PROPERTIES OF SURFACES WHOSE ASYMPTOTIC CURVES BELONG TO LINEAR COMPLEXES*

BY

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INTRODUCTION

In this paper a study is made of the geometrical properties of surfaces whose asymptotic curves belong to linear complexes. The treatment is based on the methods developed by E. J. Wilczynski in his book on projective differential geometry† and a series of memoirs published in these Transactions.‡

So far as I am aware, the only papers on this subject are a note by Sophus Lie§ and a thesis by Arnold Peter.‖ In his thesis, Peter gives the analytical details of Lie's note and establishes the theorem that "*The determination of the surfaces can be reduced to quadratures.*"

One of the theorems established below, namely, "The ruled surfaces of the problem have straight line directrices" is due to Peter. The method of proof employed here, however, is entirely different from that used by Peter and is, moreover, essentially connected with the subsequent study of non-ruled surfaces.

In my work, as well as in that of Peter, a certain quadric surface plays an important rôle. From my point of view this quadric is the locus of the directrices of the osculating ruled surfaces associated with the two families of

* Presented to the Society (Chicago), April, 1912.

† E. J. Wilczynski, *Projective Differential Geometry of Curves and Ruled Surfaces.* B. G. Teubner, Leipzig, 1906. We shall hereafter refer to this book as W.

‡ Vol. 8 (1907), pp. 223–260; vol. 9 (1908), pp. 79–120, 293–315. We shall hereafter refer to these as M_1, M_2, M_3.

§ Sophus Lie, Christiania Videnskabsselskabs Forhandlinger., 1882, Nr. 21.

‖ Arnold Peter, *Die Flächen deren Haupttangentenkurven linearen Komplexen angehören* Leipzig, Dissertation, 1895. Since the presentation of this paper to the Society, some points of its subject matter have been touched upon in two other papers, viz., one by Enrico Bompiani, Rendiconti del Circolo Matematico di Palermo, vol. 34 (1912); the other by Corrado Segre, Reale Accademia delle Scienze di Torino, vol. 49 (1913). I have recently learned that a certain aspect of the subject of this paper was treated by M. Keraval, Bulletin de la Societé Mathématique de France, vol. 39 (1911). The methods used in these papers are entirely unlike those employed here, and the results are only distantly related to those of this paper.

asymptotic curves. If the asymptotic curves of only one of the two families belong to linear complexes, this quadric is replaced by a directrix-ruled surface of a higher order. On the other hand, the quadric introduced by Peter is characteristic of the canonical differential equations of the surfaces. However there is nothing in Peter's thesis concerning the geometrical significance of this quadric, nor is there anything analogous to my discussion of the osculating ruled surfaces and the directrix-ruled surface. All of the other results in the two papers are entirely unrelated and therefore require no further comparison.* The two principal results obtained in this paper are:

A geometric construction for the surfaces of the problem (§ 4), and a normal form for their defining equations (§ 3). In the normal form of these equations the coefficients, and therefore all the invariants of the surfaces, are given explicitly as functions of the two parameters u and v of the asymptotic curves.

I take this opportunity of expressing my indebtedness to Professor Wilczynski for advice generously given me from time to time during the preparation of this paper.

§ 1. Theorems concerning ruled surfaces whose asymptotic curves belong to linear complexes

Any non-developable ruled surface may be defined by a system of differential equations of the form†

$$
(1)\qquad \begin{aligned} y'' + p_{11}\, y' + p_{12}\, z' + q_{11}\, y + q_{12}\, z &= 0, \\ z'' + p_{21}\, y' + p_{22}\, z' + q_{21}\, y + q_{22}\, z &= 0, \end{aligned}
$$

where

$$
y' = \frac{dy}{dx}, \qquad y'' = \frac{d^2 y}{dx^2}, \text{ etc.}
$$

Its generators are the lines that join those points on the integral curves C_y and C_z which correspond to the same values of the independent variable x.

The differential equations of the integral curves C_y and C_z are‡

$$
(2)\qquad \begin{aligned} y^{(4)} + 4p_1\, y^{(3)} + 6p_2\, y'' + 4p_3\, y' + p_4\, y &= 0, \\ z^{(4)} + 4q_1\, z^{(3)} + 6q_2\, z'' + 4q_3\, z' + q_4\, \dot{z} &= 0, \end{aligned}
$$

where

* Certain theorems relating to the asymptotic curves on ruled surfaces with straight line directrices have been established by Cremona, Annali di matematiche, 1867–68, Halphen, Bulletin de la Societé mathématique de France, vol. 5 (1877), and Snyder, Bulletin of the American Mathematical Society, vol. 5 (1899).

†W, p. 126.

‡ W, p. 230.

$$
\begin{aligned}
p_1 &= \frac{1}{4\Delta_1}(q_{12}\, l_{12} - p_{12}\, m_{12}), \qquad p_2 = \frac{1}{6\Delta_1}(s_{12}\, l_{12} - r_{12}\, m_{12}),\\
q_1 &= \frac{1}{4\Delta_2}(q_{21}\, l_{21} - p_{21}\, m_{21}), \qquad q_2 = \frac{1}{6\Delta_2}(s_{21}\, l_{21} - r_{21}\, m_{21}),\\
p_3 &= \frac{1}{4\Delta_1}[\, m_{12}\,(p_{12}\, r_{11} - p_{11}\, r_{12}) + l_{12}\,(p_{11}\, s_{12} - q_{12}\, r_{11}) - l_{11}\,\Delta_1\,],\\
p_4 &= \frac{1}{\Delta_1}\;[\, m_{12}\,(p_{12}\, s_{11} - q_{11}\, r_{12}) + l_{12}\,(q_{11}\, s_{12} - q_{12}\, s_{11}) - m_{11}\,\Delta_1\,],\\
q_3 &= \frac{1}{4\Delta_2}\;[\, m_{21}\,(p_{21}\, r_{22} - p_{22}\, r_{21}) + l_{21}\,(p_{22}\, s_{21} - q_{21}\, r_{22}) - l_{22}\,\Delta_2\,],\\
q_4 &= \frac{1}{\Delta_2}\;[\, m_{21}\,(p_{21}\, s_{22} - q_{22}\, r_{21}) + l_{21}\,(q_{22}\, s_{21} - q_{21}\, s_{22}) - m_{22}\,\Delta_2\,];
\end{aligned}
\tag{3}
$$

and where we shall assume $\Delta_1 \neq 0$, $\Delta_2 \neq 0$; i. e., that neither C_y nor C_z is a plane curve.

The quantities Δ_1, Δ_2, l_{ij}, m_{ij}, s_{ij} are defined as follows:

$$
\begin{aligned}
&\Delta_1 = p_{12}\, s_{12} - q_{12}\, r_{12}, \qquad \Delta_2 = p_{21}\, s_{21} - q_{21}\, r_{21},\\
&r_{11} = p_{11}^2 + p_{12}\, p_{21} - p_{11}' - q_{11}, \qquad s_{11} = p_{11}\, q_{11} + p_{12}\, q_{21} - q_{11}',\\
&r_{12} = p_{12}\,(p_{11} + p_{22}) - p_{12}' - q_{12}, \qquad s_{12} = p_{11}\, q_{12} + p_{12}\, q_{22} - q_{12}',\\
&r_{21} = p_{21}\,(p_{11} + p_{22}) - p_{21}' - q_{21}, \qquad s_{21} = p_{21}\, q_{11} + p_{22}\, q_{21} - q_{21}',\\
&r_{22} = p_{22}^2 + p_{12}\, p_{21} - p_{22}' - q_{22}, \qquad s_{22} = p_{21}\, q_{12} + p_{22}\, q_{22} - q_{22}',\\
&l_{11} = -\, p_{11}\, r_{11} - p_{21}\, r_{12} + r_{11}' + s_{11}, \qquad m_{11} = -\, r_{11}\, q_{11} - r_{12}\, q_{21} + s_{11}',\\
&l_{12} = -\, p_{12}\, r_{11} - p_{22}\, r_{12} + r_{12}' + s_{12}, \qquad m_{12} = -\, r_{11}\, q_{12} - r_{12}\, q_{22} + s_{12}',\\
&l_{21} = -\, p_{11}\, r_{21} - p_{21}\, r_{22} + r_{21}' + s_{21}, \qquad m_{21} = -\, r_{21}\, q_{11} - r_{22}\, q_{21} + s_{21}',\\
&l_{22} = -\, p_{12}\, r_{21} - p_{22}\, r_{22} + r_{22}' + s_{22}, \qquad m_{22} = -\, r_{21}\, q_{12} - r_{22}\, q_{22} + s_{22}'.
\end{aligned}
$$

The fundamental seminvariants and invariants of the integral curve C_y are given by the following expressions*:

$$
\begin{aligned}
&P_2 = p_2 - p_1' - p_1^2, \qquad P_3 = p_3 - p_1'' - 3p_1\, p_2 + 2p_1^3,\\
&P_4 = p_4 - 4p_1\, p_3 - 3p_2^2 + 12p_1^2\, p_2 - 6p_1^4 - p_1^{(3)},\\
&\theta_3 = P_3 - \tfrac{3}{2}P_2', \qquad \theta_4 = P_4 - 2P_3' + \tfrac{6}{5}P_2'' - \tfrac{6}{25}P_2^2.
\end{aligned}
\tag{4}
$$

On replacing p, P, and θ by q, Q, and θ' respectively, we obtain the corresponding expressions for the integral curve C_z.

* W, p. 239.

Let the curves C_y and C_z be two of the *curved* asymptotic lines on the integrating ruled surface S of system (1). Then we have $p_{12} = p_{21} = 0$, and we may also assume that $p_{11} = p_{22} = 0$.*

It is easy to show and, moreover, it is well known, that an asymptotic curve on a ruled surface is never a plane curve unless it is a straight line. If we leave aside the case of a quadric surface, there will be at most two asymptotic curves of the second kind which are plane curves, and these will be straight lines. It follows that the curves C_y and C_z are distinct from these since, by hypothesis, they are not plane curves. The functions q_{12} and q_{21} must therefore be different from zero.

