THE MINIMUM OF A DEFINITE INTEGRAL FOR UNILATERAL VARIATIONS IN SPACE*

BY

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For the problem of the calculus of variations with fixed end points, the curves which minimize an integral of the form

\[ J = \int F(x, y, z, x', y', z') \, ds \]

have previously been studied, provided that the minimizing arc along which the integral \( J \) is taken is free to traverse any portion of space or restricted to lie entirely within a given region \( R \). It may happen, however, that while no extremal joining the two fixed end points exists, which is entirely interior to a region \( R \), yet there may be a minimizing curve consisting partly of arcs interior to the region and partly of arcs along its boundary. This gives rise to the problem of investigating the properties which characterize a minimizing curve lying partly on a given surface \( S \), while the variations with which it is compared are restricted to lie on one side of the surface, a problem analogous to one in the plane which has been discussed by Bliss.† For this problem in space a number of necessary conditions have been derived by other writers, but the theory of the analogue of Jacobi's condition is still incomplete.

FIG. 1.

* Presented to the Society, December 27, 1913.
† These Transactions, vol. 5 (1904), pp. 477–492.

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Suppose that the arc $E$, whose minimizing properties are to be investigated, consists of three parts $E_{01}$, $E_{12}$, and $E_{23}$, as shown in the accompanying figure (Fig. 1). The arc $E_{12}$ must be an extremal of the integral $J$ on the surface $S$, while the other two, lying on the same side of $S$ and joining the end points of $E_{12}$ to the two fixed points 0 and 3, are necessarily extremals in space and tangent to the arc $E_{12}$. For the corresponding situation in the plane, the analogue of Jacobi's condition requires only that Jacobi's condition itself be satisfied by each of the arcs $E_{01}$ and $E_{23}$. For the problem here considered, however, there is in general more required than this. It is not sufficient, in order that $E$ shall minimize the integral, that no conjugate point to 0 lies on the extremal arc $E_{01}$, and no conjugate point to 2 on the arc $E_{23}$. *

It will be shown in the following pages that from the two parameter family of space extremals passing through the point 0, a set containing $E_{01}$ and depending upon a single parameter may be selected, each curve of which is tangent to the surface $S$ at a point of a curve $C$ through the point 1. Through the points of $C$ a one parameter family of extremals on $S$ can be passed, each one of which is tangent at a point of $C$ to one of the space extremals through the point 0. Again at any point of one of the single infinitude of surface extremals so determined, there is a space extremal tangent to the surface extremal and lying on the required side of the surface. These form a two parameter family of space extremals tangent to $S$.

Two new conditions, analogues of the Jacobi condition in the plane case, will be established; (1) the arc $E_{12}$ must not contain a point of contact with the enveloping curve of the one-parameter family of surface extremals above mentioned; and (2) the arc $E_{23}$ must not contain a contact point with the enveloping surface of the two parameter family of space extremals constructed as above tangent to the surface $S$.

The necessary conditions hitherto derived and the two analogues to the Jacobi condition described in the preceding paragraph, when strengthened in a manner customary in the calculus of variations, are sufficient to insure that the curve $E$ actually minimizes the integral $J$.

1. Necessary Conditions Hitherto Derived

Let the curve $E$ joining the points 0 and 3, whose minimizing properties are to be studied, have the equations

\[ (E) \quad x = x(s), \quad y = y(s), \quad z = z(s) \quad (s_0 \leq s \leq s_1), \]

where $s$ is the length of arc measured from the point 0. The curve is supposed

* See a remark by Hadamard in his Leçons sur le calcul des variations, p. 457.
not to intersect itself and is made up of three pieces, each of class $C''$, as shown in Fig. 1.

The surface $S$, on which $E_{12}$ lies, has equations of the form

$$(S) \quad x = \xi(\alpha, \beta), \quad y = \eta(\alpha, \beta), \quad z = \zeta(\alpha, \beta).$$

It has no points in common with $E_{01}$ and $E_{23}$ except 1 and 2, and is further supposed to have no multiple or singular points and to be of class $C'''$ in the vicinity of $E_{12}$.

The function $F$ in the integral $J$ is assumed to have the following properties in a region $R$ of points $(x, y, z, x', y', z')$ for which $(x, y, z)$ ranges over a neighborhood of $E$, while $(x', y', z')$ are any values not all zero:

(A) $F$ is of class $C''$;

(B) $F(x, y, z, \kappa x', \kappa y', \kappa z') = \kappa F(x, y, z, x', y', z')$ $(\kappa > 0)$;

(C) $F_1 = 0$.

The property (B) is the usual homogeneity property of the function $F$ which makes the value of the integral independent of the parametric representation of the curve along which it is taken, while (C) specifies that the problem is a regular one.†

If $E$ is to minimize $J$, then it must satisfy a number of necessary conditions which are already known and which may be stated as follows:

(I) The arcs $E_{01}$ and $E_{23}$ must be space extremals satisfying the differential equations

$$J^{(x)} = F_x - \frac{d}{ds} F_x' = 0, \quad J^{(y)} = F_y - \frac{d}{ds} F_y' = 0,$$

$$J^{(s)} = F_s - \frac{d}{ds} F_s' = 0;$$

(II) Along the arc $E_{12}$ the equations

$$\frac{J^{(x)}}{X} = \frac{J^{(y)}}{Y} = \frac{J^{(s)}}{Z} = \lambda(s)$$

must be satisfied, where $X, Y, Z$ are the direction cosines of the normal of the surface $S$ in the direction toward the side of $S$ on which $E_{01}$ and $E_{23}$ lie, while $\lambda(s)$ is merely the notation for the function of $s$ defined by the three ratios;

* An arc is said to be of class $C'(n)$ if the derivatives of the functions defining it up to and including those of order $n$ are continuous, and if the expression $\sqrt{x^2 + y^2 + z^2}$ is everywhere different from zero along it.

