THE CAUCHY PROBLEM FOR INTEGRO-DIFFERENTIAL EQUATIONS.*

BY

GRIFFITH C. EVANS

INTRODUCTION

1. A large part of the theory of integral and integro-differential equations may be reduced to the corresponding theory of algebraic and differential equations by the introduction of convenient symbolism. For direct operations this analogy is well expressed by merely considering as a product the combination

$$AB = \int A(r, \tau)\, B(\tau, s)\, d\tau, \tag{1}$$

taking the integral with constant limits a, b, or variable limits s, r, as the case may be.† In this symbolism, however, the vanishing of the "product" of two functions does not imply necessarily that one of the functions vanishes. And therefore for the treatment of the inverse operations and the construction of a complete *algebra*, it is more convenient to consider, instead of the combination written above, the following combination of certain complex quantities. Let

$$\xi = u + jU(r, s), \qquad \eta = v + jV(r, s).$$

By $\xi\eta$ we understand the quantity

$$\xi\eta = uv + j\{uV(r, s) + vU(r, s) + \int U(r, \tau)\, V(\tau, s)\, d\tau\}, \tag{2}$$

taking as the limits of the integral a, b or s, r, according as the algebra is of one kind or the other.‡ The complex unit j is merely used to separate the

* Presented to the Society September, 1913.

† V. Volterra, *Questioni generali sulle equazioni integrali ed integro-differenziali*, Rendiconti della R. Accademia dei Lincei, vol. 19 (1910), p. 169 and p. 178.

‡ In order not to introduce special parameter values into the formal consideration of the problem, it is convenient to make the assumption that for constant limits of integration the j-coefficients contain a parameter λ in such a way that $|U(r, s)| < \lambda M$, and to consider small values of λ. That convention will not generally be necessary here, since we are to introduce explicitly parameters x, y. Where the introduction of such a parameter is necessary it will be specified.

quantity ξ into two parts, which may be called respectively ordinary coefficient, denoted by Ord ξ, and j-coefficient, written j-coefficient ξ. Each coefficient may be complex in the usual sense. By $\xi = 0$ is understood, of course, the pair of equations $u = 0$, $U = 0$, and by $\xi = \eta$, the pair $u = v$, $U = V$.

If the functions U, V are permutable, *i. e.*, if

$$\int U(r, \tau)\, V(\tau, s)\, d\tau = \int V(r, \tau)\, U(\tau, s)\, d\tau,$$

then the operation defined by (2) is commutative, and ξ and η are said to be permutable.

If $u = 0$, ξ is said to be a function of nullity. We see, by referring to the condition for equality, that division by a function not of nullity is equivalent to solving a linear integral equation of the second kind of Fredholm or Volterra type, and that division by a function of nullity is often impossible. With this distinction, if our quantities are permutable among themselves we may apply *all* the laws of algebra, and if the quantities are not permutable, all except the commutative law.* The function of nullity, which includes zero as a special case, takes much the same place in this algebra as the quantity zero in the ordinary algebra. In fact the two algebras are merihedrically isomorphic.† If we do not divide by zero in the ordinary algebra, we do not need to divide by a function of nullity in the symbolic correspondence.

2. If now we introduce into the quantities u and U which define ξ extra parameters or variables $(x_1, x_2, \cdots)$ we have obviously the formula

$$\frac{\partial}{\partial x_i}(\xi\eta) = \xi\frac{\partial \eta}{\partial x_i} + \frac{\partial \xi}{\partial x_i}\eta \tag{3}$$

irrespective of the commutativity or non-commutativity of the product $\xi\eta$. To every differential equation corresponds then an integro-differential equation by means of this symbolism, and a large part of the theory of differential equations may be carried over word for word to these other equations, even when the multiplication is not commutative.