The tangents to the integral curves C_y and C_z will belong to linear complexes if, and only if, the invariants θ_3 and θ_3' vanish identically.† Let us assume, then, that system (1) has been so transformed that $p_{ij} = 0$ $(i = 1, 2; j = 1, 2)$; and let us express θ_3 and θ_3' in terms of the remaining coefficients of (1) by means of (2), (3), (4). We find the following values for θ_3 and θ_3';

$$(5) \qquad \theta_3 = q_{11}' - q_{22}' - \frac{q_{12}'}{q_{12}}(q_{11} - q_{22}), \qquad \theta_3' = q_{11}' - q_{22}' - \frac{q_{21}'}{q_{21}}(q_{11} - q_{22}).$$

Thus, the conditions $\theta_3 = \theta_3' = 0$ reduce to

$$(6) \qquad q_{11}' - q_{22}' = \frac{q_{12}'}{q_{12}}(q_{11} - q_{22}), \qquad q_{11}' - q_{22}' = \frac{q_{21}'}{q_{21}}(q_{11} - q_{22}).$$

It may happen either that C_y and C_z are the *only* asymptotic curves on the surface S which belong to linear complexes, or else that there exists a third asymptotic curve on S which has the same property. In the latter case all the asymptotic curves on S have this property and, if the function $q_{11} - q_{22}$ should happen to vanish for the original pair of asymptotic curves, we may avoid this complication by choosing another pair of fundamental curves. We may therefore assume $q_{11} - q_{22}$ to be different from zero. If, however, C_y and C_z constitute the only pair of asymptotic curves on the surface which belong to linear complexes, they both belong to the same complex and we cannot avoid the vanishing of $q_{11} - q_{22}$ if C_y and C_z are to be asymptotic curves. In this case the invariant‡

$$\theta_9 = \begin{vmatrix} u_{11} - u_{22} & u_{12} & u_{21} \\ v_{11} - v_{22} & v_{12} & v_{21} \\ w_{11} - w_{22} & w_{21} & w_{21} \end{vmatrix}$$

also vanishes while its second order minors are not all equal to zero. The

* W, pp. 114, 142.

† W, p. 254.

‡ W, pp. 96, 167.

integral ruled surface itself belongs to a linear complex but has at most one straight line directrix.

Let us return to the case when $q_{11} - q_{22}$ is different from zero. The integration of (6) gives

$$q_{11} - q_{22} = aq_{12}, \qquad q_{11} - q_{22} = bq_{21},$$

where a and b are constants different from zero. In this case not only does the invariant θ_9 vanish, but so do all of its second order minors. Therefore the surface has a pair of straight line directrices coincident or distinct according as the invariant θ_4 of the surface S vanishes or not, i. e., according as the flecnodes of each generator are coincident or distinct.* The converse of this is also true.† Hence we have the theorem:

In order that all of the asymptotic curves of a ruled surface may belong to linear complexes the surface must have two straight line directrices, distinct or coincident.

If the invariant θ_4 of the ruled surface S does not vanish, the independent variable may be so chosen as to make θ_4 equal to any non-vanishing constant.‡ We may, therefore, assume the independent variable so chosen as to make the function $q_{11} - q_{22}$ equal to unity. The equations of the surfaces S will then assume the form

$$(7) \qquad y'' + q_{11}\, y + az = 0, \qquad z'' + by + q_{22}\, z = 0,$$

where a and b are constants and where $q_{11} - q_{22} = 1$.

The differential equations (2) now become

$$y^{(4)} + (q_{11} + q_{22})\, y'' + 2q_{11}'\, y' + (q_{11}'' + q_{11}\, q_{22} - ab)\, y = 0,$$

$$z^{(4)} + (q_{11} + q_{22})\, z'' + 2q_{22}'\, z' + (q_{22}'' + q_{11}\, q_{22} - ab)\, z = 0.$$

From these it follows that *any curved asymptotic line on S is a projection of any other.*

In order to determine the ruled surfaces S all of whose asymptotic curves are twisted cubics, we must satisfy the further conditions $\theta_4 = \theta_4' = 0$.§ We thus obtain the following equations for the functions q_{11} and q_{22}:

$$(8) \qquad q_{11} - \tfrac{1}{2} = q_{22} + \tfrac{1}{2} = \varphi, \qquad \varphi'' + \tfrac{8}{5}\varphi^2 + c_1 = 0,$$

where

$$c_1 = -\tfrac{5}{8}(1 + 4ab).$$

If $\varphi' = 0$, then

$$q_{11} = k = \tfrac{1}{2} \pm \tfrac{5}{8}\sqrt{1 + 4ab}, \qquad q_{22} = k - 1,$$

so that q_{11} and q_{22} are constants.

* W, p. 169.
† W, p. 288.
‡ W, p. 117.
§ W, p. 242.

From these equations it follows that *the first derivative surface* is a projective transformation of the surface S itself, and the second derivative surface coincides with the surface S.*

The surface turns out to be the quartic

$$x_1^3 x_4 - x_2^3 x_3 = 0,\dagger$$

whose invariant θ_4 is equal to $16\,(4ab + 1)$.

It therefore follows that the invariants θ_3 and θ_4 of the curve C_y may vanish, while the invariant θ_4 of the surface S is different from zero.

If $\varphi' \neq 0$, we can integrate the equation of condition (8) by means of elliptic functions. In fact, if we multiply both members of (8) by φ' and integrate, we find

$$\varphi'^2 = -\tfrac{4}{15}(4\varphi^3 - g_2\,\varphi - g_3),$$

where $g_2 = -\frac{15}{2}C_1$ and g_3 is a constant of integration. Now it is always possible to construct a Weierstrass $\wp$-function with given invariants g_2, g_3 to satisfy this equation. We then find

$$q_{11} = \wp(u;\, g_2,\, g_3) + \tfrac{1}{2}, \qquad q_{22} = \wp\,(u;\, g_2,\, g_3) - \tfrac{1}{2}.$$

Equations (7) may be reduced to the form‡

$$(7a) \qquad \frac{d^2\eta}{du^2} + [A\wp + B]\,\eta = 0 \qquad \frac{d^2\zeta}{du^2} + [A\wp + B_1]\,\zeta = 0,$$

where A, B, B_1 are constants.

These equations are of the Picard type, and precisely those of such great importance in the analytical theory of heat. In fact, an integral of (7*a*) leads at once to an integral of the heat equation in elliptic coördinates. Since A is not of the form $n\,(n+1)$, the equations are not of the Lamé type.

If the directrices of the surface S coincide, i. e., if the invariant θ_4 vanishes, we can write equations (1) in the form

$$(9) \qquad \frac{d^2 y}{dx^2} + q\,(x)\,y = 0, \qquad \frac{d^2 z}{dx^2} + y + q\,(x)\,z = 0,$$

so that C_y is the straight line directrix and C_z is any curved asymptotic line on the surface. The condition (8) becomes

* W, pp. 187–188.

† To get this result, we transform (7) so that the fundamental curves coincide with the flecnode curves and then integrate the resulting equations.

‡ This reduction is effected by means of the following transformations:

$$u = \frac{2i}{\sqrt{15}}\,x, \qquad y = \alpha\eta + \beta\zeta, \qquad z = \gamma\eta + \delta\zeta,$$

where η and ζ are the linear factors of a certain quadratic covariant C (W, p. 124 et seq.).

$$\frac{d^2 q}{dx^2} + \tfrac{8}{5} q^2 = 0. \tag{8a}$$

If $q'' = 0$, q itself must vanish. In this case equations (9) show that the coördinates of any point on the surface S are given by the equations

$$x_1 = \alpha + \beta x^2, \qquad x_2 = \alpha x + \frac{\beta}{3} x^3, \qquad x_3 = -2\beta x, \qquad x_4 = -2\beta,$$

where α and β are arbitrary constants. Hence the equation of the surface is

$$x_3^3 + 3x_4 (x_1 x_3 - x_2 x_4) = 0,$$

which is that of a Cayley cubic scroll. If $q'' \neq 0$, equation (8a) gives on integration

$$\left(\frac{dq}{dx}\right)^2 = -\frac{4}{15}(4q^3 - g_3),$$

where g_3 is a constant. As before we construct the Weierstrass $\wp$-function satisfying this equation; so that equations (9) may be written

$$\frac{d^2 y}{du^2} + A\wp y = 0, \qquad \frac{d^2 z}{du^2} + A(\wp z + y) = 0, \tag{9a}$$

where A is a constant but not of the form $n(n+1)$.

Therefore *every ruled surface whose asymptotic curves are twisted cubics may be represented by a system of equations of the form* (7a) *or* (9a), *where* $\wp$ *is the Weierstrass elliptic function. If the coefficients of these equations reduce to constants, we obtain either a certain quartic or a Cayley cubic scroll.*

§ 2. Analytic criteria for non-ruled surfaces whose asymptotic curves belong to linear complexes

It is known that any non-developable surface S may be regarded as an integral surface of a non-involutory, completely integrable system of partial differential equations of the form*

$$y_{uu} + 2by_v + fy = 0, \qquad y_{vv} + 2a' y_u + gy = 0, \tag{10}$$

where the asymptotic curves are parametric, and where

$$y_u = \frac{\partial y}{\partial u}, \qquad y_{uu} = \frac{\partial^2 y}{\partial u^2}, \qquad \text{etc.}, \quad \cdots.$$

The integrability conditions of this system of equations are

$$\begin{gathered} a'_{uu} + g_u + 2ba'_v + 4a' b_v = 0, \qquad b_{vv} + f_v + 2a' b_u + 4ba'_u = 0, \\ g_{uu} - f_{vv} - 4fa'_u - 2a' f_u + 4gb_v + 2bg_v = 0. \end{gathered} \tag{11}$$

* M_1, p. 241.

The asymptotic curves $v =$ const. and $u =$ const. will be denoted by Γ' and Γ'' respectively.

The differential equations of Γ' and Γ'' are*

$$
\begin{aligned}
&y_{uuuu} + 4p_1' \, y_{uuu} + 6p_2' \, y_{uu} + 4p_3' \, y_u + p_4' \, y = 0, \\
&y_{vvvv} + 4p_1'' \, y_{vvv} + 6p_2'' \, y_{vv} + 4p_3'' \, y_v + p_4'' \, y = 0,
\end{aligned} \tag{12}
$$

where

$$
p_1' = -\frac{b_u}{2b}, \qquad p_2' = \frac{1}{3}(f + b_v) - \frac{1}{6}\left(\frac{b_{uu}}{b} - 2\frac{b_u^2}{b^2}\right),
$$

$$
p_3' = \frac{1}{2}\left(f_u + 4a' \, b^2 - f\frac{b_u}{b}\right),
$$

$$
p_4' = f_{uu} - f^2 + 4b^2 \, g - 2bf_v + \frac{f}{b}(2bf + 2bb_v - b_{uu}) - 2\frac{b_u}{b^2}(bf_u - b_u f).
$$

The fundamental seminvariants of Γ' have the values

$$
\begin{aligned}
P_2' &= p_2' - \frac{\partial p_1'}{\partial u} - p_1'^2, \\
P_3' &= p_3' - \frac{\partial^2 p_1'}{\partial u^2} - 3p_1' \, p_2' + 2p_1'^3, \\
P_4' &= p_4' - 4p_1' \, p_3' - 3p_2'^2 + 12p_1'^2 \, p_2' - 6p_1'^4 - \frac{\partial^3 p_1'}{\partial u^3}.
\end{aligned} \tag{13}
$$

Finally the fundamental invariants of Γ' are

$$
\theta_3' = P_3' - \frac{3}{2}\frac{\partial P_2'}{\partial u}, \qquad \theta_4' = P_4' - 2\frac{\partial \theta_3'}{\partial u} - \frac{9}{15}\left(3\frac{\partial^2 P_2'}{\partial u^2} + \frac{2}{5}P_2'^2\right). \tag{14}
$$

If we replace

$$
a', \; b, \; f, \; g, \; p_i', \; P_i', \; \theta', \; u, \; v
$$

by

$$
b, \; a', \; g, \; f, \; p_i'', \; P_i'', \; \theta'', \; v, \; u
$$

respectively, we obtain the corresponding expressions for Γ''.

