, and therefore does not approach zero for each t. We may now proceed as in the proof of (a).

* Lebesgue, loc. cit., p. 36, B. See also my paper (second footnote), p. 54.

† "$\underset{\lambda}{\text{Max}}$" means the maximum of the function with respect to the parameter λ. I use the words "maximum" and "minimum" in the sense of upper and lower limits, not of greatest and least values.

Proof of (*f*).—Using (2) and Theorem 3 for G_5', we have

$$\frac{n^2}{q_n}\left|\int_{-\frac{\pi}{2}}^{t} dt \int_{-\frac{\pi}{2}}^{t} \sin \nu v \, dv\right| = \frac{n^2}{q_n \nu^2} |\sin \nu t \pm 1| \overline{\gtrless} \frac{2n^2}{q_n \nu^2},$$

$$\frac{n^2}{q_n}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin \nu v \, dv = 0.$$

Proof of (*g*).—It would be practicable to show that for no G are the conditions of Theorem 3 satisfied when $\phi = n^2 \sin \nu v$, but it is easier to see directly that for the function $\psi(v) = v$, which belongs to all the families, $n^2 | R_n |$ remains greater than $2/9$ if $n > 1$. $\psi = v$, if $f(t) = (t/2) \sin(t/2)$ in $(-\pi, \pi)$ and $x = 0$.

$$n^2 | R_n | = n^2 \left|\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v \sin \nu v \, dv\right| = \frac{2n^2}{\nu^2} > \frac{2}{9}, \quad \text{if } n > 1.$$

8. **Legendre's Development.**—In this section n and q_n are used as hitherto, $f(t)$ is supposed defined and absolutely L-integrable in the interval $(-1, 1)$, P_n is Legendre's polynomial of order n, and A_n is the coefficient of P_n in Legendre's development of f:

$$A_n = \frac{2n+1}{2}\int_{-1}^{1} f P_n \, dt.$$

In the following lemmas, M is some fixed number independent of t and t_i.

LEMMA 1:

(*a*) $$\left|\int_{-1}^{t} P_n \, dt\right| < \frac{M}{n^{\frac{3}{2}}(1-t^2)^{\frac{1}{4}}} \qquad (t^2 \neq 1, \; n \geqq 2),$$

(*b*) $$\left|\int_{-1}^{t} P_n dt\right| < \frac{M}{n^{1.1}} \qquad (n \geqq 2).$$

LEMMA 2: *If* $P_n(t)$ *be integrated* i *times, with respect to the variables* $t_0, t_1, \cdots t_{i-1}$,

(*a*) $$\left|\int_{-1}^{t_i} \cdots \int_{-1}^{t_1} P_n \, dt_0\right| < \frac{M}{n^{i+\frac{1}{2}}(1-t_i^2)^{\frac{1}{4}}} \qquad (t_i^2 \neq 1, \; n \geqq 2i),$$

(*b*) $$\left|\int_{-1}^{t_i} \cdots \int_{-1}^{t_1} P_n \, dt_0\right| < \frac{M}{n^{i+0.1}} \qquad (n \geqq 2i).$$

LEMMA* 3:

(*a*) $$\int_{-1}^{t} P_n^2 \, dt < \frac{1}{n} \qquad (n \geqq 1),$$

(*b*) $$\int_{-1}^{t_i}\left(\int_{-1}^{t_{i-1}} \cdots \int_{-1}^{t_1} P_n \, dt_0\right)^2 dt_{i-1} < \frac{M}{n^{2i-1}} \qquad (n \geqq 2i-2).$$

* Proposition (*a*) is known.