It may be noticed, however, that this symbolism, in particular equation (3), is valid for those integro-differential equations only in which the variables of differentiation $(x_1, \cdots)$ are different from those of integration r, s. These equations were called by their inventor "*of static type*," on account of

* Evans, *L'algebra delle funzioni permutabili e non permutabili*, Rendiconti del Circolo Matematico di Palermo, vol. 34 (1912), pp. 1–28, and ibid. vol. 35, Errata Corrigi.

† Evans: *Sopra l'algebra delle funzioni permutabili*, Atti delle R. Accademia dei Lincei, ser. 5*a*, vol. 8 (1911), p. 710.

their application to the theory of slow motion.* It is the object of this paper to extend Cauchy's theorem for differential equations to integro-differential equations of this type.†

Theorem V, in the paper of Volterra first cited, deals with equations of this kind, and demonstrates the existence of what correspond, in the theory of differential equations, to the general and complete integrals of the equation. Boundary problems for this class of equations have been discussed by Volterra, Lauricella, recently by J. Pérès and others. Integro-differential equations and functional equations of non-static type have also been discussed by Volterra and others, and some corresponding existence theorems have been obtained.

Systems of symbolic differential equations with permutable coefficients

3. Let us limit ourselves to two variables of differentiation x, y besides the variables of integration r, s, and let us consider the system of symbolic equations

$$
(4)\qquad
\begin{aligned}
\frac{\partial \xi_1}{\partial y} &= F_1\left(\frac{\partial \xi_1}{\partial x}, \cdots, \frac{\partial \xi_n}{\partial x}; \xi_1, \cdots, \xi_n; y, x \mid r, s\right),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
\frac{\partial \xi_n}{\partial y} &= F_n\left(\frac{\partial \xi_1}{\partial x}, \cdots, \frac{\partial \xi_n}{\partial x}; \xi_1, \cdots, \xi_n; y, x \mid r, s\right).
\end{aligned}
$$

The variables r, s are to be restricted to a single region T of the complex plane, such that if r and s are any two points of T the whole of the straight line joining them lies in T. If for $x, y, \xi_1, \cdots, \xi_n, \partial\xi_1/\partial x, \cdots, \partial x_n/\partial x$ we substitute variables $x_1, \cdots, x_{2n+2}$, the functions $F_1, \cdots, F_n$ are to be analytic in these variables in a $(2n+2)$-dimensional neighborhood of the origin, which may be chosen in such a way as to be independent of r, s. The coefficients in the developments of these F_i are continuous functions of r, s in T, permutable among themselves.

The developments in (4) will be convergent and have meaning provided that the ordinary coefficients of the ξ_i, with their first derivatives in x, lie in the given $(2n+2)$-dimensional neighborhood.‡

* V. Volterra: *Sulle equazioni della elettrodinamica*, Rendiconti della R. Accademia dei Lincei, vol. 23 (1909), p. 209.

† In the interim of the writing of this paper and its publication, an article has appeared by M. Paul Lévy, in which the notion of characteristic is applied to another kind of functional equation. See: *Sur l'intégration des équations aux dérivées fonctionelles partielles*, Rendiconti del Circolo Matematico di Palermo, vol. 37 (1914), pp. 113–169.

‡ See: *A theorem of convergence*, Rendiconti del Circolo Matematico di Palermo, vol. 34 (1912), p. 10. To insure this convergence when the limits of integration are constant, parameters $\lambda_1, \cdots, \lambda_n$ must be introduced into the j-coefficients of $\xi_1, \cdots, \xi_n$. In Theorem 1, however, on account of the initial conditions these parameters are provided for by the variable y.