† For the definition of $F_1$ and a "regular problem" see a paper by Mason and Bliss, these Transactions, vol. 9 (1908), p. 444. This paper will be referred to hereafter as M. and B.
(III) Along $E_{12}$ the inequality $\lambda(s) \leq 0$ must hold;
(IV) The function
\[
E(x, y, z; x', y', z'; p, q, r) = F(x, y, z, p, q, r) - pF_x'(x, y, z, x', y', z') - qF_y'(x, y, z, x', y', z') - rF_z'(x, y, z, x', y', z')
\]

must be positive or zero for all values $(x, y, z, x', y', z')$ on the arc $E_{03}$ and for arbitrary values $(p, q, r) \neq (0, 0, 0)$;
(V) At the points 1 and 2, $E_{01}$ and $E_{23}$ must be tangent to the arc $E_{12}$;
(VI) The arc $E_{01}$ must not contain a point conjugate to 0 between 0 and 1.*

From the property IV it follows that the quadratic form
\[
Q(x, y, z; x', y', z'; p, q, r) = F_{x'x'}p^2 + F_{y'y'}q^2 + F_{z'z'}r^2 + 2F_{x'y'}qp + 2F_{x'z'}qr + 2F_{y'z'}rp
\]

must be positive or zero for the arguments described for $E$.† The sign of $F_1$ must therefore be positive, and it can be shown that $Q \geq 0$ for all values $(x, y, z, x', y', z')$ in the region $R$ and for arbitrary directions $(p, q, r)$.$‡$

Furthermore $Q$ vanishes only when the directions $x', y', z'$ and $p, q, r$ are in the same line.§ The same properties hold for the $E$-function, with the further restriction that $E$ vanishes only when $x', y', z'$ and $p, q, r$ are identical (not opposite) directions.||

The condition II can also be stated in terms of the integrand
\[
f(\alpha, \beta, \alpha', \beta') = F(\xi, \eta, \xi' + \xi_\alpha \beta', \eta + \eta_\beta \beta', \xi_\alpha \alpha' + \xi_\beta \beta')
\]
of the integral
\[
J = \int f(\alpha, \beta, \alpha', \beta') \, ds
\]

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* For the conditions (I), (IV), (VI) on the arcs $E_{01}$ and $E_{12}$, see M. and B., loc. cit., pages 443, 455, 453, respectively. For (II), (III), (V), see Hadamard, loc. cit., pages 133, 179, 456.

† The argument is like that of M. and B., p. 456, for the functions $F_{x'x'}, F_{y'y'}, F_{z'z'}$.

‡ M. and B., p. 456, formulas (40).


|| The proof of this statement may be made by means of the formulas in M. and B., but more elegantly by the use of the results of Behaghel, Mathematische Annalen, vol. 73 (1913), p. 596. The functions $A, B, C$ used by him may become infinite for directions parallel to the axes and the validity of his proof is questionable in such cases, though his result is correct. The justification for these statements will appear in the near future in a paper by Bliss.
taken along curves on the surface $S$. Along $E_{12}$ the parameters $\alpha$ and $\beta$ are functions
\[ \alpha = \alpha(s), \quad \beta = \beta(s) \quad (s_1 \leq s \leq s_2) \]
which are of class $C''$. After an easy calculation the Euler equations for
the integral (3) are seen to have the form
\[
\begin{align*}
    f_\alpha - \frac{d}{ds} f_\alpha' &= J^{(2)} \xi_\alpha + J^{(2)} \eta_\alpha + J^{(2)} \zeta_\alpha = 0, \\
    f_\beta - \frac{d}{ds} f_\beta' &= J^{(2)} \xi_\beta + J^{(2)} \eta_\beta + J^{(2)} \zeta_\beta = 0,
\end{align*}
\]
and these are satisfied along $E_{12}$ on account of the condition II.

Furthermore the function*
\[ f_1 = \frac{f_\alpha' \alpha' - f_\alpha \alpha'^2 - f_\beta' \beta'^2}{\alpha'^2} \]
has the values
\[ f_1 = \frac{Q(x, y, z; x', y', z'; \xi_\alpha, \eta_\alpha, \zeta_\alpha)}{\alpha'^2}, \]
where
\[ x' = \xi_\alpha \alpha' + \xi_\beta \beta', \quad y' = \eta_\alpha \alpha' + \eta_\beta \beta', \quad z' = \xi_\alpha \alpha' + \xi_\beta \beta'. \]

One of these expressions is always well defined and different from zero, since the derivatives $x'$, $y'$, $z'$ can coincide with at most one of the coordinate directions on the surface. The problem of minimizing the integral (3) on the surface is therefore a regular one, and $E_{12}$ is one of its extremals.

If a space extremal satisfying the condition I is tangent to a surface extremal satisfying II, then at the point of contact there is a relation between their curvatures which will be of service later. Let the elements of the second order on the two extremals at the point of contact be, respectively,
\[ (x, y, z, x', y', z', x'', y'', z''), \]
and
\[ (x, y, z, x', y', z', x'', y'', z''). \]

From equations (1) and (2) it follows that
\[
\begin{align*}
    \lambda X &= J^{(2)} - \widetilde{J}^{(2)}, \\
    \lambda Y &= J^{(2)} - \widetilde{J}^{(2)}, \\
    \lambda Z &= J^{(2)} - \widetilde{J}^{(2)},
\end{align*}
\]
and
\[
\begin{align*}
    \lambda [X(x'' - \tilde{x}'') + Y(y'' - \tilde{y}'') + Z(z'' - \tilde{z}'')] \\
    &= (x'' - \tilde{x}'') (J^{(2)} - \widetilde{J}^{(2)}) + (y'' - \tilde{y}'') (J^{(2)} - \widetilde{J}^{(2)}) + (z'' - \tilde{z}'') (J^{(2)} - \widetilde{J}^{(2)}) \\
    &= Q(x, y, z; x', y', z'; x'' - \tilde{x}'', y'' - \tilde{y}'', z'' - \tilde{z}'').
\end{align*}
\]