LEMMA* 4

(a) $$\int_{-1}^{t} | P_n | \, dt < \frac{M}{\sqrt{n}} \qquad (n \geqq 1),$$

(b) $$\int_{-1}^{t_i} \left| \int_{-1}^{t_{i-1}} \cdots \int_{-1}^{t_1} P_n \, dt_0 \right| dt_{i-1} < \frac{M}{n^{i-\frac{1}{2}}} \qquad (n \geqq 2i - 1).$$

Proof of Lemma 1.—From the known equation

$$(2n + 1) P_n = \frac{d}{dt}(P_{n+1}) - \frac{d}{dt}(P_{n-1}),$$

we derive

(1) $$\left| \int_{-1}^{t} P_n \, dt \right| = \left| \int_{-1}^{1} \right| + \left| \int_{1}^{t} \right| = \frac{1}{2n + 1} | P_{n+1}(t) - P_{n-1}(t) | \quad (n > 0).$$

Hobson† has proved the inequality

(2) $$(1 - t^2)^{\frac{1}{4}} | P_n(t) | < \frac{M}{\sqrt{n}} \qquad (n > 0).$$

This with (1) proves that

$$\left| \int_{-1}^{t} P_n \, dt \right| < \frac{M}{2n + 1}\left(\frac{1}{\sqrt{n + 1}} + \frac{1}{\sqrt{n - 1}} \right) \frac{1}{(1 - t^2)^{\frac{1}{4}}} < \frac{3M}{n^{\frac{3}{2}} (1 - t^2)^{\frac{1}{4}}}$$
$$(t^2 \neq 1, \; n \geqq 2),$$

which is (a). To establish (b) we note first that‡

(3) $$\left| \int_{-1}^{-1+x} \frac{dt}{(1 - t^2)^{\frac{1}{4}}} \right|, \quad \left| \int_{1-x}^{1} \frac{dt}{(1 - t^2)^{\frac{1}{4}}} \right| \gtreqless \frac{4}{3} x^{\frac{3}{4}} \qquad (0 \leqq x \leqq 1).$$

Case 1: $-1 + 1/n^{0.8} \leqq t \leqq 1 - 1/n^{0.8}$.

(4) $$\left| \int_{-1}^{t} P_n \, dt \right| \gtreqless \int_{-1}^{-1+1/n^{0.8}} | P_n | \, dt + \left| \int_{-1+1/n^{0.8}}^{t} P_n \, dt \right| = \mathrm{I} + \mathrm{II}.$$

By (2) and (3),

(5) $$\mathrm{I} < \frac{4}{3}\left(\frac{1}{n^{0.8}} \right)^{\frac{3}{4}} \frac{M}{\sqrt{n}} = \frac{4M}{3n^{1.1}} \qquad (n > 0).$$

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* I do not need this, but I put it down as a simple corollary of the others; (a) is known.

† Hobson, Proceedings of the London Mathematical Society, ser. 2, vol. 7 (1909), p. 25.

‡ For example, the first of these equals, if we put $t = z - 1$,

$$\left| \int_0^x \frac{dz}{(2z - z^2)^{\frac{1}{4}}} \right| \gtreqless \left| \int_0^x \frac{dz}{z^{\frac{1}{4}}} \right| = \frac{4}{3} x^{\frac{3}{4}}, \text{ if } 0 \leqq x \leqq 1.$$

By (2), since $1 - t^2 \geqq 1/n^{1.6}$, and

$$\int_a^b P_n\,dt = \frac{1}{2n+1}\,[P_{n+1}(t) - P_{n-1}(t)]_a^b,$$

$$\text{(6)}\qquad \mathrm{II} < \frac{1}{2n+1}\cdot\frac{4M}{\sqrt{n-1}}\cdot n^{0.4} < \frac{4M}{2n}\cdot\sqrt{\frac{2}{n}}\cdot n^{0.4} < \frac{4M}{n^{1.1}}\qquad (n>1).$$

The inequalities (5) and (6) establish Case 1.

Case 2: $-1 \leqq t \leqq -1 + 1/n^{0.8}$.

$$\left|\int_{-1}^{t} P_n dt\right| \leqq \int_{-1}^{-1+1/n^{0.8}} |P_n|\,dt,$$

and by (5) this is less than $M/n^{1.1}$ if $n > 0$. Likewise, if

Case 3: $1 - 1/n^{0.8} \leqq t \leqq 1$,

$$\left|\int_{-1}^{t} P_n dt\right| \leqq \left|\int_{-1}^{1-1/n^{0.8}} P_n dt\right| + \int_{1-1/n^{0.8}}^{t} |P_n|\,dt < \frac{2M}{n^{1.1}}\qquad (n>0),$$

by Case 1, (2), and (3).