THEOREM 1. *There is one and only one system of solutions* $\xi_1, \cdots, \xi_n$ *of the system of equations* (4) *analytic in* x *and* y *in the neighborhood of the origin, and such that for* $y = 0$ *the conditions* $\xi_1 = 0, \cdots, \xi_n = 0$ *are satisfied. These solutions are themselves permutable with the coefficients of* $F_1, \cdots, F_n$ *and with each other.*

From the fact that the region of analyticity of the F_i is independent of r, s we are able to get inequalities for the values of the coefficients in the development of the F_i. We have then a complete identity of operations with the corresponding theory for differential equations; whence is established, step by step, the permutability of every coefficient in the development of the ξ_i with those of the F_i, and also the fact that they are permutable with each other, because they are, symbolically, integral rational expressions in the coefficients of the F_i. Likewise is established the uniqueness of the formal solutions so developed. The convergence of the developments follow immediately from the convergence of the corresponding power series for the solution of the analogous differential equations if we replace the length of the rectangle of convergence ρ, in the y direction, in the theory of differential equations, by the distanec ρ / c, where c is some quantity greater than unity, and greater also than the interval of integration, ab, or rs, as the case may be.

In fact, if

$$| \xi | = | u + jU | = | u | + | U | < A,$$

$$| \eta | = | v + jV | = | v | + | V | < B,$$

it follows that

$$| \xi\eta | < ABc,$$

and these are the inequalities that it is necessary to use in the demonstration of the convergence of the j-coefficients and ordinary coefficients in the developments of the ξ_i.

4. Let us say that a function $u(x \mid r, s)$

$$u(x \mid r, s) = \sum_0^\infty {}_i A_i(r, s)(x - x_0)^i$$

permutably analytic, or permutably analytic in x, if it is analytic in x and all its coefficients are mutually permutable functions of r, s. We notice that if such a function is permutably analytic about the point x_0, it is permutably analytic when developed about any other point x_1 in the region of analytic extension; for the new coefficients are linear expressions in terms of the old, the coefficients in these expressions being monomials in $x_1 - x_0$.

A necessary and sufficient condition that $u(x \mid r, s)$ be permutably analytic in a given region is that it be analytic in that region and satisfy the condition

$$\int u(x_1 \mid r, \tau)\, u(x_2 \mid \tau, s)\, d\tau = \int u(x_2 \mid r, \tau)\, u(x_1 \mid \tau, s)\, d\tau,$$

where the limits of integration are constant, or s, r, as we deal with one kind of permutability or the other, and where x_1 and x_2 are any two values within the region of analyticity.*

The solutions $\xi_1, \cdots, \xi_n$ *in Theorem* 1 *are permutably analytic.*

We say that two functions are *analytically permutable*, or analytically permutable with each other, when every coefficient in the development of one is permutable with each of the coefficients of the other. As before, this definition is independent of the point x_0, or (x_0, y_0), about which the functions are developed.

Two functions are permutable if they are analytically permutable.

A necessary and sufficient condition that $u(x \mid r, s)$ and $v(x \mid r, s)$ be analytically permutable in x is that they be analytic in x in two regions σ_1 and σ_2 respectively, and that

$$\int u(x_1 \mid r, \tau)\, v(x_2 \mid \tau, s)\, dt = \int v(x_2 \mid r, \tau)\, u(x_1 \mid \tau, s)\, d\tau,$$

where x_1 is any value in σ_1 and x_2 any value in σ_2.

5. We may generalize Theorem 1 to the case where $\xi_1, \cdots, \xi_n$ are, for $y = 0$, arbitrary functions of x, r, s provided that these functions are permutably analytic in x, and analytically permutable with each other and with the coefficients in the developments of the F_i.† The F_i are assumed to be developable throughout a $(2n + 2)$-dimensional neighborhood that includes the ordinary coefficients of the given values of the ξ_i. *There is a unique system of solutions, and these solutions are permutably analytic, and analytically permutable with each other and with the coefficients of the* F_i.