* See Bolza, Vorlesungen über Variationsrechnung, p. 196.
This last expression can vanish, however, only when
\[ x'' - x'' = \kappa x', \quad y'' - y'' = \kappa y', \quad z'' - z'' = \kappa z', \]
where \( \kappa \) is a value which may be zero. In case it does vanish the equations (5) show that \( \lambda = 0 \), as a result of the homogeneity properties of the function \( F \).\footnote{M. and B., p. 441, formula (5).} Hence the following lemma is true:

**Lemma 1:** If \( \lambda \neq 0 \) at the point of contact of a surface extremal with a space extremal, then at that point the inequality
\[ Xx'' + Yy'' + Zz'' + Xi' + Yy' + Zz' \]
holds; or, in other words, the projections of the first curvatures, \( 1/r \) and \( 1/r' \), of the two extremals on the normal to the surface are different.

From this point in the paper it will be assumed that the arc \( E_{03} \) satisfies the conditions I, II, IV, V and the modified conditions
(III') \( \lambda(s) > 0 \) along \( E_{12} \);
(VI') the arc \( E_{01} \) does not contain a point conjugate to 0 between 0 and 1 or at 1.\footnote{The condition VI' is necessary except in one singular case. See M. and B., p. 453.}

On the basis of these assumptions further conditions, analogous to Jacobi's conditions for the usual form of the problem, will be derived.

2. **The Three Families of Extremals Associated with \( E_{03} \)**

The totality of solutions of the differential equations (1) passing through 0 contains two arbitrary parameters. As a result of the hypotheses made upon the function \( F \) they may be represented in the form
\[ x = \phi(s, u, v, w), \quad y = \psi(s, u, v, w), \quad z = \chi(s, u, v, w), \]
\[ u^2 + v^2 + w^2 = 1, \]
the parameters \( u, v, w \) being the direction cosines of the positive tangent to the extremal at the point 0. The particular extremal \( E_{01} \) is contained in this set for \( u = u_0, v = v_0, w = w_0 \), and the functions \( \phi, \psi, \chi, \phi_s, \psi_s, \chi_s \) are of class \( C' \) in neighborhoods of the two sets of parameter values
\[ s = 0, \quad u^2 + v^2 + w^2 = 1; \]
\[ 0 \leq s \leq s_1, \quad u = u_0, \quad v = v_0, \quad w = w_0 \]
of which the second set defines the curve \( E_{01} \). At the point 0, where \( s = 0 \), the following identities in \( u, v, w \) hold
\[ x_0 = \phi(0, u, v, w), \quad y_0 = \psi(0, u, v, w), \quad z_0 = \chi(0, u, v, w). \]
Let the notation $\Delta(s, u, v, w)$ designate the determinant

$$
\Delta(s, u, v, w) = \begin{vmatrix}
\phi_s & \phi_u & \phi_v & \phi_w \\
\psi_s & \psi_u & \psi_v & \psi_w \\
\chi_s & \chi_u & \chi_v & \chi_w \\
0 & u & v & w
\end{vmatrix}.
$$

On account of the condition VI' it follows* that

$$(7) \quad \Delta(s, u_0, v_0, w_0) \neq 0 \quad (0 < s \equiv s_1).$$

The equations

$$
\xi(\alpha, \beta) + \omega X = \phi(s, u, v, w), \\
\eta(\alpha, \beta) + \omega Y = \psi(s, u, v, w), \\
\zeta(\alpha, \beta) + \omega Z = \chi(s, u, v, w),
$$

have an initial solution $(\alpha, \beta, \omega; s, u, v, w) = (\alpha_1, \beta_1, 0; s_1, u_0, v_0, w_0)$ corresponding to the point 1, at which the functional determinant for $\alpha, \beta, \omega$ is different from zero. Hence these equations have solutions

$$(8) \quad \alpha = \alpha(s, u, v, w), \quad \beta = \beta(s, u, v, w), \quad \omega = \omega(s, u, v, w)$$

of class $C''$ near $(s_1, u_0, v_0, w_0)$ and reducing to $(\alpha_1, \beta_1, 0)$ at these values. The derivatives of $\omega$ at the point 1 are readily calculated to be

$$(9) \quad \omega_s = X\phi_s + Y\psi_s + Z\chi_s, \quad \omega_u = X\phi_u + Y\psi_u + Z\chi_u, \quad \omega_v = X\phi_v + Y\psi_v + Z\chi_v, \quad \omega_w = X\phi_w + Y\psi_w + Z\chi_w,$$

$$\omega_{st} = X\phi_{st} + Y\psi_{st} + Z\chi_{st} - 2M\alpha_s \beta_s - N\beta_s^2 \quad \omega_{uv} = X\phi_{uv} + Y\psi_{uv} + Z\chi_{uv},$$

where the notations in the last line are the same as those of the preceding section. The set $(x'', y'', z'')$ belongs to $E_{01}$ at the point 1, and the set $(\bar{x}'', \bar{y}'', z'')$ to $E_{12}$.

The solutions $(s, u, v, w)$ of the equations

$$(10) \quad \omega(s, u, v, w) = 0, \quad \omega_s(s, u, v, w) = 0, \quad u^2 + v^2 + w^2 = 1$$

determine points where the extremal (6) is tangent to the surface $S$. On account of the tangency of $E_{01}$ and $E_{12}$, it is evident that among the solutions are the values $(s_1, u_0, v_0, w_0)$. On the other hand the determinants of the matrix

$$
\begin{vmatrix}
\omega_{ss} & \omega_{su} & \omega_{sv} & \omega_{sw} \\
\omega_s & \omega_u & \omega_v & \omega_w \\
0 & u & v & w
\end{vmatrix},
$$

* See M. and B., §3 and p. 453.
† For the definitions of $L, M, N$ and their relations to the first curvature, see Scheffers, *Anwendung der Differential und Integral-Rechnung auf Geometrie*, vol. 2, page 494 (D).
cannot all vanish at \((s_1, u_0, v_0, w_0)\). At these values \(\omega_s = 0\) since \(E_{01}\) is
tangent to the surface \(S\); but from equations (9) and the property (7) it follows
readily that the determinants of the last two rows can not all vanish; and
from III' and the lemma of the preceding section, \(\omega_s\) is different from zero.
The equations (10) therefore have solutions for \(s\) and two of the variables
\(u, v, w\) in terms of the third, passing through the values \((s_1, u_0, v_0, w_0)\) and
of class \(C\) near them.