Proof of Lemma 2.—Lemma 2 is proved by means of Lemma 1 and equalities like the following:

$$\int_{-1}^{t_2} dt_1 \int_{-1}^{t_1} P_n\,dt_0 = \frac{1}{2n+1}\left[\frac{1}{2n+3}P[_{n+2}(t_2) - P_n(t_2)] - \frac{1}{2n-1}[P_n(t_2) - P_{n-2}(t_2)]\right],$$

$$\left|\int_{-1}^{t_2} dt_1 \int_{-1}^{t_1} P_n\,dt_0\right| = \left|\int_{-1}^{t_2} \frac{1}{2n+1}[P_{n+1}(t_1) - P_{n-1}(t_1)]\,dt_1\right| \leqq \frac{1}{2n+1}\left(\left|\int_{-1}^{t_2} P_{n+1}\,dt_1\right| + \left|\int_{-1}^{t_2} P_{n-1}\,dt_1\right|\right).$$

Proof of Lemma 3(*b*).—By (*a*) of Lemma 2,

$$\int_{-1}^{t_i} (\qquad)^2\cdot dt_{i-1} < \int_{-1}^{1}\frac{M^2}{n^{2i-1}\sqrt{(1-t_{i-1}^2)}}\,dt_{i-1} = \frac{\pi M^2}{n^{2i-1}}\quad (n \geqq 2i-2).$$

Proof of Lemma 4(*b*).—By (*a*) of Lemma 2, and (3),

$$\int_{-1}^{t_i} |\qquad|\,dt_{i-1} < \int_{-1}^{1}\frac{M}{n^{i-\frac{1}{2}}(1-t_{i-1}^2)^{\frac{1}{4}}}\,dt_{i-1} = \int_{-1}^{0} + \int_{0}^{1} < \frac{2M}{n^{i-\frac{1}{2}}}\,\frac{4}{3} < \frac{4M}{n^{i-\frac{1}{2}}}$$

$$(n \geqq 2i-2).$$

The following statements are now easily established.

(*a*) *For every absolutely L-integrable function, A_n/n approaches zero, but there exists such a function for which $q_n A_n/n$ does not approach zero.*

(*b*) *For every function whose square is L-integrable, $A_n / \sqrt{n}$ approaches zero, but there exists such a function for which $q_n A_n / \sqrt{n}$ does not approach zero.*

(*c*) *There exists a continuous* function for which $q_n A_n$ does not approach zero.*

(*d*) *For every function of limited variation, $n^{0.1} A_n$ approaches zero.*

(*e*) *For every function of $G_2^{(i)}$, $n^{i-\frac{1}{2}} A_n$ approaches zero.*

(*f*) *For every function of $G_5^{(i)}$, $n^{i+0.1} A_n$ approaches zero.*

Proof of (*a*).—The first part of (*a*) follows from Theorem 1 for F_1, ϕ being† P_n, for P_n is limited, and by Lemma 1 (*b*) its integral from -1 to t approaches zero. But if ϕ be $q_n P_n$, condition (1°) of that theorem is not satisfied.

Proof of (*b*).—Using Theorem 1 for F_2, $\phi = \sqrt{n} P_n$, and Lemma 1 (*b*), we have

$$\int_{-1}^{1} \frac{n^2 P_n^2}{n} dt = \frac{2n}{2n+1} < 1 \qquad (n \gtreqless 1),$$

$$\int_{-1}^{t} \frac{n P_n}{\sqrt{n}} dt < \frac{M}{n^{0.6}}.$$

Proof of (*c*).—Condition (1°) of Theorem 1 for F_{4a} is not satisfied by $\phi = q_n n P_n$, for

$$q_n \int_{-1}^{1} n \,|\, P_n \,|\, dt \gtreqless q_n n \int_{-1}^{1} P_n^2 \, dt = \frac{2n q_n}{2n+1}.$$