In the subsequent applications of these equations we shall generally assume that $a' \not\equiv 0$, $b \not\equiv 0$, so that ruled surfaces are to be excluded unless the contrary is stated.

Let us consider the integral surfaces S for which the asymptotic curves Γ' and Γ'' belong to linear complexes. For these surfaces the invariants θ_3' and θ_3'' must vanish identically. By direct calculation from equations (12), (13), (14), these conditions are found to reduce to

* M_2, pp. 90, 93.

$$(15)\qquad \begin{aligned} \Omega' &\equiv 4a'b - \frac{\partial^2 \log a'}{\partial u\,\partial v} = 0, \\ \Omega'' &\equiv 4a'b - \frac{\partial^2 \log b}{\partial u\,\partial v} = 0, \end{aligned}$$

whence, by subtraction,

$$(16)\qquad \frac{\partial^2 \log (a'/b)}{\partial u\,\partial v} = 0.$$

Hence, by integration,

$$(17)\qquad b = \varphi(u)\,\theta(v)\,a'.$$

The transformation

$$(18)\qquad y = \lambda(u, v)\,\bar{y}, \qquad \bar{u} = \alpha(u), \qquad \bar{v} = \beta(v)$$

leaves the parametric curves of equations (10) unchanged. Moreover the function $\lambda(u, v)$ may be chosen in such a way that the transformed system of equations shall again have the same (canonical) form as the system (10). The transformed coefficients $\bar{a}'$, $\bar{b}$, $\bar{f}$, $\bar{g}$ will then be given by the equations

$$(19)\qquad \begin{aligned} \bar{a}' &= \frac{\alpha_u}{\beta_v^2}\,a', \qquad & \bar{f} &= \frac{1}{\alpha_u^2}(f - b\zeta - \tfrac{1}{2}\mu), \\ \bar{b} &= \frac{\beta_v}{\alpha_u^2}\,b, \qquad & g &= \frac{1}{\beta_v^2}(g - a'\eta - \tfrac{1}{2}\nu), \end{aligned}$$

where

$$\eta = \frac{\alpha_{uu}}{\alpha_u}, \qquad \zeta = \frac{\beta_{vv}}{\beta_v}, \qquad \mu = \eta_u - \tfrac{1}{2}\eta^2, \qquad \nu = \zeta_v - \zeta^2.*$$

In particular, the transformation

$$y = \frac{1}{\sqrt{\dfrac{d\bar{u}}{du}\cdot\dfrac{d\bar{v}}{dv}}}\cdot\bar{y}, \qquad \bar{u} = \int \sqrt[3]{\varphi(u)}\,du, \qquad \bar{v} = \int \frac{1}{\sqrt[3]{\theta(v)}}\,dv,$$

will replace equations (10) by a system of the same form in which the fundamental invariants a' and b are equal to each other. Let us assume that this transformation has been effected already, so that we have

$$a' = b.$$

The conditions

$$\Omega' = \Omega'' = 0$$

will then reduce to

$$(20)\qquad \frac{\partial^2 \log b}{\partial u\,\partial v} = 4b^2.$$

If we put $b = e^{\phi/2}$, the above equation becomes

* M_1, pp. 249–250.

$$\frac{\partial^2 \phi}{\partial u \, \partial v} = 8e^{\phi}. \tag{20a}$$

But (20a) is the form known as Liouville's equation.* Its general integral is

$$e^{\phi} = \frac{1}{4} \frac{U' V'}{(U + V)^2},$$

where U and V are functions of the single variables u and v respectively, and where the accents indicate differentiation with respect to these variables. Hence the general integral of (20) is

$$a' = b = \frac{1}{2} \frac{\sqrt{U' V'}}{(U + V)}, \tag{21}$$

where we assume $U' \not\equiv 0$, $V' \not\equiv 0$ when we wish to exclude ruled surfaces. We shall refer to equations (10) as being in the *normal form*† when the invariants a' and b are given by (21). We therefore have the theorem:

Both families of asymptotic curves of a non-ruled surface belong to linear complexes, if and only if, the differential equations of the surface can be put in the normal form characterized by the values (21) *for the invariants a' and b.*‡

§ 3. Properties of the osculating ruled surfaces, and explicit determination of the differential equations of the surfaces of § 2

The differential equations of R_1 (one of the osculating ruled surfaces of the first kind) are§

$$\begin{aligned} y_{vv} + p_{11} y_v + p_{12} z_v + q_{11} y + q_{12} z &= 0, \\ z_{vv} + p_{21} y_v + p_{22} z_v + q_{21} y + q_{22} z &= 0, \end{aligned} \tag{22}$$

where $z = y_u$ and

$$p_{11} = p_{12} = p_{22} = 0, \qquad q_{11} = g, \qquad q_{12} = 2a',$$

$$p_{21} = -4a' b, \qquad q_{21} = g_u - 2a' f, \qquad q_{22} = g + 2a'_u.$$

Since p_{12} is equal to zero, Γ'' is an asymptotic curve on R_1 as well as on the surface S.

The invariants of weights four and nine of R_1 are

$$\theta = (u_{11} - u_{22})^2 + 4u_{12} u_{21}, \tag{23}$$

and

$$\theta_9 = \begin{vmatrix} u_{11} - u_{22} & u_{12} & u_{21} \\ v_{11} - v_{22} & v_{12} & v_{21} \\ w_{11} - w_{22} & w_{12} & w_{21} \end{vmatrix},$$

* Darboux *Leçons sur la Théorie des Surfaces*, t. IV, pp. 419–424.

† The complete *normal form* will be given later.

‡ The invariants Ω' and Ω'' vanish for these surfaces only.

§ M_2, p. 81 et seq.

where

$$u_{11} - u_{22} = 8a'_u, \qquad u_{12} = -8a', \qquad u_{21} = 4a'_{uu} + 8a'(f + b_v),$$

$$v_{11} - v_{22} = 16(a'_{uv} - 4a'^2 b), \qquad v_{12} = -16a'_v,$$

$$v_{21} = 8(a'_{uuv} - 4a' a'_u b + 2a'_v f + 2a' f_v + 2a'_v b_v + 2a' b_{vv})$$

$$= 8[a'_{uuv} + 2a'_v(f + b_v) - 4a'^2 b_u - 12a' ba'_u],$$

$$w_{11} - w_{22} = 32(a'_{uvv} - 4a'^2 b_v - 12a' ba'_v), \qquad w_{12} = -32a'_{vv},$$

$$w_{21} = 16[a'_{uuvv} + 2a'_{vv}(f + b_v) - 4a'^2 b_{uv} - 12a'(a'_v b_u + a'_u b_v)$$

$$- 16a'(ba'_{uv} - a'^2 b^2) - 20a'_u a'_v b],$$

the final forms of u_{21}, v_{21}, w_{21} resulting from an application of the integrability conditions (11).

Analogous equations may be obtained for R_2 (an osculating ruled surface of the second kind).

If we differentiate the conditions $\Omega' = 0$, $\Omega'' = 0$, we obtain the following equations:

$$(24)\qquad \begin{aligned}
&a'a'_{uuv} - a'_v a'_{uu} - 12a'^2 ba'_u - 4a'^3 b_u = 0,\\
&a'a'_{uvv} - a'_u a'_{vv} - 12a'^2 ba'_v - 4a'^3 b_v = 0,\\
&b b_{uuv} - b_v b_{uu} - 12a' b^2 b_u - 4b^3 a'_u = 0,\\
&b b_{uvv} - b_u b_{vv} - 12a' b^2 b_v - 4b^3 a'_v = 0,\\
&a'a'_{uuvv} - a'_{uu} a'_{vv} - 24a' ba'_u a'_v - 12a'^2(a'_u b_v + a'_v b_u)\\
&\qquad - 12a'^2 ba'_{uv} - 4a'^3 b_{uv} = 0,\\
&bb_{uuvv} - b_{uu} b_{vv} - 24a' bb_u b_v - 12b^2(a'_u b_v + a'_v b_u)\\
&\qquad - 12a' b^2 b_{uv} - 4b^3 a'_{uv} = 0,
\end{aligned}$$

which enable us to reduce the elements of θ_9 to

$$u_{11} - u_{22} = 8a'_u, \qquad u_{12} = -8a', \qquad u_{21} = 4[a'_{uu} + 2a'(f + b_v)],$$

$$v_{11} - v_{22} = 16\frac{a'_u a'_v}{a'}, \quad v_{12} = -16a_v, \quad v_{21} = \frac{8a'_v}{a'}[a'_{uu} + 2a'(f + b_v)],$$

$$w_{11} - w_{22} = 32\frac{a'_u a'_{vv}}{a'}, \quad w_{12} = -32a'_{vv}, \quad w_{21} = \frac{16a'_{vv}}{a'}[a'_{uu} + 2a'(f + b_v)].$$

If we substitute these simplified expressions in θ_9, we find that θ_9 and all of its second order minors vanish. In the same way we find that θ'_9, the invariant of

weight nine for R_2, and all of its second order minors vanish. Consequently we have the theorem:

*If both families of asymptotic curves of a surface S belong to linear complexes, then both families of osculating ruled surfaces R_1 and R_2 have straight line directrices, distinct or coincident according as the corresponding invariant θ or θ' is different from zero or not.**

We shall need several other formulæ connected with the theory of the osculating ruled surfaces. In developing these formulæ we shall indicate another proof of the above theorem.