These solutions and the equations (8) determine a one parameter family of
values
\[
\alpha = \alpha(\tau), \quad \beta = \beta(\tau), \quad s = \sigma(\tau),
\]
\[
u = \nu(\tau), \quad v = v(\tau), \quad w = w(\tau)\]
of class \(C\) near \(\tau = 0\), which in turn determine a curve
\[(C)\]
\[
\alpha = \alpha(\tau), \quad \beta = \beta(\tau)
\]
on the surface \(S\) and a one parameter family
\[
x = \phi[s, u(\tau), v(\tau), w(\tau)] = \phi(s, \tau),
\]
\[
y = \psi[s, u(\tau), v(\tau), w(\tau)] = \psi(s, \tau),
\]
\[
z = \chi[s, u(\tau), v(\tau), w(\tau)] = \chi(s, \tau),
\]
of the extremals (6). Through each point of \(C\) there passes a unique extremal
of the set tangent to the surface \(S\). The functions \(\phi(s, \tau)\) and \(\phi_s(s, \tau)\) are
of class \(C\) near the values \(s_0 \leq s \leq s_1, \tau = 0\), which define the arc \(E_{01}\), with
similar properties for \(\psi\) and \(\chi\).

On account of the hypotheses on the function \(F\) and the resulting properties
of \(f\), it is possible to prove the existence of a one parameter family of extremals
on the surface \(S\) satisfying the differential equations II or (4), and such that
each extremal passes through a point of \(C\), and has the direction determined
by the corresponding space extremal of the set (11) tangent to \(S\) at that
point. Let the length of arc along the surface extremal be denoted by \(\sigma\) and
be measured so that \(\sigma - \sigma(\tau)\) is the length of arc from \(C\).

There exists, then, a one parameter family of surface extremals with equations
on the surface of the form
\[(12)\]
\[
\alpha = A(\sigma, \tau), \quad \beta = B(\sigma, \tau),
\]
or, in space, of the form
\[
x = \xi[A(\sigma, \tau), B(\sigma, \tau)] = \xi(\sigma, \tau),
\]
\[
y = \eta[A(\sigma, \tau), B(\sigma, \tau)] = \eta(\sigma, \tau),
\]
\[
z = \xi[A(\sigma, \tau), B(\sigma, \tau)] = \xi(\sigma, \tau),
\]

* The parameter \(\tau\) will be \(u - u_0, v - v_0\) or \(w - w_c\).
where the functions \( \xi(\sigma, \tau) \) and \( \xi_{e}(\sigma, \tau) \) are of class \( C' \) near the values
\[
s_{1} \leq \sigma \leq s_{2}, \quad \tau = 0,
\]
defining the arc \( E_{12} \). Along the curve \( C \) the relations
\[
\phi [\sigma(\tau), \tau] = \xi [\sigma(\tau), \tau], \quad \phi_{e} [\sigma(\tau), \tau] = \xi_{e} [\sigma(\tau), \tau],
\]
with similar ones for \( \psi \) and \( \chi \), are identities in \( \tau \). They express the fact that at a point of \( C \) the surface extremal (12) and the space extremal (11) are tangent to each other.

Through each point of an extremal on the surface \( S \) determined by the equations (12), there passes a space extremal whose direction at the point of intersection coincides with that of the extremal on the surface. This extremal must satisfy the differential equations I. If its length of arc \( s \) is measured from the initial value \( \sigma \) at the point of contact with the surface, the existence theorems for differential equations and the properties of the function \( F \) justify the following theorem:

**Theorem.** There exists a two parameter family of space extremals tangent to the extremals (13) on the surface \( S \), and having equations of the form
\[
(14) \quad x = \Phi (s, \sigma, \tau), \quad y = \Psi (s, \sigma, \tau), \quad z = X(s, \sigma, \tau),
\]
where \( \Phi \) and \( \Phi_{e} \) are of class \( C' \) near the values
\[
s = \sigma, \quad s_{1} \leq \sigma \leq s_{2}, \quad \tau = 0,
\]
\[
s_{2} \leq s \leq s_{3}, \quad \sigma = s_{2}, \quad \tau = 0,
\]
defining, respectively, the arcs \( E_{12} \) and \( E_{23} \). Similar properties hold for \( \Psi \) and \( X \).

The identities
\[
(15) \quad \Phi(\sigma, \sigma, \tau) = \xi(\sigma, \tau), \quad \Phi_{e}(\sigma, \sigma, \tau) = \xi_{e}(\sigma, \tau),
\]
with similar ones for the other functions, express the tangency of (14) with the surface extremal (13) at their point of intersection.