Systems of symbolic differential equations whose coefficients are not necessarily permutable

6. As has been remarked by M. Pérès in regard to certain integral equations,‡ the method of procedure may be extended to take care of the case where the coefficients in the analytic developments are not necessarily permutable among themselves. We have, as before, a unique formal determination of the coefficients, and the convergence of the resulting series is established§ by

* Compare the concept of "continuous group of permutable functions," V. Volterra: *Sopra equazioni di tipo integrale*, Proceedings of the Fifth International Congress of Mathematicians, Cambridge, 1913, vol. 1, p. 405.

† When the limits of integration are constants, the given values of the j-coefficients of $\xi_1, \cdots, \xi_n$ are supposed to contain parameters $\lambda_1, \cdots, \lambda_n$ respectively, unless the F_i are entire functions of the $2n$ arguments consisting of the ξ_i and their derivatives.

‡ J. Pérès: *Sulle equazioni integrali*, Rendiconti della R. Accademia dei Lincei, ser. 5*a*, vol. 22 (1913), pp. 66–70.

§ The system (4) may be replaced by one linear in the derivatives in regard to x by the introduction of n more equations and n more unknowns. The dominant series is most simply stated in terms of the linear system.

means of exactly the same dominant series that is used in Theorem 1. In fact, the less-or-equal relation between the terms of series and dominant series (all of whose coefficients are positive) holds even to the partial terms whose sum is the coefficient of $x^m y^n$.

Let F be analytic in its $2n+2$ arguments in a $(2n+2)$-dimensional neighborhood which includes the points determined by the ordinary coefficients of the functions $\xi_1(x \mid r, s), \cdots, \xi_n^0(x \mid r, s)$, functions which are analytic in x, in the neighborhood of $x = 0$, with coefficients that in T are continuous functions of r, s. We have then the following theorem:

THEOREM 2. *There is one and only one system of solutions $\xi_1, \cdots, \xi_n$ of* (4) *under the given conditions,* which are analytic at the origin in x and y, and which, for $y = 0$, take the given values $\xi_1^0, \cdots, \xi_n^0$.*

By means of the process of analytic extension we may extend our theorem to the case where the given functions are analytic along a certain given segment of the x-axis.

7. Two of Volterra's integro-differential equations, viz.,

$$\nabla^2 u(x, y \mid r, s) + \int_y^x \left\{ A_1(r, \tau) \frac{\partial^2 u(x, y \mid \tau, s)}{\partial x^2} + A_2(r, \tau) \frac{\partial^2 u(x, y \mid \tau, s)}{\partial y^2} \right\} d\tau = 0 \tag{5}$$

and

$$\nabla^2 u(x, y \mid r, s) + \lambda \int_a^b \left\{ A_1(r, \tau) \frac{\partial^2 u(x, y \mid \tau, s)}{\partial x^2} + A_2(r, \tau) \frac{\partial^2 u(x, y \mid \tau, s)}{\partial y^2} \right\} d\tau = 0 \tag{6}$$

have the common symbolic form

$$\alpha_1 \frac{\partial^2 \xi}{\partial x^2} + \alpha_2 \frac{\partial^2 \xi}{\partial y^2} = 0, \tag{7}$$

in which are to be considered only the solutions of nullity.

For small values of λ, the equation (7) may be written in the form

$$\frac{\partial^2 \xi}{\partial y^2} = \alpha \frac{\partial^2 \xi}{\partial x^2}, \tag{8}$$

in which

$$\alpha = -\frac{1}{\alpha_2} \alpha_1.$$

In close relation to (8) stands the system of equations

* As in § 5, parameters $\lambda_1, \cdots, \lambda_n$ must be introduced when the limits of integration are constant.

$$(9) \qquad \frac{\partial \xi_2}{\partial y} = \alpha \frac{\partial \xi_1}{\partial x}, \qquad \frac{\partial \xi}{\partial y} = \xi_2, \qquad \frac{\partial \xi_1}{\partial y} = \frac{\partial \xi_2}{\partial x}.$$