Proof of (*d*).—Putting $\phi = n^{1.1} P_n$ in Theorem 1 for F_5, we have, by Lemma 1 (*b*),

$$\left| \int_{-1}^{t} n^{1.1} P_n \, dt \right| < M \qquad (n \gtreqless 2),$$

and, by Lemma 1 (*a*),

$$\left| \int_{-1}^{t} n^{1.1} P_n \, dt \right| < \frac{n^{1.1} M}{n^{1.5}(1-t^2)^{\frac{1}{4}}} \quad \text{if } t^2 \neq 1 \qquad (n \gtreqless 2),$$

$$= 0 \quad \text{if } t^2 = 1 \qquad (n \gtreqless 1).$$

Proof of (*e*).—Consider G_2'. (The reasoning is analogous for the other G_2's.) Applying the first condition of Theorem 3 for G_2', and using $\phi = n^{\frac{1}{2}} P_n$, we have, by Lemma 3 (*b*), and Lemma 2 (*b*),

$$\int_{-1}^{1} \left(\int_{-1}^{t_1} n^{\frac{1}{2}} P_n \, dt_0 \right)^2 dt_1 < \frac{n^3 M}{n^3} = M \qquad (n \gtreqless 2),$$

$$\int_{-1}^{t_2} \left(\int_{-1}^{t_1} n^{\frac{1}{2}} P_n \, dt_0 \right) dt_1 < \frac{n^{\frac{1}{2}} M}{n^{2.1}} \qquad (n \gtreqless 4),$$

$$\int_{-1}^{1} n^{\frac{1}{2}} P_n \, dt_0 = 0 \qquad (n \gtreqless 1).$$

* But I cannot show that for every continuous function A_n remains limited.

† Exactly, $\phi = [(2n+1)/2n] P_n$, but this approaches the same limit as P_n. Similar remarks hold for the other proofs.

Proof of (f).—Consider for example G_5'. Using Theorem 3 for G_5' and $\phi = n^{2.1} P_n$, and Lemma 2 (b) and (a), we have

$$\left| \int_{-1}^{t_2} \left(\int_{-1}^{t_1} n^{2.1} P_n \, dt_0 \right) dt_1 \right| < \frac{n^{2.1} M}{n^{2.1}} = M \qquad (n \gtreqless 4),$$

$$< \frac{n^{2.1} M}{n^{2.5} (1 - t_2^2)^{\frac{1}{4}}} \quad \text{if } t_2^2 \neq 1 \quad (n \gtreqless 4),$$

$$= 0 \qquad \text{if } t_2^2 = 1 \quad (n \gtreqless 2),$$

the last result being obtained by use of the proof of Lemma 2. Similarly,

$$\int_{-1}^{1} n^{2.1} P_n \, dt_0 = 0 \qquad (n \gtreqless 1).$$

COROLLARY 1: *If f is the integral of a function which has limited variation in the interval $(-1, 1)$, its Legendre development converges absolutely uniformly in the same interval.*

For, by (f), the absolute value series may be compared with the series whose general term is $1/n^{1.1}$.

COROLLARY 2: *If f has limited variation in $(-1, 1)$, the series whose general term* is $|A_n| P_n^2$ converges uniformly in any interval within $(-1, 1)$.*

For, by (2), in any such interval nP^2 remains uniformly limited, and therefore, by (d), the series in question may be compared with the series whose general term is $1/n^{1.1}$.

We may now apply to the developments of Legendre the same reasoning as was used in the case of Fourier's developments, and obtain the following results concerning their rates of convergence:†

COROLLARY 3: *Let R_n be the remainder of the formal development of f at the point x, and let $\psi(t)$ be the fraction $[f(t) - f(x)]/(t - x)$.*

(a) *If ψ is absolutely L-integrable, R_n / n approaches zero.*

(b) *If ψ^2 is L-integrable, $R_n / \sqrt{n}$ approaches zero.*

(c) *If ψ has limited variation, $n^{0.1} R_n$ approaches zero.*

(d) *If ψ belongs to $G_5^{(i)}$, $n^{i+0.1} R_n$ approaches zero.*

9. **Definite Integrals.**—Much of this section, in a somewhat different form, has been established by W. H. Young‡ independently of Lebesgue's theorems, but by means of reasoning closely analogous to that of Lebesgue. The present paper shows that the results are really only corollaries of Lebesgue's

* À fortiori this is true of the series whose general term is $A_n^2 P_n^2$.