If P_η and P_ζ are the flecnodes on the generator $P_y P_z$ of R_1, we have†

$$y = 16a'(\eta + \zeta), \qquad z = (8a'_u + \sqrt{\theta})\,\eta + (8a'_u - \sqrt{\theta})\,\zeta.$$

When we make this transformation the equations of R_1, referred to its flecnode curves, are found to be

$$(25)\qquad \begin{aligned} &\eta_{vv} + \pi_{11}\,\eta_v + \pi_{12}\,\zeta_v + \kappa_{11}\,\eta + \kappa_{12}\,\zeta = 0,\\ &\zeta_{vv} + \pi_{21}\,\eta_v + \pi_{22}\,\zeta_v + \kappa_{21}\,\eta + \kappa_{22}\,\zeta = 0, \end{aligned}$$

where

$$\pi_{11} = \frac{a'_v}{a'} + \frac{C}{\sqrt{\theta}} + \frac{\theta_v}{2\theta}, \qquad \pi_{12} = \frac{a'_v}{a'} + \frac{C}{\sqrt{\theta}} - \frac{\theta_v}{2\theta},$$

$$\pi_{21} = \frac{a'_v}{a'} - \frac{C}{\sqrt{\theta}} - \frac{\theta_v}{2\theta}, \qquad \pi_{22} = \frac{a'_v}{a'} - \frac{C}{\sqrt{\theta}} + \frac{\theta_v}{2\theta},$$

$$C = 8a'_{uv} - 8\frac{a'_u\,a'_v}{a'} - 32a'^2\,b.$$

But since we are dealing with the case when $\Omega' = 0$, we find $C = 0$. Consequently

$$\pi_{12} = \pi_{21} = \frac{a'_v}{a'} - \frac{\theta_v}{2\theta} = \frac{2^5}{a'\theta}\,[a'(a'\,a'_{uuv} - a'_{uu}\,a'_v - 12a'^2\,ba'_u - 4a'^3\,b) + a'_u(a'_u\,a'_v - a'\,a'_{uv} + 4a'^3\,b)] = 0,$$

$$\pi_{11} = \pi_{22} = 2\frac{a'_v}{a'},$$

and therefore

$$\theta = \varphi(u)\,a'^2,$$

where φ is an arbitrary function of u. We find further

* Several months after this paper was presented to the American Mathematical Society, a memoir containing this theorem was published by Enrico Bompiani, Rendiconti del Circolo Matematico di Palermo, vol. 34 (1912).

† M_2, p. 83 et seq.

$$\kappa_{11} = \frac{1}{\Delta}\left[-4a'\theta - 32ga'\sqrt{\theta} - 32a'a_u'\sqrt{\theta} - \frac{8}{\sqrt{\theta}}\left(2a_{vv}'\theta + a'\theta_{vv} - \frac{a'\theta_v^2}{2\theta}\right)\right],$$

$$\kappa_{12} = \frac{-8}{\Delta\sqrt{\theta}}\left[2a_{vv}'\theta + a'\theta_{vv} - \frac{a'\theta_v}{2\theta}\right],^* \qquad \text{etc.,}$$

which reduce to

$$\kappa_{11} = a_u' + g + \frac{\sqrt{\theta}}{8}, \qquad \kappa_{12} = 0, \qquad \kappa_{21} = 0, \qquad \kappa_{22} = a_u' + g - \frac{\sqrt{\theta}}{8}.$$

The equations of the osculating ruled surface R_1 may therefore be reduced to the form

$$(25a) \qquad \begin{aligned} &\eta_{vv} + \frac{\theta_v}{\theta}\eta_v + \left(a_u' + g + \frac{\sqrt{\theta}}{8}\right)\eta = 0, \\ &\zeta_{vv} + \frac{\theta_v}{\theta}\zeta_v + \left(a_u' + g - \frac{\sqrt{\theta}}{8}\right)\zeta = 0, \end{aligned}$$

where $\theta / a'^2 = \varphi(u)$, a function of u alone.

The corresponding equations for R_2 may be obtained from the above by replacing θ, a', g by θ', b, f respectively, and interchanging u and v. Combining these results we have the following theorem:

If the invariants θ and θ' are different from zero, and if both families of asymptotic curves on S belong to linear complexes, the invariants θ / a'^2 and θ' / b^2† are functions of the single variables u and v respectively. The equations of the osculating ruled surfaces may be reduced to the form (25a), *which makes evident the already established theorem that the osculating ruled surfaces have straight line directrices.*

Suppose $\theta = 0$. Let P_ζ coincide with the double flecnode on $P_y P_z$. We make the transformation

$$y = \eta, \qquad z = \frac{a_u'}{2a'}\eta - \zeta,‡$$

so that the equations of R_1 referred to Γ'' and its flecnode curve are§

$$(26) \qquad \begin{aligned} &\eta_{vv} + \frac{b_v^2 - 2bb_{vv}}{4b^2}\eta - 2a'\zeta = 0, \\ &\zeta_{vv} + \frac{b_v^2 - 2bb_{vv}}{4b^2}\zeta = 0, \end{aligned}$$

which are characteristic of ruled surfaces with coincident directrices.

* In order to obtain these results we must use relations (24), (11) and the relation

$$2a_{vv}'\theta + a'\theta_{vv} - a_v'\theta_v = 0$$

obtained by differentiating $\pi_{12} = \pi_{21} = 0$ with respect to v.

† $\theta' / b^2 = \psi(v)$, ψ being an arbitrary function of v.

‡ M_2, p. 86.

§ The reductions are effected by means of (24), (11), and the condition $\theta = 0$. The equations of R_2 referred to Γ' and its flecnode curve have a similar form when $\theta' = 0$.

We can now complete the determination of the *normal form* of the equations of S. We have

$$a' = b, \qquad \frac{\partial^2 \log b}{\partial u \partial v} = 4b^2,$$

and the integrability conditions (11), which are reducible to

$$b_{uu} + g_u + 6bb_v = 0, \qquad b_{vv} + f_v + 6bb_u = 0,$$

$$g_{uu} - f_{vv} - 4fb_u - 2bf_u + 4gb_v + 2bg_v = 0.$$

We have further (from the previous theorem and the definition of θ and θ'),

$$\theta = 64\,[b_u^2 - 2b\,(b_{uu} + 2b \cdot \overline{f + b_v})] = \varphi(u)\,b^2,$$

$$\theta' = 64\,[b_v^2 - 2b\,(b_{vv} + 2b \cdot \overline{g + b_u})] = \psi(v)\,b^2.$$

Whence

$$f = -b_v - \frac{1}{2}\frac{\partial^2 \log b}{\partial u^2} - \frac{1}{4}\frac{b_u^2}{b^2} + \varphi_1(u),$$

$$g = -b_u - \frac{1}{2}\frac{\partial^2 \log b}{\partial v^2} - \frac{1}{4}\frac{b_v^2}{b^2} + \varphi_2(v),$$

where

$$\varphi_1(u) = -\frac{\varphi(u)}{256}, \qquad \varphi_2(v) = -\frac{\psi(v)}{256}.$$

Substituting these values of f and g in the integrability conditions, we find that the first two are satisfied identically and the third becomes

$$-4b_u\,\varphi_1(u) + 4b_v\,\varphi_2(v) - 2b\,\varphi_1'(u) + 2b\,\varphi_2'(v) = 0.$$

Now b is given by equation (21), so that

$$\frac{b_u}{b} = \frac{1}{2}\frac{U''}{U'} - \frac{U'}{U+V}, \qquad \frac{b_v}{b} = \frac{1}{2}\frac{V''}{V'} - \frac{V'}{U+V}.$$

We must therefore have

$$2\frac{\varphi_1(u)\,U' - \varphi_2(v)\,V'}{U+V} + \varphi_2(v)\frac{V''}{V'} - \varphi_1(u)\frac{U''}{U'} = \varphi_1'(u) - \varphi_2'(v).$$

Thus the left hand member of this equation reduces to a function of u minus a function of v. Now U' and V' can not vanish, for then b also would vanish. Hence U and V are not constants, and we must have

$$\frac{\varphi_1(u)\,U' - \varphi_2(v)\,V'}{U+V} = H_1(u) - H_2(v)$$

or

$$(\alpha) \quad \varphi_1(u)\,U' - \varphi_2(v)\,V' = UH_1(u) - VH_2(v) + VH_1(u) - UH_2(v),$$

where φ_i, H_i are functions to be determined. These must be such that $VH_1(u) - UH_2(v)$ is of the form $G_1(u) - G_2(v)$, that is we must have

$$\frac{\partial^2}{\partial u \partial v}(VH_1(u) - UH_2(v)) = 0,$$

whence

$$\frac{1}{U'}\frac{dH_1}{du} = \frac{1}{V'}\frac{dH_2}{dv}.$$

Since the left hand member is a function of u alone and the right hand member is a function of v alone, their equality implies

$$\frac{1}{U'}\frac{dH_1}{du} = k_0, \quad \frac{1}{V'}\frac{dH_2}{dv} = k_0,$$

where k_0 is a constant. Therefore

$$H_1(u) = k_0 U + k_1, \qquad H_2(v) = k_0 V + k_2,$$

where k_1 and k_2 are additional constants. If these values for H_1 and H_2 be substituted in (α), we find

$$\varphi_1(u) = \frac{aU^2 + bU + c}{U'}, \qquad \varphi_2(v) = \frac{aV^2 - bV + c}{V'},$$

where $a = k_0$, $b = k_1 - k_2$, and c is a further constant. If we substitute these values of $\varphi_1(u)$ and $\varphi_2(v)$ in the expressions for f and g, we shall find

$$(27)\quad \begin{aligned} f = &-\frac{1}{2}\frac{\sqrt{U'V'}}{(U+V)}\left(\frac{V''}{2V'} - \frac{V'}{(U+V)}\right) - \frac{1}{4}\left(\frac{U''}{U'} - \frac{3}{4}\left(\frac{U''}{U'}\right)^2\right) \\ &+ \frac{3}{4}\left(\frac{U''}{U+V} - \frac{U'^2}{(U+V)^2}\right) + \frac{aU^2 + bU + c}{U'}, \\ g = &-\frac{1}{2}\frac{\sqrt{U'V'}}{(U+V)}\left(\frac{U''}{2U'} - \frac{U'}{(U+V)}\right) - \frac{1}{4}\left(\frac{V''}{V'} - \frac{3}{4}\left(\frac{V''}{V'}\right)^2\right) \\ &+ \frac{3}{4}\left(\frac{V''}{U+V} - \frac{V'^2}{(U+V)^2}\right) + \frac{aV^2 - bV + c}{V'}. \end{aligned}$$

If the osculating ruled surfaces are characterized by equations of the form (26), it is readily seen that $a = b = c = 0$, in the above equations. A discussion of some of the geometrical configurations corresponding to this case will be given below. We may recapitulate these results in the following theorem:

Every non-ruled surface all of whose asymptotic curves belong to linear complexes is defined by a system of equations of the form (10), *where the fundamental invariants a', b and seminvariants f, g are given explicitly as functions of u and v*

by equations (21) *and* (27), *and conversely. Since all the invariants of* S *are functions of* a', b, f, g *and their derivatives, they can be determined as explicit functions of* u *and* v.