Under the boundary conditions

$$(9') \quad (\xi)_{y=0} = \xi^0(x, r, s), \quad (\xi_2)_{y=0} = \xi_2^0(x, r, s), \quad (\xi_1)_{y=0} = \frac{\partial}{\partial x}\xi^0(x, r, s),$$

the system (9) has one and only one analytic solution.*

But the system (9) under the conditions (9′) is equivalent to the system (8) under the conditions

$$(8') \qquad (\xi)_{y=0} = \xi^0(x, r, s), \qquad (\partial \xi / \partial y)_{y=0} = \xi_2^0(x, r, s),$$

and (8), then, under these conditions, has one and only one analytic solution. If ξ^0 and ξ_2^0 are functions of nullity, the solution is a function of nullity, and (8) is equivalent to (5) or (6).

Equation (5) needs no further discussion. In equation (6), however, we made the assumption that λ was small. If λ is not small (6) is equivalent to (8) under the condition that λ is not a special parameter value for $A_2(r, s)$. *Equation* (6) *has therefore a unique analytic solution under the given conditions, provided that* λ *is not a special parameter value for* $A_2(r, s)$.

THE CORRESPONDENCE BETWEEN SYMBOLIC DIFFERENTIAL AND INTEGRO-DIFFERENTIAL EQUATIONS

8. Let us consider the equation

$$(10) \qquad \Phi\left(U, \frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial^2 U}{\partial x^2}, \cdots; y, x \mid r, s\right) = 0,$$

of which the left hand member is an integral rational expression in U and its derivatives up to the nth order, with coefficients that are analytic in x and y throughout a neighborhood σ, independent of r, s, of a given analytic curve σ. The coefficients in the analytic developments of these coefficients are continuous functions of r, s in T. Multiplication of U by itself or by any of its derivatives, or of any of these by a function of r, s is to be interpreted as combination according to the formula (1) [§ 1]. Multiplication of U or any of its derivatives by a function not involving r, s may be interpreted in the ordinary way; or, on the other hand, that function may be regarded as implicitly a function of r, s, according as we care to consider one equation or another. In this way we generate an integro-differential equation which includes that discussed by Volterra in the Theorem V already referred to.

Equation (10) may be rewritten as a symbolic differential equation. In

* It is not necessary to introduce parameters λ_i into these boundary conditions (see note to § 5).

fact if we replace in it U and its derivatives by $jU = \xi$ and its derivatives, and replace every function involving r, s, explicitly, or by definition, by j times that function, leaving the functions not involving r, s as they are, and if we then collect terms, we shall have a symbolic differential equation whose solutions of nullity are the solutions of (10), and vice-versa. Let us write this equation as

$$\Psi\left(\xi, \frac{\partial \xi}{dx}, \frac{\partial \xi}{dy}, \frac{\partial^2 \xi}{\partial x^2}, \cdots; y, x \mid r, s\right) = 0. \tag{11}$$

9. If the curve σ is the x-axis, and (11) can be solved for $\partial^n \xi / \partial y^n$, the equation (11) may be replaced by a system of equations of the type of (4), and Theorem 1 or Theorem 2 may be applied. We thus are able to restate for integro-differential equations of the form (10) the ordinary Cauchy existence theorems for differential equations.

If the coefficients in the analytic development of (10) are permutable among themselves, the solution of (10) is permutably analytic, and analytically permutable with the coefficients, provided that the same statement is true of the initial conditions.

Characteristics of integro-differential equations *

10. If the curve σ is not the x-axis but is some analytic curve $y = \varphi(x)$, we may still determine a solution of the equation (10), or, what is the same thing, a solution of nullity of equation (11), by assigning proper values for the solution and its derivatives up to and including those of order $n - 1$ along the curve. For, by a transformation of variable, this case may be reduced to that already discussed. For certain families of curves, however, as with differential equations, this transformation is impossible; the solution is not uniquely determined by means of the given conditions. These families of curves, to which correspond the characteristics of differential equations, are the subject of this last section.