The identities (15) give rise to a number of further relations important in the following pages. By differentiating the first one and applying the second, it follows that
\[
\Phi_{e}(\sigma, \sigma, \tau) = 0, \quad \Phi_{e}(\sigma, \sigma, \tau) = \xi_{e}(\sigma, \tau),
\]
and by differentiating the second,
\[
\Phi_{e}(\sigma, \sigma, \tau) = \xi_{e}(\sigma, \tau) - \Phi_{e}(\sigma, \sigma, \tau) = \ddot{z} - x'^{2}
\]
with similar relations for \( \Psi \) and \( X \). The determinant
\[
\Delta(s, \sigma, \tau) = \begin{vmatrix}
\Phi_{e} & \Phi_{e} & \Phi_{e} \\
\Phi_{e} & \Phi_{e} & \Phi_{e} \\
X_{e} & X_{e} & X_{e}
\end{vmatrix}
\]
has therefore the derivative
\[
\Delta_s(\sigma, \sigma, \tau) = \begin{vmatrix} \Phi_s & \Phi_{sr} & \Phi_r \\ \Psi_s & \Psi_{sr} & \Psi_r \\ X_s & X_{sr} & X_r \end{vmatrix} = \begin{vmatrix} x'' - x'' \xi_s \xi_r \\ y'' - y'' \eta_s \eta_r \\ z'' - z'' \zeta_s \zeta_r \end{vmatrix}
\]
for the values \((\sigma, \sigma, \tau)\) along the surface \(S\). But at a point where the functional determinant of the functions \(A\) and \(B\) is different from zero, the relations (13) show that this has the value
\[
\Delta_s(\sigma, \sigma, \tau) = K \{X(x'' - x'') + Y(y'' - y'') + Z(z'' - z'')\}
\]
where \(K\) is a factor of proportionality which does not vanish. According to the lemma of § 1, the following result may be stated:

**Lemma 2.** The derivative \(\Delta_s(\sigma, \sigma, \tau)\) is different from zero at any point of the curve \(E_{13}\) on the surface \(S\) at which the determinant
\[
\begin{vmatrix} A_s & A_r \\ B_s & B_r \end{vmatrix}
\]
is different from zero.

### 3. Auxiliary Theorems

The computations of the following pages can be considerably simplified by the use of some auxiliary theorems, the first of which may be stated as follows:

**Theorem.** If a family of curves
\[
(17) \quad x = \phi(s, \lambda), \quad y = \psi(s, \lambda), \quad z = \chi(s, \lambda)
\]
satisfies the relation
\[
J^{(a)} \phi_s + J^{(a)} \psi_s + J^{(a)} \chi_s = 0,
\]
then the derivative of the integral
\[
\int_{s_0}^{s_1} F(\phi, \psi, \chi, \phi_s, \psi_s, \chi_s) \, ds,
\]
when \(s_0, s_1,\) and \(\lambda\) are functions \(s_0(v), s_1(v), \lambda(v)\), is
\[
[F_{x_s} x_v + F_{y_s} y_v + F_{z_s} z_v]_0^1.
\]
The arguments of the derivatives of \(F\) are the same as those in the integral, and
\[
(18) \quad x_v = \phi_s s_v + \phi_\lambda \lambda_v, \quad y_v = \psi_s s_v + \psi_\lambda \lambda_v, \quad z_v = \chi_s s_v + \chi_\lambda \lambda_v.
\]
This theorem follows at once by known methods\(^*\) provided that the func-

\(^*\) See M. and B., §2.
tions \( s_0, s_1, \lambda \) are of class \( C' \) for \( |v| \leq \epsilon \), and \( \phi, \phi_* \) of class \( C' \) in the region
\[
\begin{align*}
    s_0(v) &\equiv s \equiv s_1(v), \\
    \lambda &\equiv \lambda(v) \quad (|v| \leq \epsilon),
\end{align*}
\]
with similar properties for \( \psi \) and \( \chi \).

**Theorem.** If the curves of a second family
\[
(19) \quad x = \Phi(t, \mu), \quad y = \Psi(t, \mu), \quad z = X(t, \mu)
\]
adjoint those of the first family for parameter values
\[
\begin{align*}
    s &= s_1(v), \\
    \lambda &= \lambda(v) \quad ; \\
    t &= t_1(v), \\
    \mu &= \mu(v),
\end{align*}
\]
and if corresponding arcs of the two families have the same direction at their common point, then the function
\[
(20) \quad G(v) = \int s_0(\Phi, \Psi, \chi, \phi, \phi_*, \psi_*, \psi_*) \, ds + \int t_1(\Phi, \Psi, \chi, \phi, \phi_*, \psi_*, \psi_*) \, dt,
\]
in which \( t_2 \) is a function of \( v \) as well as \( t_1 \) and \( \mu \), has the derivative
\[
(21) \quad G'(v) = \left[ F_{x'} x_0 + F_{y'} y_0 + F_{z'} z_0 \right]_0.
\]
The expressions in the bracket must be formed at the point 0 for the first family, and at the point 2 for the second, as described in the first theorem.

The derivative of \( G(v) \) has the value
\[
\left[ F_{x'} x_0 + F_{y'} y_0 + F_{z'} z_0 \right]_0 + \left[ F_{x'} x_0 + F_{y'} y_0 + F_{z'} z_0 \right]_0,
\]
from the first theorem. At the point 1 the two brackets cancel on account of the homogeneity properties of \( F \) and the relations
\[
\begin{align*}
    \phi \left[ s_1(v), \lambda(v) \right] &= \Phi \left[ t_1(v), \mu(v) \right], \\
    \phi_* \left[ s_1(v), \lambda(v) \right] &= \kappa(v) \Phi_* \left[ t_1(v), \mu(v) \right],
\end{align*}
\]
with similar ones for the other functions, which express the intersection and the tangency of the two families. It is understood that the continuity properties of the two families are like those described above for the first.

A similar theorem evidently holds when the number of families is more than two. If the curves of the first family all pass through a fixed point 0 for \( s = s_0(v), \lambda = \lambda(v) \), the bracket in (21) vanishes at 0 since the same is true of the derivatives of (17). If the curves of the second family similarly pass through a fixed point 2 for \( t = t_0(v), \mu = \mu(v) \), then \( G(v) \) is a constant.

The family (19) may be independent of \( \mu \), in which case it will be simply an enveloping curve of the family (17). If in this case \( t_0(v) \) is a constant, the second of the integrals (20) will be simply the value of \( J \) taken along the envelope from 1 to a fixed point 2, and the function \( G(v) \) will again be a constant.
4. Jacobi's Conditions for Unilateral Variations

As a result of the assumptions made on the curve $E_{02}$ at the end of §1, it is possible to prove the following further necessary conditions for a minimum:

(VII) The curve $E_{03}$ must not have a point of contact between 1 and 2 with an enveloping curve $d$ of the family (13) of extremals on the surface $S$.