† Cf. D. Jackson, *On the Degree of Convergence*, etc., these Transactions, vol. 13 (1912), pp. 305–318; and *On Approximation by Trigonometric Sums*, etc., ibid., pp. 491–515.

‡ Young, Proceedings of the London Mathematical Society, ser. 2, vol. 9 (1910), pp. 463–485.

fundamental theorem (§ 2), and possesses the advantage that the conditions stated here are thus shown to be necessary as well as sufficient. Young has nothing to do with the family F_6, and in Theorem 6 I consider several families where Young considers but one. His account includes, however, two topics which mine omits: non-absolutely convergent L-integrals, and integrals over infinite intervals.

Let us consider the evaluation of the definite integral

$$(1) \qquad \int_k^l f(t)\, g(t)\, dt,$$

by means of the common method of expanding $g(t)$ in a series,

$$(2) \qquad g(t) = u_1(t) + u_2(t) + \cdots$$

(the sum of the first n terms of which shall be called $s_n(t)$), multiplying it through by $f(t)$, and integrating termwise from k to l, thus:

$$(3) \qquad \int_k^l fg dt = \int_k^l fu_1\, dt + \int_k^l fu_2\, dt + \cdots.$$

Concerning the validity of this process we have the following

THEOREM 5: *Let the various integrals of* (3) *exist as absolutely convergent L-integrals. A necessary and sufficient condition that* (3) *shall be valid for all functions f belonging to one of the families F of Theorems* 1 *and* 2, *is that $g - s_n$ shall satisfy the conditions imposed on ϕ in that part of those theorems that applies to the F in question.*

*In the cases** *of F_1, F_2, and F_3, the condition stated remains sufficient if* (2°) *of Theorem* 1 *be replaced by the condition that* (2) *be valid in* (k, l), *except perhaps at a null set.*

For it is necessary and sufficient that

$$\int_k^l (g - s_n) f\, dt$$

approach zero, and therefore Theorem 1 (with corollary) and Theorem 2 are applicable.

Continuing the same problem and notation, let us now suppose that it is not convenient to express g or its 1st, 2d, . . . , or $(i - 1)$th integral $(i \gtreqless 1)$ by means of a series, but that its ith integral may be so expressed, thus:

$$(1) \qquad J(t_i) = \int_k^{t_i} \cdots \int_k^{t_1} g\, dt_0 = U_1(t_i) + U_2(t_i) + \cdots.$$

Writing

$$(2) \qquad S_n = U_1 + \cdots + U_n, \qquad R_n = J - S_n,$$

* Stated in another paper (see second footnote), pp. 63, 64.

and denoting differentiation by primes, we have, except at most at a null set,

$$J^{(i)} = g = s_n + r_n. \tag{3}$$

We now give the conditions that

$$\int_k^l fg\,dt = \lim_{n=\infty} \int_k^l s_n\,dt\,; \tag{4}$$

that is, that (1) may be differentiated termwise i times, then multiplied through by f and integrated termwise from k to l, giving the integral of fg.

THEOREM 6: *Let* ($a°$) *the U's have i derivatives, and let all the integrals involved exist as absolutely convergent L-integrals. Let* ($b°$) $\lim_n R_n^{(p)}(k) = \lim_n R_n^{(p)}(l) = 0,\ p = 0, 1, \cdots, i-1$. *Then a necessary and sufficient condition that* (4) *be valid, for all functions belonging to the family* $G_j^{(i)}$, *is that* R_n *satisfy the conditions on* ϕ *in Theorem* 1, *or* 2, *for* $F_j^{(i)}$.