§ 4. Properties of the linear complexes of the asymptotic curves, and a geometric construction for the surfaces of the problem

The directrices δ_1 and δ_2 of the surface R_1 are found to have the equations*

$$(28)\qquad \begin{aligned} 2^4 a' x_1 + Ax_2 + 2^5 a'^2 bx_4 &= 0, \\ 2^4 a' x_3 + Ax_4 &= 0, \end{aligned}$$

and

$$\begin{aligned} 2^4 a' x_1 + Bx_2 + 2^5 a'^2 bx_4 &= 0, \\ 2^4 a' x_3 + Bx_4 &= 0, \end{aligned}$$

where

$$A = 8a'_u - \sqrt{\theta}, \qquad B = 8a'_u + \sqrt{\theta}.$$

The equations of the directrices δ'_1 and δ'_2 of R_2 are found to be

$$(28a)\qquad \begin{aligned} 2^4 bx_1 + A_1 x_3 + 2^5 a' b^2 x_4 &= 0, \\ 2^4 bx_2 + A_1 x_4 &= 0, \end{aligned}$$

and

$$\begin{aligned} 2^4 bx_1 + B_1 x_3 + 2^5 a' b^2 x_4 &= 0, \\ 2^4 bx_2 + B_1 x_4 &= 0, \end{aligned}$$

where

$$A_1 = 8b_v - \sqrt{\theta'}, \qquad B_1 = 8b_v + \sqrt{\theta'}.$$

These equations are of fundamental importance in the succeeding investigation.

It is clear that, when Ω' and Ω'' vanish, the osculating linear complexes of the curves Γ' and Γ'' coincide with the linear complexes which contain all of the tangents to these curves. Hence, referred to the moving tetraedron $P_y P_z P_\rho P_\sigma$, the linear complexes determined by Γ' and Γ'' are given by the equations†

$$(29)\qquad \begin{aligned} &(a)\quad C' \equiv -b_v\,\omega_{34} - b\omega_{14} + b\omega_{23} = 0, \\ &(b)\quad C'' \equiv -a'_u\,\omega_{42} + a'\,\omega_{14} + a'\,\omega_{23} = 0. \end{aligned}$$

The invariants of C' and C'' are $-b^2$ and a'^2 respectively.

We shall now determine the linear complexes to which belong the asymptotic curves through the neighbouring points $P_{u+du,\,v}$ and $P_{u,\,v+dv}$. Their

* $P_y P_z P_\rho P_\sigma$ is the tetrahedron of reference. In this system the coördinates of the point $P \equiv \alpha y + \beta z + \gamma\rho + \delta\sigma$ are, by definition, $(\alpha, \beta, \gamma, \delta)$.

† M_2, pp. 92–94.

equations will, of course, have the same form as (29), the tetrahedra of reference being those associated with the points considered. Now when u receives an increment du, v remaining constant, the semi-covariants y, z, ρ, σ change. Let $\bar{y}$, $\bar{z}$, $\bar{\rho}$, $\bar{\sigma}$ be their new values.* We shall find

$$
(30)\qquad
\begin{aligned}
\bar{y} &= y + z\,du + \cdots,\\
\bar{z} &= z - (fy + 2b\rho)\,du + \cdots,\\
\bar{\rho} &= \rho + \sigma\,du + \cdots,\\
\bar{\sigma} &= \sigma + [(2bg - f_v)\,y + 4a'\,bz - (2b_v + f)\,\rho]\,du + \cdots.
\end{aligned}
$$

Let the coördinates of any point Q referred to T and $\bar{T}$ be (x_1, x_2, x_3, x_4) and $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)$ respectively. Then the point Q is given by each of the expressions

$$
x_1\,y + x_2\,z + x_3\,\rho + x_4\,\sigma, \qquad \bar{x}_1\,\bar{y} + \bar{x}_2\,\bar{z} + \bar{x}_3\,\bar{\rho} + \bar{x}_4\,\bar{\sigma},
$$

which must therefore be identical except for a factor. Making use of this fact and equations (30), we find (on neglecting du^2 and higher powers of du)

$$
(31)\qquad
\begin{aligned}
\omega x_1 &= \bar{x}_1 + (-f\bar{x}_2 + \alpha\bar{x}_4)\,du,\\
\omega x_2 &= \bar{x}_1\,du + \bar{x}_2 + 4a'\,b\bar{x}_4\,du,\\
\omega x_3 &= -2b\bar{x}_2\,du + \bar{x}_3 - \beta\bar{x}_4\,du,\\
\omega x_4 &= \bar{x}_3\,du + \bar{x}_4.
\end{aligned}
$$

Whence

$$
\begin{aligned}
\omega'\,\bar{x}_1 &= x_1 + (fx_2 - \alpha x_4)\,du,\\
\omega'\,\bar{x}_2 &= -x_1\,du + x_2 - 4a'\,bx_4\,du,\\
\omega'\,\bar{x}_3 &= 2bx_2\,du + x_3 + \beta x_4\,du,\\
\omega'\,\bar{x}_4 &= -x_3\,du + x_4,
\end{aligned}
$$

where ω and ω' are factors of proportionality and where

$$
\alpha = 2bg - f_v, \qquad \beta = 2b_v + f.
$$

If ω_{ij} and $\bar{\omega}_{ij}$ are the coördinates of a line referred to T and $\bar{T}$ respectively, we shall have

$$
\begin{aligned}
\bar{\omega}_{14} &= \omega_{14} + (-\omega_{13} - f\omega_{42})\,du,\\
\bar{\omega}_{23} &= \omega_{23} + (-\omega_{14} + 4a'\,b\omega_{34} - \beta\omega_{42})\,du,\\
\bar{\omega}_{42} &= \omega_{42} + (\omega_{14} + \omega_{23})\,du.
\end{aligned}
$$

* The tetrahedron $T \equiv P_y\,P_z\,P_\rho\,P_\sigma$ is displaced to $\bar{T} \equiv P_{\bar{y}}\,P_{\bar{z}}\,P_{\bar{\rho}}\,P_{\bar{\sigma}}$.

The linear complex C''_{u+du}, containing the asymptotic curve Γ''_{u+du}, is given by the equation

$$(32) \qquad -\bar{a}'_u\,\bar{\omega}_{42} + \bar{a}'\,(\bar{\omega}_{14} + \bar{\omega}_{23}) = 0,$$

referred to $\bar{T}$, where

$$\bar{a}' = a' + a'_u\,du + \cdots,$$

$$\bar{a}'_u = a'_u + a'_{uu}\,du + \cdots.$$

Therefore the equation of C''_{u+du}, referred to T, is

$$(32a) \qquad \begin{aligned}&(-a'_u\,\omega_{42} + a'\,\omega_{14} + a'\,\omega_{23})\\ &\qquad + (-2a'\,\omega_{13} + 4a'^2\,b\omega_{34} + \delta\omega_{42})\,du = 0,\end{aligned}$$

where

$$\delta = -a'_{uu} - 2a\,(f + b_v) = \frac{\theta - 2^6\,a'^2_u}{2^7\,a'}.$$

We shall need the invariant $\bar{I}''$ of the complex

$$-2a'\,\omega_{13} + 4a'^2\,b\omega_{34} + \delta\omega_{42} = 0,$$

and the mutual invariant $(I'', \bar{I}'')$ of this complex and C''_u. We find

$$\bar{I}'' = -2a'\,\delta, \qquad (I'', \bar{I}'') = 2a'\,a'_u.$$

The complexes C''_u an dC''_{u+du} determine the pencil

$$(33) \qquad \lambda\,(-a'_u\,\omega_{42} + a'\omega_{14} + a'\,\omega_{23}) + \mu\,(-2a'\,\omega_{13} + 4a'^2\,b\omega_{34} + \delta\omega_{42}) = 0.$$

The special complexes of this pencil are given by the values of λ and μ which satisfy the equation

$$\lambda^2\,I'' + \lambda\mu\,(I'', \bar{I}'') + \mu^2\,\bar{I}'' = 0.$$

Hence

$$\frac{\lambda}{\mu} = \frac{-a_u \pm \dfrac{\sqrt{\theta}}{2^3}}{a'}.$$

Introducing these values of λ/μ into equation (33), we find the line coördinates of the directrices of the special complexes of the pencil to be

$$(34) \qquad \begin{aligned}&\omega_{12} = 4a'^3\,b, \qquad \omega_{13} = a'^2_u \mp \frac{a'_u\sqrt{\theta}}{2^3} + a'\,\delta, \qquad \omega_{14} = a'\left(-a'_u \pm \frac{\sqrt{\theta}}{2^3}\right),\\ &\omega_{23} = a'\left(-a'_u \pm \frac{\sqrt{\theta}}{2^3}\right), \qquad \omega_{34} = 0, \qquad \omega_{42} = -2a'^2.\end{aligned}$$

Their point equations are

$$(34a) \qquad \begin{aligned}&2^4\,a'\,x_1 + (8a'_u \mp \sqrt{\theta})\,x_2 + 2^5\,a'^2\,bx_4 = 0,\\ &2^4\,a'\,x_3 + (8a'_u \mp \sqrt{\theta})\,x_4 = 0.\end{aligned}$$

If we substitute the coördinates (34) in equation (29a), we find that the directrices of the congruence determined by the complexes C''_u and C'_{u+du} (i. e., according to equation (28) the directrices of R_1) belong to the complex C'_v. Similarly we prove that the directrices of R_2 belong to C''_u. Hence the theorem:

The directrices of the osculating ruled surfaces R_1 are contained in the complexes C'_v determined by Γ', and the directrices of R_2 are contained in the complexes C''_u determined by Γ''.

It is of interest to obtain this result in another way. Let H be the common osculating hyperboloid of R_1 and R_2. Its equation is

$$x_1 x_4 - x_2 x_3 + 2a' b x_4^2 = 0.^* \tag{35}$$

The lines (34a) and the corresponding lines of R_2 are easily seen to be on H. The complex C''_u contains three generators of the first kind on H and therefore all the generators of this set. The complex C'_v contains three and therefore all the generators of the second set on H. The directrices (δ_1, δ_2) and (δ'_1, δ'_2) are among these generators, as we have just seen. This proves the theorem.