Without any essential loss of generality, for our treatment, we can assume that equations (10) and (11) are of the second order. We shall suppose them, for the present, to be linear in the derivatives of highest order. Equation (11) may then be rewritten as

$$\alpha_{11} \frac{\partial^2 \xi}{\partial x^2} + \alpha_{12} \frac{\partial^2 \xi}{\partial x \partial y} + \alpha_{22} \frac{\partial^2 \xi}{\partial y^2} + \lambda = 0. \tag{12}$$

In this equation α_{11}, α_{12}, α_{22} and λ are functions of ξ, $\partial \xi / \partial x$, $\partial \xi / \partial y$, x, y, r,

* This treatment may be compared with that in Hadamard's *Leçons sur la propogation des ondes*, chap. 7, Paris, 1913. The equations (12), (13), (14), (15), (17), (18′), below, correspond to the equations (1), (7), (8), (8′) and (12), (11), (13) respectively in the chapter cited.

s of the type specified in § 8. We assume that on the curve $y = \varphi(x)$ we are given ξ and $\partial\xi / \partial y$, or what amounts to the same thing, ξ, $\partial\xi / \partial x$ and $\partial\xi / \partial y$ as analytic functions of x in the region σ, with coefficients continuous functions of r, s in T.*

Precisely as in the theory of differential equations, we have from (12), by expressing $\partial^2 \xi / \partial x^2$ and $\partial^2 \xi / \partial x \partial y$ for points on $y = \varphi(x)$ in terms of $\partial\xi / \partial y$ and $\partial^2 \xi / \partial y^2$, as the equation for the determination of $\partial^2 \xi / \partial y^2 = \xi_{22}$

$$\Gamma \xi_{22} + \Pi = 0. \tag{13}$$

In this equation

$$\begin{aligned} \Gamma &= \alpha_{11}\left(\frac{dy}{dx}\right)^2 - \alpha_{12}\left(\frac{dy}{dx}\right) + \alpha_{22}, \\ \Pi &= \alpha_{11}\left(\frac{d\xi_1}{dx} - \frac{dy}{dx}\frac{d\xi_2}{dx}\right) + \alpha_{12}\frac{d\xi_2}{dx} + \lambda, \end{aligned} \tag{14}$$

where d / dx refers to differentiation along the curve, and ξ_1 and ξ_2 denote $\partial\xi / \partial x$ and $\partial\xi / \partial y$ respectively. In these formulæ it is important to preserve the order of all quantities that involve the j, so as not to necessitate the introduction of the hypothesis of permutability.

The equation (13) *enables us to determine* ξ_{22}, *unless* Γ *is a function of nullity at some point of the curve* $y = \varphi(x)$; *i. e., unless*

$$\operatorname{Ord} \Gamma = 0. \tag{15}$$

The curves defined by the differential equation (15) may be called the *ordinary characteristics* of the integro-differential equation (10). On account of the way (11) is formed from (10), equation (15), which involves no j-coefficients, must be independent of the solution $\xi = jU$. *The ordinary characteristics are independent of the solution of the equation* (10).

11. If we make an analytic transformation of the independent variables x, y that reduces the curve $y = \varphi(x)$ to the x-axis, it is immediately verifiable that a sufficient condition that the transformation of the equation (11) be solvable for $\partial^n \xi / \partial y^n$ is that the curve $y = \varphi(x)$ be nowhere tangent to an ordinary characteristic. That it is not also a necessary condition depends on the fact that it is often possible to divide by a function of nullity.

THEOREM 3. *If* $y = \varphi(x)$ *is not tangent to any ordinary characteristic, the given values of* U *and* $\partial U / \partial y$ *uniquely determine an analytic solution of* (10).