In the cases of $G_1^{(i)}$, $G_2^{(i)}$, *and* $G_3^{(i)}$, *the conditions remain sufficient if it be required that* (1) *be valid in* (k, l), *except perhaps at a null set, instead of that* R_n *satisfy* (2°) *of Theorem* 1.

Proof.—From ($a°$) and successive applications of (d), § 4, it follows that

$$\begin{aligned} \int_k^{t_{i-p}} \cdots \int_k^{t_1} r_n\,dt_0 &= R_n^{(p)}(t_{i-p}) + [\,P_0 R_n^{(p)}(k) + \cdots + P_{i-p-1} R_n^{i-1}(k)\,] \\ &= R_n^{(p)} + \eta \qquad\qquad (p = 0, \cdots, i-1), \end{aligned} \tag{5}$$

where P_s is a polynomial of degree s in t_{i-p}. By ($b°$), η approaches zero uniformly with respect to t_{i-p}. It is necessary and sufficient that

$$\lim_{n=\infty} \int_k^l fr_n\,dt = 0, \tag{6}$$

and for this the conditions of Theorem 3 relative to $G_j^{(i)}$ are necessary and sufficient. To see that the theorem is true it is now only necessary to write out the conditions involved in that theorem and in our present hypothesis in full. This will be done for the case of $G_1^{(i)}$ only. Theorem 3 says that the first of the following integrals shall remain uniformly limited if n is large enough (except perhaps at a null set), and that the others shall approach zero.

$$\int_k^{t_i} \cdots \int_k^{t_1} r_n\,dt_0, \tag{7}$$

$$\int_k^{t_{i+1}} \cdots \int_k^{t_1} r_n\,dt_0, \tag{8}$$

$$\int_k^l \int_k^{t_{i-1}} \cdots \int_k^{t_1} r_n\,dt_0,\quad \int_k \int_k^{t_{i-2}} \cdots \int_k^{t_1} r_n\,dt_0,\quad \cdots,\quad \int_k^l r_n\,dt_0. \tag{9}$$

Sufficiency.—By ($b°$) $\lim_{n=\infty} R_n^{(p)}(l) = 0$, $p = 0, \cdots, i-1$, and this with (5) shows that (9) approach zero. By hypothesis and (1°) of Theorem 1, $| R_n(t_i) | < M$, $n > n_M$, except perhaps at a null set, and this with (5) shows that the same is true of (7), and again, by (2°) of Theorem 1,

$$\lim_{n=\infty} \int_k^{t_i+1} R_n \, dt_i = 0,$$

and therefore, since η approaches zero uniformly, (8) approaches zero.

Necessity.—If either (1°) or (2°) of Theorem 1 is not satisfied by R_n, it follows from (5) that either (7) or (8), both necessary conditions, is not satisfied.

The corollary of Theorem 1 applies to the second part of the theorem.

10. **Termwise Integration.**—We might now consider the following similar problem. Let g be differentiable i times, and let its ith derivative be represented by a series. Under what conditions may this series be integrated termwise i times, then multiplied through by f and again integrated termwise, giving the integral of fg? But this method of evaluating a definite integral would not ordinarily be of advantage, because it is integration and not differentiation that usually leads to simpler functions, though it does frequently happen that it is no *harder* to evaluate an integral in this way than in one of the others, for what is lost in making the expansion more difficult is gained from the extra generality of the function f by which its integral may be multiplied. However, I wish to present the matter in another form. As already noted, a necessary and sufficient condition that termwise integration of a convergent sequence $\{ \phi(n) \}$ be permissible is (1°) of Theorem 3. Let us suppose that ϕ satisfies not only this condition but also the severer condition (1°) of Theorem 1. Is anything more than termwise integration permissible? An answer to this question is found in Theorem 7, for which I introduce the following definition and lemmas.