Let v remain fixed and let u vary. Then R_2 remains fixed as P_y describes the asymptotic curve Γ'_v on S. Hence the directrices (δ'_1, δ'_2) of R_2 belong to all of the complexes C''_u. But R_2 is any surface of the one parameter family of osculating ruled surfaces of the second kind. Hence all the directrices of R_2 are generators of a ruled surface S_2 contained in all of the complexes C''_u of S. Similarly the directrices of R_1 are generators of a ruled surface S_1 contained in all of the complexes C'_v of S. We shall now prove that S_1 is identical with S_2 and therefore a quadric, the two sets of directrices being complementary reguli.

Since the four directrices (δ_1, δ_2) and (δ'_1, δ'_2) of the osculating ruled surfaces R_1 and R_2 determined by Γ' and Γ'' are situated upon the osculating hyperboloid H of the point P_y, and since this is true for every point of Γ' when v is fixed, and for every point of Γ'' when u is fixed, it follows that all the directrices of R_1 intersect all the directrices of R_2. Therefore the generators of S_1 and S_2 constitute complementary reguli on the same quadric surface. We shall call this surface the *directrix quadric* and denote it, for brevity, by the letter Q.

If the integrating surface S is not ruled, all the complexes C' can not be those of a pencil of complexes. For, if all the complexes C' were contained in a pencil, the congruence determined by C'_v and C'_{v+dv} would have the same pair of directrices (δ'_1, δ'_2) for every v. Therefore all the osculating ruled surfaces

* M_2, p. 82.

of one family would belong to the same linear congruence, viz., the one having (δ_1', δ_2') as directrices. But this implies that S itself is a ruled surface of the congruence. Similarly we show that the complexes C'' are not all contained in a pencil.

Therefore *if all the linear complexes of the asymptotic curves of either family belong to a pencil, the surface must be ruled.*

If Γ' or Γ'' belong to a one parameter family of linear complexes not contained in a net, then these complexes could not intersect in more than two lines.* Therefore, there could not be more than one pair of directrices for the osculating ruled surfaces of the other family. But by the above theorem the surface S would, in this case, be ruled. We therefore have the theorem:

If both families of asymptotic curves of a non-ruled surface S belong to linear complexes, these complexes must form two one-parameter families belonging to two involutory nets.†

This theorem, together with one which we shall now prove, leads to an important result, namely, *a geometric construction for the surfaces having the property in question.*

The osculating hyperboloid $H_{u+du,\,v}$ of the surfaces R_1 and R_2 determined by the asymptotic curves through the point $P_{u+du,\,v}$ has the equation

$$\bar{x}_1\,\bar{x}_4 - \bar{x}_2\,\bar{x}_3 + 2\bar{a}'\,\bar{b}\bar{x}_4^2 = 0,$$

referred to $\bar{T}$. This becomes, when referred to T,

$$(x_1\,x_4 - x_2\,x_3 + 2a'\,bx_4^2) + (-\,2bx_2^2 - 2b_v\,x_2\,x_4 + \gamma x_4^2)\,du = 0,$$

where

$$\gamma = 2\,[\,(a'\,b)_u - bg\,].$$

Now $H_{u,\,v}$ and $H_{u+du,\,v}$ intersect along two generators of R_2 consecutive to $P_y\,P_\rho$, and two generators of the second set upon H in the planes

$$2^4\,bx_2 - (8b_v \pm \sqrt{\theta'})\,x_4 = 0\ddagger. \tag{36}$$

The tangent plane to $H_{u,\,v}$ at the point $(y_1, 0, y_3, 0)$ is

$$x_2\,y_3 - x_4\,y_1 = 0. \tag{36a}$$

Therefore the generators of intersection of $H_{u,\,v}$ and $H_{u+du,\,v}$, apart from those consecutive to $P_y\,P_\rho$, pass through the points

$$\omega\eta = (8b_v - \sqrt{\theta'})\,y - 16b\rho, \qquad \omega\zeta = -\,(8b_v + \sqrt{\theta'})\,y + 16b\rho. \tag{36b}$$

These two generators are the flecnode tangents to R_2 at the flecnodes on $P_y\,P_\rho$.

* The complexes of a hypernet have, at most, two lines in common.

† The mutual invariant (I'', I') vanishes and therefore the complexes are in involution.

‡ This follows from the definition of θ' and the second integrability condition.

Let us now assume that the asymptotic curves Γ'' belong to a one parameter family of linear complexes contained in a net, but not contained in a pencil. The complexes of this net have in common a regulus ρ'' of a quadric Q, as we have seen. The directrices $(\delta_1(u, v), \delta_2(u, v))$ of R_1 are situated upon the complementary regulus ρ' of Q, and also upon the osculating hyperboloid $H_{u,v}$. Hence Q and $H_{u,v}$ intersect in another pair of generators $(g_1(u), g_2(u))$ of Q and $H_{u,v}$ which belong to the regulus ρ''. Again, the directrices $(\delta_1(u+du, v), \delta_2(u+du, v))$ of $R_{1_{u+du,v}}$ are on Q and belong to ρ'; hence Q and $H_{u+du,v}$ intersect in another pair of generators $(g_1(u+du), g_2(u+du))$ which belong to ρ''. Now the generators, not consecutive to $P_y P_\rho$, common to $H_{u,v}$ and $H_{u+du,v}$ are given by equations (36), (36*a*), (36*b*). These being on $H_{u,v}$ and $H_{u+du,v}$ must intersect the four generators $(\delta_1(u, v), \delta_2(u, v))$ and $(\delta_1(u+du, v), \delta_2(u+du, v))$ of Q and are, therefore, generators of the regulus ρ''. If we call these lines (l_1, l_2), it follows that the pairs $(g_1(u), g_2(u))$ and $(g_1(u+du), g_2(u+du))$ both coincide with the pair (l_1, l_2). For otherwise the total intersection of Q and $H_{u,v}$ would be of higher order than the fourth, i. e., Q and $H_{u,v}$ would coincide and S would be itself a quadric surface. In precisely the same way we see that $H_{u+du,v}$, etc., $\cdots$, intersect in two generators belonging to ρ'' of Q. These must, therefore, be (l_1, l_2). Proceeding in this manner we find that all of the osculating hyperboloids of $R_2(v_0)$ (say) have in common a fixed pair of lines of the regulus ρ'' of Q, viz., the pair (l_1, l_2), and therefore R_2 has a pair of straight line directrices* $(\delta_1(u, v_0), \delta_2(u, v_0))$. Hence all of its asymptotic curves belong to linear complexes. In particular Γ' belongs to a linear complex, and this is true for every value of v. We therefore have the theorem:

If the asymptotic curves of one family of a non-ruled surface belong to a one-parameter family of linear complexes contained in a net, then the asymptotic curves of the second family must also belong to a one-parameter family of linear complexes contained in an involutory net.

To construct the surfaces having the property in question we may now proceed as follows: Select a one-parameter family of pairs of lines $(\delta_1(u), \delta_2(u))$ on Q. Each of these pairs of lines determines a pencil of linear complexes, $(\delta_1(u), \delta_2(u))$ being the axes of the special complexes of this pencil. In each of these pencils pick out (according to some arbitrary law) one complex $C(u)$. We thus obtain a one-parameter family of complexes contained in a net. Every line which intersects $(\delta_1(u), \delta_2(u))$ belongs to $C(u)$. Let us begin with $u = u_0$. Let $R_1(u_0)$ be an arbitrary ruled surface having $(\delta_1(u_0), \delta_2(u_0))$ as directrices. $R_1(u_0)$ will then belong to $C(u_0)$, and one of its asymptotic curves $\Gamma(u_0)$ intersecting every generator twice will likewise belong to

* $(\delta_1(u, v_0), \delta_2(u, v_0))$ are the lines (l_1, l_2).

the complex $C(u_0)$. Let $g_{u_0}^{(0)}$, $g_{u_0}^{(1)}$, $g_{u_0}^{(2)}$, etc., be the generators of $R_1(u_0)$ corresponding to $v = v_0$, $v_0 + dv$, $v_0 + 2dv$, etc., and intersecting $\Gamma(u_0)$ in the points A_0, A_1, A_2, etc.

Now let $(\delta_1(u_0 + du), \delta_2(u_0 + du))$ be the pair of lines of Q which correspond to $u = u_0 + du$, and $C(u_0 + du)$ the corresponding linear complex. Let P_0 be a point of $g_{u_0}^{(0)}$ ultimately to be allowed to approach the corresponding point A_0 of $\Gamma(u_0)$ as its limit. The line $g_{u_0+du}^{(0)}$ through P_0 which intersects $(\delta_1(u_0 + du), \delta_2(u_0 + du))$ belongs to the complex $C(u_0 + du)$. Hence the null-plane of P_0 in the complex $C(u_0 + du)$ contains $g_{u_0+du}^{(0)}$. Let its intersection with $g_{u_0}^{(1)}$ be called P_1. Then $P_0 P_1$ also belongs to $C(u_0 + du)$. Denote by $g_{u_0+du}^{(1)}$ the line through P_1 which intersects $(\delta_1(u_0 + du), \delta_2(u_0 + du))$, and by P_2 the point where $g_{u_0}^{(2)}$ is cut by the null-plane of P_1 in the null-system of $C(u_0 + du)$. Then $g_{u_0+du}^{(1)}$ and $P_1 P_2$ are lines of the complex $C(u_0 + du)$. As we continue this process and proceed to the limit (i. e., $dv \doteq 0$) we obtain a curve on $R_1(u_0 + du)$ whose tangents obviously belong to the complex $C(u_0 + du)$. Moreover, the null-planes of the points of this curve in the null-system of the complex $C(u_0 + du)$ are clearly tangent to the surface $R_1(u_0 + du)$. But for any curve of a linear complex the osculating plane at any point of the curve coincides with its null-plane. Therefore the curve $R_1 R_2 R_3 \cdots$ is an asymptotic curve of $R_1(u_0 + du)$.

By starting anew with the lines $(\delta_1(u_0 + 2du), \delta_2(u_0 + 2du))$ and the surface $R_1(u_0 + du)$, etc., we can construct a new ruled surface $R_1(u_0 + 2du)$ with an asymptotic curve on it belonging to the complex $C(u_0 + 2du)$, etc. If now we allow du to approach zero, we (finally) obtain a surface S as the envelope of the ruled surfaces constructed in this way; and the asymptotic curves of one family on S clearly belong to the given one parameter family of linear complexes. But by a previous theorem it follows that the asymptotic curves of the second family on S will also belong to linear complexes.