The proof will be made here for the case where the enveloping curve $d$ does not have a singular point at its point of contact with $E_{02}$. In that case a stronger condition holds, namely,

(VII') The curve $E_{03}$ must not have a point of contact with an enveloping curve $d$ either between 1 and 2 or at 2.

If the curve $d$ has a singular point at its contact point with $E_{02}$, the condition VII still holds and can be proved by other methods; but the case where $d$ touches $E$ at the point 2 requires, as usual, a special investigation.

The equation which defines $d$ on the surface $S$ is

$$\Delta (\sigma, \tau) = \begin{vmatrix} A_\sigma & A_\tau \\ B_\sigma & B_\tau \end{vmatrix} = 0,$$

where $A$ and $B$ are the functions (12) defining the extremals on the surface. If this is satisfied at a point $(\sigma, \tau) = (\sigma_0, 0)$ of the arc $E_{12}$, the derivative $A''$ will be different from zero at that point, as is known from the theory of the plane problem applied to the integral (3). The last equation has therefore a solution

$$\sigma = \Sigma (u), \quad \tau = T (u),$$

reducing to $(\sigma_0, 0)$ for $u = 0$ and of class $C'$ near that value. The equations of the envelope in space will be

$$(d) \quad x = \xi [\Sigma (u), T (u)], \quad y = \eta [\Sigma (u), T (u)], \quad z = \xi [\Sigma (u), T (u)].$$

The parameter $u$ is chosen as one of the values $\pm \tau$, but so as to make the positive direction along $d$ the same as that of the surface extremal tangent to it.

For an arbitrary value of $u$ the corresponding value of $\tau$ defines an extremal $E_{04}$ of the set (11) which is tangent to the corresponding extremal $E_{45}$ of the set (13) at a point 4 of the curve $C$ as shown in Fig. 2. Further, $E_{45}$ is tangent to $d$ at a point 5, and $d_{66}$ touches $E_{03}$ at the point 6. The families (11), (13), and $d$ are then of the form described in §3 when $u = v$, and the function

$$G (u) = J (E_{04}) + J (E_{45}) + J (d_{66}) + J (E_{63})$$

is a constant, where $E_{45}$ denotes the arc of $E_{03}$ between 6 and 3. An argument precisely like that used for problems in the plane, shows that $d_{66}$ can not be an extremal on the surface, and since

$$G (0) = J (E_{03})$$

it follows that $E_{03}$ can not minimize $J$ if VII is not satisfied.
If the condition VII' is satisfied together with those postulated at the end of §1, a final necessary condition is the following:

(VIII) The curve $E_{03}$ must not have a point of contact between 2 and 3 with an enveloping surface $D$ of the family (14) of extremals tangent to $S$.

Fig. 2.

Here again the proof will be made for the case where $D$ does not have a singular point at its contact with $E_{03}$, and the stronger condition holding under those circumstances is

(VIII') The curve $E_{03}$ must not have a point of contact with $D$ either between 2 and 3 or at 3.

The points on the enveloping surface are defined by the equation

$$A(s, (\tau, t) = 0.$$ (22)

On account of Lemma 2 of §2, the determinant is different from zero on the arc $E_{23}$ near the point 2, though it vanishes at 2 itself. If the equation is satisfied at a point 7 of the arc $E_{23}$ for which $(s, \sigma, \tau) = (s_1, s_2, 0)$, and further defines a non-singular enveloping surface $D$, then the argument of Mason and Bliss* shows that there is a curve on $D$,

$$x = x(u), \quad y = y(u), \quad z = z(u),$$ (d)

passing through the point 7 for $u = 0$, and which is tangent to a one parameter family of the extremals (15) at points defined by equations of the form

$$s = S(u), \quad \sigma = \Sigma(u), \quad \tau = T(u),$$

* See M. and B., §3.
where $S, \Sigma, T$ reduce to $s_1, s_2, 0$ for $u = 0$. The parameter $u$ can be so chosen that the positive direction on the curve $d$ coincides with the positive direction of the extremal (15) at the point defined by these functions.

An arbitrarily chosen value $u$ defines an extremal $E_{04}$ of the set (11) tangent to an extremal $E_{45}$ of the set (12) at a point $4$ of the curve $C$; and $E_{45}$ is in turn tangent at a point $5$ to the extremal $E_{56}$ defined by the equations

$$x = \Phi(s, \Sigma, T), \quad y = \Psi(s, \Sigma, T), \quad z = X(s, \Sigma, T).$$

Finally this last curve is tangent to $d_{67}$ at a point $6$. When $u$ is set equal to $v$, the families (11), (13), (23), and $d_{67}$ are of the kind described in §3, and the function

$$G(u) = J(E_{04}) + J(E_{45}) + J(E_{56}) + J(d_{67}) + J(E_{72})$$

is a constant. Since

$$G(0) = J(E_{03})$$

and $d$ can not be an extremal, it follows that $E_{03}$ can not minimize $J$ if VIII is not satisfied.

5. The Existence of a Field

If the arc $E_{03}$ satisfies the stronger set of eight conditions described in §§1, 4, it will surely minimize the integral $J$ with respect to variations sufficiently near to it and lying on one side of the surface $S$. But before proceeding to the proof of this fact it is necessary to show how a field may be constructed about the minimizing arc.