Definition: Let us denote by $E_j^{(i)}(x)$ the family of functions f such that, for a certain preassigned x in (k, l), $f(t)(t-x)^i$ belongs to F_j, i being a positive integer or zero.*

Lemma 1: *In order that the integral of Theorem* 1 *shall exist and approach zero for all functions f belonging to the family $E_j^{(i)}$, it is necessary and sufficient that the conditions of Theorem* 1 *for F_j be satisfied when ϕ is replaced by $\phi / (t-x)^i$.*

This is a simple corollary of Theorem 1.

Lemma 2: *If $g(t)$ is continuous in (k, l), the ith iterated integral $J(t_i)$ may be expressed thus:*

$$(1) \qquad J(t_i) = \int_x^{t_i} \cdots \int_x^{t_1} g \, dt_0 = \frac{(t_i - x)^i}{i!} g(\tau),$$

where τ is a point of the interval (x, t_i).

* E. g., $t^{-\frac{1}{2}}$ belongs to $E_1'(0)$ in $(0, 1)$. As far as Lemma 1 is concerned i might be any real number.

For, by successive applications of (*a*), § 4, we may learn that, for each p $(p = 0, 1, \cdots, i - 1)$ and at each point of (k, l),

$$\frac{d^pJ}{dt^p} = \int_x^{t_{i-p}} \cdots \int_x^{t_1} g\, dt_0, \qquad \frac{d^iJ}{dt^i} = g. \tag{2}$$

By a well-known theorem we may therefore write

$$J(t_i) = J(x) + \frac{dJ}{dt}\bigg|_x \frac{t_i - x}{1!} + \cdots + \frac{(t_i - x)^i}{i!} g(\tau),$$

and, evidently, by (1) and (2), all the terms on the right vanish except the last.

THEOREM 7: *If, in the interval* (k, l), (1°) *of Theorem* 1 *is satisfied, and* $\{\phi(n)\}$ *is a convergent sequence having continuous terms and a continuous limit, it is permissible* (*a*) *to integrate* $\{\phi(n)\}$ *termwise* i *times from* x *to* t, *and* (*b*) *to multiply both sides of the resulting equation by any function* f *belonging to* $E_1^{(i)}(x)$, *and again to integrate termwise from* a *to* b, a, b, *and* x *being any points of the interval, thus:*

$$\lim_{n=\infty} \int_x^{t_i} \cdots \int_x^{t_1} \phi(n)\, dt_0 = \int_x^{t_i} \cdots \int_x \lim_{n=\infty} \phi(n)\, dt_0, \tag{a}$$

$$\lim_{n=\infty} \int_a^b f(t_i)\, dt_i \int_x^{t_i} \cdots \int_x^{t_1} \phi(n)\, dt_0 = \int_a^b f(t_i)\, dt_i \int_x^{t_i} \cdots \int_x^{t_1} \lim_{n=\infty} \phi(n)\, dt_0. \tag{b}$$

Proof.—Without loss of generality we may suppose that $\lim \phi(n) = 0$, for otherwise the limit of the sequence $\{\phi(n) - \lim \phi(n)\}$ is zero, and this latter sequence possesses the properties ascribed to $\{\phi\}$ above.

Case 1: $(i = 0)$. We have to show merely that, for f in $(E_1^0 = F_1)$,

$$\lim_{n=\infty} \int_a^b f\phi(n)\, dt = 0; \tag{b}$$

and this has been established already in the corollary to Theorem 1.

Case 2: $(i > 0)$. Let us suppose that (*b*) is true when i is replaced by $r - 1$; if we choose $f = 1$, the result is (*a*) for $i = r$. Hence (*a*) may be obtained by induction from (*b*) for Case 1. To establish (*b*) for Case 2 it is, by Lemmas 1 and 2, necessary and sufficient that

$$\lim_{n=\infty} \int_a^b f(t_i)\, \phi(n, \tau) \frac{(t_i - x)^i}{i!} dt_i = 0.$$

From the corollary to Theorem 1 and Lemma 1, it is clear that for this it is sufficient that $\phi(n, \tau)$ be uniformly limited in (a, b), and that for each τ its limit be zero. Both these conditions are satisfied by hypothesis.

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