12. It may happen that all the curves in the plane are ordinary characteristics of the equation (10). In other words Ord Γ may vanish identically. A

* The j-coefficients of these functions of r, s and of the functions and values arbitrarily assigned are supposed to contain parameters λ, if we are dealing with integration with constant limits.

necessary and sufficient condition that Ord Γ be identically zero is that we have identically

$$\text{Ord } \alpha_{11} = \text{Ord } \alpha_{12} = \text{Ord } \alpha_{22} = 0. \tag{16}$$

We may still, however, be able to determine solutions of the equation by giving arbitrary values of U and $\partial U / \partial y$ along $y = \varphi(x)$.

As an illustration of this fact, let us restrict ourselves to an integro-differential equation in which the limits of integration are s, r, and let

$$\alpha_{ij} = jA_{ij}(x, y \mid r, s) + jB_{ij}(x, y \mid r, s)\beta_{ij}, \quad \lambda = jC(x, y \mid r, s)\gamma,$$

where β_{ij}, γ are functions of ξ, $\partial\xi / \partial x$, $\partial\xi / \partial y$, x, y, r, s of the kind already described, and A_{ij}, B_{ij}, C and their first derivatives in regard to r are continuous functions of r, s and analytic functions of x, y, which for $s = r$ become merely functions a_{ij}, b_{ij}, c, of x, y alone. We assume, moreover, that the β_{ij} are functions of nullity.

To investigate this equation, differentiate it with regard to r. It will then, as is directly verifiable, take the form

$$\begin{aligned}\left\{ a_{11} + j\frac{\partial A_{11}}{\partial r} + \left(b_{11} + j\frac{\partial B_{11}}{\partial r}\right)\beta_{11} \right\}\frac{\partial^2\xi}{\partial x^2} \\ + \left\{ a_{12} + j\frac{\partial A_{12}}{\partial r} + \left(b_{12} + j\frac{\partial B_{12}}{\partial r}\right)\beta_{12} \right\}\frac{\partial^2\xi}{\partial x\partial y} \\ + \left\{ a_{22} + j\frac{\partial A_{22}}{\partial r} + \left(b_{22} + j\frac{\partial B_{22}}{\partial r}\right)\beta_{22} \right\}\frac{\partial^2\xi}{\partial y^2} + \lambda' = 0.\end{aligned}$$

The solutions of nullity of this last equation are solutions of nullity of the original one, and vice-versa. But this last equation has definite ordinary characteristics unless a_{11}, a_{12}, a_{22} are all zero.

13. Let us now suppose that we have an integro-differential equation (12) for which the ordinary characteristics are defined, and that $y = \varphi(x)$ is one of them. Let us consider how much is arbitrary in the solution.

The equation (13) shows that it is not generally possible to assign values of ξ and ξ_2 arbitrarily along an ordinary characteristic. For if, for instance, we restrict ourselves to variable limits of integration r, s and take account of the fact that since $y = \varphi(x)$ is an ordinary characteristic Γ is a function of nullity, it follows that in general $\Gamma\xi_{22}$ vanishes to a higher order than Π along $r = s$, and equation (13) is not satisfied. By properly choosing the values of ξ and ξ_2, however, we may still have solutions.

By successive differentiation of (12) and elimination of all partial derivatives except those with regard to y, by means of the relation $y = \varphi(x)$, we have the equation

$$\Gamma\xi_{222} + \Pi_1 = 0, \qquad \Gamma\xi_{2222} + \Pi_2 = 0, \qquad \cdots, \tag{17}$$

in which

$$\Pi_i = \left(\alpha_{12} - 2\alpha_{11}\frac{dy}{dx}\right)\frac{d}{dx}\left(\frac{\partial^{i+1}\xi}{\partial y^{i+1}}\right) + \mu_i, \tag{18}$$

where μ_i does not involve differentiation along $y = \varphi(x)$ of $\partial^{i+1}\xi / \partial y^{i+1}$. Hence, if all the derivatives of ξ of order up to i are known, Π_i becomes an integro-differential expression of the first order in $\partial^{i+1}\xi / \partial y^{i+1}$. We may also write

$$\Pi = \alpha_{11}\left(\frac{d^2\xi}{dx^2} - 2\frac{dy}{dx}\frac{d\xi_2}{dx} - \frac{d^2y}{dx^2}\xi_2\right) + \alpha_{12}\frac{d\xi_2}{dx} + \lambda. \tag{18'}$$