For this purpose consider first the family (14) which contains the arcs $E_{12}$ and $E_{23}$, respectively, for the parameter values

$$s = \sigma, \quad s_1 \leq \sigma \leq s_2, \quad \tau = 0;$$

$$s_2 \leq s \leq s_3, \quad \sigma = s_2, \quad \tau = 0.$$

The symbol $\Pi_i$ will be used to denote the set of points $(s, \sigma, \tau)$ which satisfy either of the following sets of conditions:

$$\sigma \leq s \leq \sigma + \epsilon, \quad |\tau| \leq \epsilon;$$

$$\sigma + \epsilon \leq s \leq s_3 + \epsilon, \quad |\sigma - s_2| \leq \epsilon, \quad |\tau| \leq \epsilon.$$

If the constant $\epsilon$ is chosen sufficiently small, any two distinct points of $\Pi_i$ will be transformed by the equations (14) into distinct points $(x, y, z)$; and in the interior of $\Pi_i$, the determinant (22) will be everywhere different from zero.

To prove this it will first be shown that in a suitably chosen neighborhood of a point $\pi_0(s_0, \sigma_0, \tau_0)$ satisfying the second of the conditions (II), no two distinct points $\pi(s, \sigma, \tau), \pi'(s', \sigma', \tau')$ of the region $\Pi_i$ can define
the same point \((x, y, z)\) by means of the equations (14). By Taylor's formula,*

\[(24) \quad \Phi(s', \sigma', \tau') - \Phi(s, \sigma, \tau) = (s' - s) A_1 + (\sigma' - \sigma) A_2 + (\tau' - \tau) A_3\]

where, for example,

\[(25) \quad A_2 = \int_0^1 \Phi_s(s'', \sigma'', \tau'') \, du,\]

\[s'' = s + u(s' - s), \quad \sigma'' = \sigma + u(\sigma' - \sigma), \quad \tau'' = \tau + u(\tau' - \tau),\]

and the other coefficients have similar values. The expression (24) and the similar ones for \(\Psi\) and \(X\) can vanish simultaneously only when the determinant of the coefficients vanishes. But this is impossible near \(\pi_0\) since the determinant (22) is different from zero at \(\pi_0\) and the derivatives of \(\Phi, \Psi, X\) are continuous.

Near a point \(\pi_0\) satisfying the first of the conditions \(\Pi\) the expression (25) for \(A_2\) can be put in the form

\[A_2 = \int_0 (s'' - \sigma'') \int_0 \Phi_{s\sigma}(s'' - \sigma, \sigma'', \tau''), \quad s'', \sigma'', \tau''\] \[du \, dv,\]

since \(\Phi_{s\sigma}\) vanishes when its first two arguments are the same. The point \((s'', \sigma'', \tau'')\) is always in \(\Pi\), when \(\pi\) and \(\pi'\) are in \(\Pi\), and \(0 \leq u \leq 1\), and hence \(s'' - \sigma''\) is never negative. By a double application of the mean value theorem for a definite integral

\[A_2 = (s_1 - \sigma_1) \Phi_{s\sigma},\]

where the arguments of \(\Phi_{s\sigma}\) contain properly chosen intermediate values for \(u\) and \(v\), and \((s_1, \sigma_1, \tau_1)\) is the middle point of the segment \(\pi \pi'\).

Near a point \(\pi_0\) satisfying the second of the conditions \(\Pi\) the difference (24) can therefore be put into the form

\[(26) \quad (s' - s) A_1 + (\sigma' - \sigma) (s_1 - \sigma_1) A_2 + (\tau' - \tau) A_3,\]

where \(A_1, A_2, A_3\) approach continuously the values of \(\Phi_s, \Phi_{s\sigma}, \Phi_{s\tau}\) at the point \(\pi_0\) when \(\pi\) and \(\pi'\) approach \(\pi_0\). This expression and the similar ones for \(\Psi\) and \(X\) can vanish simultaneously, for \(\pi\) and \(\pi'\) sufficiently near to \(\pi_0\), only when

\[s' - s = \tau' - \tau = (\sigma' - \sigma) (s_1 - \sigma_1) = 0,\]

since at the point \(\pi_0\) the determinant (22) is different from zero. But if \(s' = s\) and \(\tau' = \tau\), the values of \(\sigma'\) and \(\sigma\) must be distinct since \(\pi\) and \(\pi'\) are distinct. The difference \(s_1 - \sigma_1\) can not vanish since \(s_1 = s = s'\) is greater than the larger of \(\sigma\) and \(\sigma'\) on account of the conditions \(\Pi\). Hence near \(\pi_0\) the

difference (24) and the similar ones for $\Psi$ and $X$ cannot all vanish if $\pi$ and $\pi'$ are distinct.

Suppose now that it should not be possible to choose a constant $\epsilon$ such as is described in the first part of the italicized statement, and consider a sequence $\{\epsilon_k\}$ of decreasing positive constants having zero as its limit. For any choice of $k$ there would exist at least one pair of distinct points,

$$(s, \sigma, \tau)_k, \quad (s', \sigma', \tau')_k$$

in the region $\Pi_k$, and defining the same point $(x, y, z)_k$ by means of the equations (14). The two sequences (27) would have points of condensation

$$(s_0, \sigma_0, \tau_0), \quad (s'_0, \sigma'_0, \tau'_0)$$

necessarily lying in the region $\Pi$, and these two points must be the same. Otherwise they would define distinct points of the arc $E_{13}$, and on account of the continuity of the functions (14) the pairs of values (27) which lie sufficiently near to the values (28) could not define the same point $(x, y, z)_k$.

Neither could the pair of sets of values (27) define the same point $(x, y, z)_k$ in case the two condensation points (28) were identical, as has been seen in the paragraphs just preceding.

A similar argument shows that $\epsilon$ can be chosen so as to satisfy the last part of the italicized statement. Otherwise there would be a sequence $(s, \sigma, \tau)_k$ of interior points of the respective regions $\Pi_k$, at each of which the determinant (22) vanishes. Such a sequence must have a condensation point $\pi_0$ in $\Pi$. But $\pi_0$ could not satisfy the second of the conditions (22), since at such points $\Delta(s, \sigma, \tau)$ is different from zero. Neither could it satisfy the first condition; for near such a point $\pi_0$ the value of $\Delta(s, \sigma, \tau)$ is different from zero, on account of Lemma 2 and because for $s > \sigma$

$$\Delta(s, \sigma, \tau) = (s - \sigma)\Delta_s [s + \theta(s - \sigma), \sigma, \tau] \neq 0.$$  

The points of the sequence $(s, \sigma, \tau)_k$ which condense on $\pi_0$ have $s > \sigma$, since they are interior points, and hence the existence of the sequence is contradicted.