If the asymptotic curves of a single family belong to linear complexes (i. e., if $\Omega' = 0$, $\Omega'' \neq 0$), we select a one parameter family of pairs of lines $(\delta_1(u), \delta_2(u))$ on a directrix ruled surface of the third or higher order so as to preclude the possibility of the second family of asymptotic curves belonging to linear complexes. The remaining steps in this construction are precisely the same as in the case just considered and need not be repeated. Our construction shows that this directrix ruled surface may be chosen arbitrarily.

§ 5. The lines common to the complexes C'_v, C''_u, C'_{v+dv}, C''_{u+du}*

The complexes C'_v, C''_u, C'_{v+du}, C''_{u+du} have in common the same lines as the complexes

* Cf. M_2, p. 97.

$$- a'_u \omega_{42} + a' \omega_{14} + a' \omega_{23} = 0,$$

$$- 2a' \omega_{13} + 4a'^2 b\omega_{34} + \delta\omega_{42} = 0,$$

$$- b_v \omega_{34} - b\omega_{14} + b\omega_{23} = 0,$$

$$+ 2b\omega_{12} + 4a' b^2 \omega_{42} + \delta' \omega_{34} = 0,$$

where

$$\delta = \frac{\theta - 2^6 a'^2_u}{2^7 a'}, \qquad \delta' = \frac{\theta' - 2^6 b_v^2}{2^7 b}.$$

All of the lines common to these complexes intersect the lines

$$2a' bx_1 + a'_u bx_2 + a' b_v x_3 = 0, \qquad x_4 = 0;$$

$$2bx_2 + b_v x_4 = 0, \qquad 2a' x_3 + a'_u x_4 = 0,$$

which are the directrices d and d' of the first and second kinds of the point $P_{y(u,\,v)}$.* If

$$p = - a'_u y + 2a' z, \qquad q = - b_v y + 2b\rho,$$

then $\lambda p + \mu q$ is an arbitrary point on the directrix of the first kind. Similarly, if we write

$$r = - a' b_v z - ba'_u \rho + 2a' b\sigma,$$

then $\lambda' y + \mu' r$ is an arbitrary point on the directrix of the second kind. The coördinates of the line joining these points are

$$\omega_{12} = - 2a' \lambda\lambda' + a' a'_u b_v \lambda\mu' + a' b_v^2 \mu\mu',$$

$$\omega_{13} = - 2b\mu\lambda' + ba'^2_u \lambda\mu' + bb_v a'_u \mu\mu',$$

$$\omega_{14} = - 2a' ba'_u \lambda\mu' - 2a' bb_v \mu\mu',$$

$$\omega_{23} = - 2a' ba'_u \lambda\mu' + 2a' bb_v \mu\mu',$$

$$\omega_{34} = 4a' b^2 \mu\mu', \qquad \omega_{42} = - 4a'^2 b\lambda\mu'.$$

In order that this line may belong to the complexes C'_{v+dv} and C''_{u+du}, λ, μ, λ', μ' must satisfy the equations

$$\theta\lambda\mu' - 2^7 \lambda' \mu + 2^6 (a'_u b_v - 2^3 a'^2 b^2) \mu\mu' = 0,$$

$$2^7 \lambda\lambda' - 2^6 (a'_u b_v - 2^3 a'^2 b^2) \lambda\mu' - \theta' \mu\mu' = 0.$$

Hence

$$\frac{\lambda}{\mu} = \pm \sqrt{\frac{\theta'}{\theta}}, \qquad \frac{\lambda'}{\mu'} = \frac{(a'_u b_v - 2^3 a'^2 b^2) \pm \dfrac{\sqrt{\theta\theta'}}{2^3}}{2}.$$

* M_2, pp. 95, 96.

Thus we have two lines* common to these four complexes, and they are obviously the diagonals of the skew quadrilateral $(\delta_1, \delta_2, \delta_1', \delta_2')$. Since $\lambda/\mu = \pm\sqrt{\theta'/\theta}$, l_1 and l_2 intersect the directrix d in two points which are harmonic conjugates with respect to p and q. Again d' intersects the osculating hyperboloid H in the points $P(1, 0, 0, 0)$ and $Q(x_1, x_2, x_3, x_4)$, where

$$x_1 = (a_u' b_v - 2^3 a'^2 b^2), \qquad x_2 = -2a' b_v, \qquad x_3 = -2a_u' b, \qquad x_4 = 4a' b.$$

Hence l_1 and l_2 intersect d' in two points which are harmonic conjugates with respect to P and Q. We therefore have the theorem:

If the asymptotic curves of a non-ruled surface belong to linear complexes, and if the directrices of R_1 and R_2 are distinct, the diagonals of the net-work of skew quadrilaterals formed by the directrices of R_1 and R_2 on the directrix quadric are the lines common to C_v', C_u'', C_{v+dv}', C_{u+du}''. These lines intersect the directrices of the first kind in points which are harmonic conjugates with respect to the intersections of the latter with the asymptotic tangents. They intersect the directrices of the second kind in points which are harmonic conjugates with respect to the intersections of the latter with the osculating hyperboloid.

§ 6. Applications to the theory of ruled surfaces

If the integrating surface S is a ruled surface not a quadric, we may assume a' equal to zero, and b different from zero. Then from equations (11) and

$$\Omega' \equiv b_{uv} - \frac{b_u b_v}{b},$$

it follows that we can find by quadratures a transformation of the type (18) which replaces equations (10) by the following system:

$$y_{uu} + 2y_v + fy = 0, \qquad y_{vv} - ay = 0, \tag{37}$$

where a is a constant and f is a function of u alone, since, by equations (11), $g_u = g_v = f_v = 0$. Therefore *any non-developable ruled surface (not a quadric) whose asymptotic curves belong to linear complexes may be represented by a system of equations of the form* (37), *and conversely, every such system of equations defines a projectively equivalent family of non-developable surfaces having the property in question.*

From (37) we find that

$$\theta' = 2^8 a, \qquad \theta_4 = \frac{2}{5} f_{uu} + \frac{16}{25} f^2 - 4a.\dagger$$

* We shall refer to these as l_1 and l_2.

† θ' denotes the invariant of weight four for the surface S and θ_4 the invariant of weight four for Γ''.

If we impose the additional condition that the asymptotic curves be twisted cubics, we find precisely the same equations as we obtained from the point of view of ordinary differential equations. Corresponding to the first case under (9) we find

$$\theta' = a = f = 0.$$

Equations (37) now become

$$y_{uu} + 2y_v = 0, \qquad y_{vv} = 0.$$

On integrating the second equation we find

$$y = U_1 v + U_2,$$

where U_1 and U_2 are functions of u alone. Substituting this expression for y in the first equation and integrating the resulting equations

$$\frac{d^2 U_1}{du^2} = 0, \qquad \frac{d^2 U_2}{du^2} + 2U_1 = 0,$$

we find

$$y = (au + b)v + \left(-\frac{au^3}{3} - bu^2 + cu + d \right),$$

where a, b, c and d are arbitrary constants. Therefore the equation of the surface S is

$$3y_4 (y_2 y_3 - y_1 y_4) - 2y_3^3 = 0,$$

which is the equation of a Cayley cubic scroll. If we notice that the conditions imposed upon equations (37) in this case are precisely that *the invariants θ', C', and h shall vanish, we have a simple direct proof of the theorem* that* "If these invariants vanish identically, the surface must be a Cayley cubic scroll." By the classical method we find the line $y_3 = y_4 = 0$ is the nodal line, and the point $P(1, 0, 0, 0)$ is the pinch-point.

In his third memoir,† Wilczynski has proved an interesting and important theorem for the case when θ' and a certain invariant $\mathfrak{B}$‡ are each different from zero. Now for the surfaces of the type we are considering [i. e., $\Omega' = 0$] the osculating linear complexes are indeterminate, and the invariant $\mathfrak{B}$ vanishes whether θ' vanishes or not. The theorem just referred to undergoes extensive modifications in this case. In fact, we can find the locus of the pinch-points and singular tangent planes not only of the ∞^1 osculating Cayley cubic scrolls belonging to a single generator g of S but also of all the ∞^2 Cayley cubic scrolls associated with S.

An arbitrary point P_Y on the generator g of S will be represented by

* M_3, p. 307.

† M_3, p. 315.

‡ $\mathfrak{B} = -2^6 ((b\theta'_u - 2b_u\theta')^2 - 4b^2 c'^2 \theta')$.

$$Y = y + l\rho,$$

where l is an arbitrary constant.

Since the invariant Ω' vanishes, we may assume that the equations of S have been reduced to the form (37). We shall find

$$Y_{uu} + 2Y_v + fY = 0, \qquad Y_{vv} - aY = 0. \tag{38}$$

The semi-covariants of this system of equations are

$$Y = y + l\rho, \qquad Z = z + l\sigma, \qquad P = aly + \rho, \qquad \Sigma = alz + \sigma.$$

The pinch-point π of the Cayley cubic scroll, osculating S at the point P_Y, is given by the expression

$$\pi = 2^3 \Sigma = 2^3 (alz + \sigma).$$

Hence as P_Y moves along the generator g, the pinch-point of the osculating Cayley cubic scroll moves along the line joining the semi-covariant points P_z and P_σ. We shall now set up the equations of S in the form (1). They are found to be

$$Y_{uu} + fy + 4\rho = 0, \qquad \rho_{uu} + 4ay + f\rho = 0. \tag{39}$$

The semi-covariants of this system are

$$r = 2z, \qquad s = 2\sigma, \qquad y = y, \qquad \rho = \rho.$$

For uniformity we shall use the notation of the theory of ruled surfaces and write these equations

$$y_{uu} + fy + 4z = 0, \qquad z_{uu} + 4ay + fz = 0, \tag{39a}$$

and denote their semi-covariants by y, z, ρ, σ. If we denote the semi-covariants of (37) by y', z', ρ', σ' we shall find

$$y' = y, \qquad z' = \frac{\rho}{2}, \qquad \rho' = z, \qquad \sigma' = \frac{\sigma}{2}.$$

Therefore the equations of transformation between the coördinate systems determined by $P_{y'} P_{z'} P_{\rho'} P_{\sigma'}$ and $P_y P_z P_\rho P_\sigma$ are

$$\begin{aligned} &\omega' x_1' = x_1, \qquad \omega' x_2' = 2x_3, \qquad \omega' x_3' = x_2, \qquad \omega' x_4' = 2x_4, \\ &\omega x_1 = 2x_1', \qquad \omega x_2 = 2x_3', \qquad w x_3 = x_2', \qquad \omega x_4 = x_4', \end{aligned} \tag{40}$$

where ω and ω' are factors of proportionality. From these equations we find that the coördinates of the pinch-point π of the Cayley cubic scroll osculating S at P_y are

$$x_1 = x_2 = 0, \qquad x_3 : x_4 = a,$$

the coördinate system being that determined by $P_y\, P_z\, P_\rho\, P_\sigma$. Equations (37) show that the invariant θ' is a constant; so that the first derived ruled surface of S is the principal surface* of the flecnode congruence of S. We therefore have the theorem:

The locus of the pinch-points of the osculating Cayley cubic scrolls of a ruled surface whose asymptotic curves belong to linear complexes is the principal surface of the flecnode congruence of the surface itself.