Let us assume that we are able to choose our arbitrary values so that

$$j\text{-coefficient } \Gamma = 0. \tag{19}$$

From (13) it follows that necessarily

$$\Pi = 0, \tag{20}$$

and from (17),

$$\Pi_1 = 0, \qquad \Pi_2 = 0, \qquad \cdots. \tag{21}$$

We must then, if we choose values to satisfy (19), so choose them that they also satisfy (20) and (21).

We may regard (19) and (20) as simultaneous equations that hold along $y = \varphi(x)$ for ξ and ξ_2. In fact, if in the expression for j-coefficient Γ we substitute for the ξ_1 that appears in α_{11}, α_{12}, α_{22}, by means of the formula

$$\xi_1 = \frac{d\xi}{dx} - \frac{dy}{dx}\xi_2,$$

we get for (19) an integro-differential equation of order zero in ξ_2 and of the first order in ξ. Considered as an equation to determine ξ_2, it is an integral equation, and may generally be solved for ξ_2. On substituting this value in (20), we have an integro-differential equation of the second order to determine ξ, simpler than those discussed here, since it involves differentiation with regard to but one variable. In general, then, *if* (19) *holds we may assign at one point of the characteristic arbitrary values of* ξ *and* $d\xi / dx$, *whence will be determined the values of* ξ *and* ξ_2 *all along the curves.* The equations $\Pi_1 = 0$, $\Pi_2 = 0$, $\cdots$ now become integro-differential equations of the first order to determine ξ_{22}, ξ_{222}, $\cdots$ respectively. Therefore *we may assign arbitrarily the values of* ξ_{22}, ξ_{222}, $\cdots$ *at one point of the characteristic, whence their values will be determined all along the curve.*

14. These results may be generalized in various ways. The extension is immediate to the case where instead of x, y we have n variables x_1, x_2, $\cdots$, x_n. The characteristics become spaces of $n - 1$ dimensions,

$$x_n = f_i(x_1, x_2, \cdots, x_{n-1}),$$

and provided that the given $(n-1)$-space

$$x_n = \varphi(x_1, x_2, \cdots, x_{n-1})$$

contains no $(n-1)$-dimensional element of a characteristic, arbitrarily assigned values of ξ and $\partial\xi / \partial x_n = \xi_n$ determine a solution of the integro-differential equation throughout the n-space. In a characteristic space the values of ξ and $\partial\xi / \partial x_n$ may no longer be chosen arbitrarily. If they are taken so that (19) is satisfied, then along an $(n-2)$-space

$$x_{n-1} = \Psi(x_1, x_2, \cdots, x_{n-2})$$

values of ξ, $\partial\xi / \partial x_{n-1}$, ξ_{nn}, $\cdots$ may be chosen arbitrarily, their values being then determined by the integro-differential equation throughout the given $(n-1)$-space. Here by $\partial\xi / \partial x_{n-1}$ we mean the total derivative in the $(n-1)$-space. In fact, the equation (19) may be rewritten as an integral equation in ξ_n, as before it was rewritten for ξ_2, and the result of substituting in (20) is an integro-differential equation of the second order in ξ. This equation and the equations (21) are integro-differential equations of the kind treated in this article, referred to $n-1$ variables of differentiation. Thus a formal development for a solution is obtained.

In the extension of the results for equations of the second order to those of higher order, or from equations linear in the derivatives of highest order to those that are not, there is no difference in the case that we are now treating from the similar extensions in the case of differential equations.

BOSTON, *July* 26, 1913.