An interesting special case arises when $f(u)$ reduces to a constant. It follows from the general theory of ruled surfaces that the surface S then gives rise to a surface of pinch-points S_1 which is a projective transformation of S; and S_1 gives rise to a surface of pinch-points which is the surface S itself. As P_Y describes an asymptotic curve on S, the corresponding pinch-point describes an asymptotic curve on the surface of pinch-points.

From equations (40) it follows that the nodal line of the Cayley cubic scroll of the point P_Y intersects the generator g in the point

$$Q \equiv at_1\, y + t_2\, z,$$

where $t_1 : t_2 = l$. The flecnodes on g are the points

$$\eta = \sqrt{ay} - z, \qquad \zeta = \sqrt{ay} + z.$$

The cross ratio of the points (P, Q, η, ζ) is -1. The singular tangent plane of the Cayley cubic scroll is tangent to the osculating hyperboloid at the pinch-point.† Its coördinates are

$$(t_2,\ -at_1,\ 0,\ 0).$$

If we denote that generator of S_1 (the surface of pinch-points), which corresponds to g, by g', we see that the singular tangent planes of the Cayley cubic scrolls of g form a pencil whose axis is g'.

Therefore *as the point P moves along the generator g of S, the singular tangent planes of the osculating Cayley cubic scrolls rotate about the corresponding generator of the locus of pinch-points as axis. If $f(u)$ reduces to a constant the point P, the point Q where the nodal line of the Cayley cubic scroll of P intersects the generator g, and the flecnodes η and ζ on g form a harmonic group.*

§ 7. The directrix curves on the surfaces S of the problem can be determined by quadratures

The linear complexes determined by the asymptotic curves through P_y have a congruence in common. The directrix d (the directrix of the first kind)

* W, pp. 216, 218, 220.

† M_3, p. 303.

of this congruence is situated in the tangent plane of P_y; the directrix d' (the directrix of the second kind) passes through P_y. As P_y moves over S, d and d' generate the directrix congruences of the first and second kinds respectively.* The differential equation of the directrix curves on S, which correspond to the developables of these congruences, is

$$bLdu^2 + 2Mdudv - a'Ndv^2 = 0, \tag{41}$$

where

$$L = -2a'(2a'bf + 2a'bb_v + ba'_{uu}) + ba'^2_u = \frac{\theta}{64},$$

$$M = 2a'b(a'b_{uv} - ba'_{uv}) + 2(b^2 a'_u a'_v - a'^2 b_u b_v),$$

$$N = -2b(2a'bg + 2a'ba'_u + a'b_{vv}) + a'b_v^2 = \frac{\theta'}{64}.$$

Since Γ' and Γ'' belong to linear complexes, we have

$$\Omega' = \Omega'' = 0,$$

and therefore

$$M = 2[a'^2(bb_{uv} - b_u b_v) - b^2(a'a'_{uv} - a'_u a'_v)] = 0.$$

Hence equation (41) becomes

$$b\theta du^2 - a'\theta' dv^2 = 0. \tag{41a}$$

If we introduce the transformation†

$$\bar{u} = \alpha(u) = \int \sqrt[3]{\frac{\theta_1(u)}{\varphi_1(u)}}\,du, \qquad \bar{v} = \beta(v) = \int \sqrt[3]{\frac{\theta_2(v)}{\varphi_2(v)}}\,dv, \qquad y = \frac{1}{\sqrt{\alpha_u \beta_v}}\bar{y},$$

we see, from equations (17), (18), (19) and a previous theorem, that (41a) becomes

$$d\bar{u}^2 - d\bar{v}^2 = 0,$$

which gives (on integration)

$$\bar{u} + \bar{v} = \text{const.}, \qquad \bar{u} - \bar{v} = \text{const.},$$

as the finite equations of the two families of directrix curves on S. *Since M vanishes the directrix curves form a conjugate system on the surface S.*

If Ω', Ω'', θ, θ' all vanish (i. e., if the normal forms of the seminvariants f and g are characterized by the vanishing of the constants a, b, c), equation (41) *is satisfied for every direction $du : dv$.* In this case we find that the point

* M_2, p. 114 et seq.

† The origin of this transformation is clear if we recall that $b/a' = \theta_1(u)/\theta_2(v)$, $\theta/a'^2 = \varphi_1(u)$, $\theta'/b^2 = \varphi_2(v)$.

$$P = \lambda\,(y + y_u\,du + y_v\,dv) + \mu\,(\tau + \tau_u\,du + \tau_v\,dv)^*$$

is on the line

$$2bx_2 - b_v\,x_4 = 0\,, \qquad 2a'\,x_3 + a'_u\,x_4 = 0$$

for every value of $du : dv$, if

$$\lambda : \mu = -\,(2bQ + b_v S) : 2b = -\,(2a'\,R + a'_u\,S') : 2a'\,.$$

It follows that under these circumstances all the directrices of the second kind in the vicinity of a point P have a point in common. Referred to T the coördinates of this point are found to be

$$(42) \quad x_1 = -\,(2bQ + b_v S)\,, \quad x_2 = -\,2a'\,bb_v\,, \quad x_3 = -\,2a'_u b^2\,, \quad x_4 = 4a'\,b^2\,.$$

Using the fact that the invariants Ω' and Ω'' vanish, we find

$$x_1 : x_2 : x_3 : x_4 = 2^3\,a'^2\,b^2 - a'_u\,b_v : 2a'\,b_v : 2a'_u\,b : -\,4a'\,b\,.$$

To prove that this point is fixed in space we consider the following expression

$$\pi = -\,(2bQ + b_v\,S)y - 2a'\,bb_v\,z - 2a'_u\,b^2\,\rho + 4a'\,b^2\,\sigma$$

for the point (42), and calculate

$$\pi + \frac{\partial \pi}{\partial u}\,du + \frac{\partial \pi}{\partial v}\,dv\,,$$

making use of the relations

$$\Omega' = \Omega'' = \theta = \theta' = 0\,.$$

We find

$$\pi + \frac{\partial \pi}{\partial u}\,du + \frac{\partial \pi}{\partial v}\,dv = \left(1 + \frac{a_u\,b + 4a'\,b_u}{2a'\,b}\,du + \frac{a'\,b_v + 4a'_v\,b}{2a'\,b}\,dv\right)\pi\,,$$

which shows that π is a fixed point.

Let us now consider the directrices of the first kind. The point

$$P = \lambda\,(r + r_u\,du + r_v\,dv) + \mu\,(s + s_u\,du + s_v\,dv)$$

will be on the line joining

$$r = -\,a'_u\,y + 2a'\,z \quad \text{and} \quad s = -\,b_v\,y + 2b\rho\,,$$

if the equations

$$(8a'^2\,b^2 + a'_u\,b_v)\,(\lambda a'\,dv + \mu bdu) = 0\,, \qquad \lambda a'\,dv + \mu bdu = 0\,,$$

are satisfied. If we choose

$$\lambda : \mu = -\,b\,du : a'\,dv\,,$$

* $\tau = -\,a'\,b_v\,z - ba'_u\,\rho + 2a'\,b\sigma\,, \quad Q = 8a'^2\,b^2 - a'\,b_{uv} - a'_u\,b_v\,, \quad S = a'_u\,b + 2a'\,b_u\,,$
$S' = a'\,b_v + 2a'_u\,b\,, \qquad R = -\,(2\,a'\,bf + 2a'\,bb_v + ba'_{uu} + a'_u\,b_u)\,.$

we see that two consecutive directrices intersect for every value of the ratio $du : dv$. It therefore follows that all the directrices of the first kind in the vicinity of the point P_y lie in the same plane. We proceed to determine this plane. By using the conditions

$$\Omega' = \Omega'' = \theta = \theta' = 0$$

we can show that the four points

$$r, \; s, \; r + r_u\, du + r_v\, dv, \; s + s_u\, du + s_v\, dv$$

are coplanar. After some reductions we find the equation of their plane, which we shall call the *directrix plane,* to be

$$2^2\, a'\, bx_1 + 2a'_u\, bx_2 + 2a'\, b_v\, x_3 + (2^3\, a'^2\, b^2 + a'_u\, b_v)\, x_4 = 0\,.$$

It can be readily shown that this *directrix plane* is the null-plane of π in both of the complexes C' and C''. If we introduce the transformation*

$$\frac{\lambda\mu}{k} x_1 = -\alpha\mu X - \lambda\beta Y + \alpha\beta Z + \lambda\mu\omega\,, \qquad \frac{\lambda\mu}{k} x_3 = \lambda Y - \alpha Z\,,$$

$$\frac{\lambda\mu}{k} x_2 = \mu X - \beta Z\,, \qquad \frac{\lambda\mu}{k} x_4 = Z\,,$$

which refers the plane to the canonical tetrahedron of the point P_y, the above equation becomes

$$Z + 8\omega = 0\,,$$

and the point π referred to this tetrahedron becomes

$$X = Y = 0\,, \qquad Z = -2^3\,, \qquad \omega = 1\,.$$

These results may be combined in the following theorem:

If both families of asymptotic curves of the surface S belong to linear complexes and if the invariants θ and θ' vanish, the directrix congruences degenerate. The directrix congruence of the first kind consists of the net of lines in the directrix plane, while the directrix congruence of the second kind consists of the sheaf of lines through the null-point of the directrix plane in the complexes C' and C''.

There remains to be considered the case of a degenerate *directrix quadric.* This case is closely allied to that of the degenerate directrix congruence referred to above. There also exist a number of other curves and congruences organically connected with the configurations of this problem. The discussion of all of these questions will be reserved for a future occasion.

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* M_2, p. 104.