If the region $\Pi$, is chosen as described above, its image in the $xyz$-space, after transformation by equations (14), is a region $\mathcal{F}_2$, a continuum and its boundary, whose boundary points are the images of the boundary points of $\Pi$. The correspondence between $\mathcal{F}_2$ and $\Pi$, is one-to-one. The single valued functions

$$(s, \sigma, \tau) = s(x, y, z), \quad \sigma = \sigma(x, y, z), \quad \tau = \tau(x, y, z)$$

so defined are continuous over $\mathcal{F}_2$, and in the interior of $\mathcal{F}_2$ they have continuous derivatives.

The first part of this statement follows at once from some theorems proved.
by Bliss,* since the images of the interior and boundary of \( \Pi \) are distinct. The preceding paragraphs of this paper show that the correspondence between \( \mathcal{F}_1 \) and \( \Pi \) is one-to-one. An interior point of \( \mathcal{F}_2 \) corresponds to an interior point of \( \Pi \), at which \( \Delta(s, \sigma, \tau) \neq 0 \), and the fundamental theorem of the implicit function theory shows at once that the functions (29) are of class \( C' \) at such points. If a sequence of points \((x, y, z)\) of \( \mathcal{F}_2 \) approaches a boundary point \((x', y', z')\) of \( \mathcal{F}_2 \), the corresponding values \((s, \sigma, \tau)\) defined by the equations (29) must approach the values \((s', \sigma', \tau')\) which define \((x', y', z')\) by means of equations (14), as may readily be seen from the continuity of the functions \( \Phi, \Psi, \chi \). Hence the functions (29) are continuous everywhere in \( \mathcal{F}_2 \).

There exists a field \( \mathcal{F}_1 \) about the arc \( E_{01} \) through each point of which there passes a unique extremal of the set (6) defined by equations of the form

\[
\begin{align*}
    s &= s(x, y, z), & u &= u(x, y, z), \\
    v &= v(x, y, z), & w &= w(x, y, z).
\end{align*}
\]

These functions are continuous everywhere in \( \mathcal{F}_1 \) and have continuous derivatives except at the point 0.†

The symbol \( \mathcal{F}_{12} \) will be used to denote the set of points \((x, y, z)\) consisting of the interior of \( \mathcal{F}_2 \), the points of \( S \) in \( \mathcal{F}_2 \), and the points interior to \( \mathcal{F}_1 \). Let \( V \) be a continuous curve

\[
(V) \quad x = \xi(t), \quad y = \eta(t), \quad z = \zeta(t) \quad (4 \leq t \leq 4)
\]

consisting of a finite number of non-singular arcs of class \( C' \), joining the points 0 and 3, and lying in the region \( \mathcal{F}_{12} \) on the same side of \( S \) with the arc \( E_{01} \).

If \( \mathcal{F}_1 \) is taken sufficiently small the only points where \( V \) can enter \( \mathcal{F}_2 \) are on the boundary \( B \) of \( \mathcal{F}_2 \) which corresponds to the parameter values

\[
(31) \quad \sigma \leq s < \sigma + \epsilon, \quad \sigma = s(t), \quad |\tau| < \epsilon.
\]

A neighborhood of the point 1 can be chosen so small that the only boundary points of \( \mathcal{F}_2 \) in it are points of the surface \( S \) and points on the boundary \( B \) defined by the conditions (31). Further, \( \mathcal{F}_1 \) can be taken so small that the only points common to it and \( \mathcal{F}_2 \) are in the same neighborhood. Under these circumstances the curve \( V \) can enter \( \mathcal{F}_2 \) only at points of \( B \) or at points of \( S \)

† For the existence of the field in the neighborhood of the point 0, see Bliss and Mason, Fields of extremals in space, these Transactions, vol. 11 (1910), p. 328; also references at the beginning of the same paper; also Sasuè, Mathematica Annalen, vol. 71 (1911), p. 380. The existence of the field about the rest of the arc follows by the usual methods. A proof analogous to that just preceding in the text is more elegant than any o these.

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distinct from $B$. The latter are, however, excluded since $V$ lies entirely on one side of $S$ near the point $1$.

6. **Sufficient Conditions for a Minimum**

It is proposed to prove in this section that an arc $E_{03}$ of the kind described at the end of §5, will give the integral $J$ a value at least equal to $J(E_{03})$. To each value of $t$ defining a point $6$ of $V_{03}$ in $S_2$, there will correspond a unique curve

$$E_{06} = E_{04} + E_{45} + E_{56},$$

shown in Fig. 1, where $E_{04}$, $E_{45}$, $E_{56}$ belong respectively to the families (11), (13), and (14); while if $t$ defines a point $6$ which does not lie in $S_2$ it will determine a unique extremal $E_{06}$ of the family (6).

The integral $J(E_{06})$, is a continuous function of $t$ which has the derivative

$$\frac{d}{dt} J(E_{06}) = F_x \cdot x' + F_y \cdot y' + F_z \cdot z' |^6.\ast$$

In this expression the arguments of the derivatives of $F$ are the values $(x, y, z, x', y', z')$ which define the point and direction of $E_{06}$ at the point $6$, while $x'$, $y'$, $z'$ are the derivatives at the same point of the functions $(V)$ defining $V$.

To prove this consider the one parameter family of curves $E_{06}$ determined by the points of $V_{03}$ interior to $S_3$. It is determined by substituting the functions defining $V$ in the expressions (29) for $s$, $\sigma$, $\tau$ in terms of $x$, $y$, $z$. The functions $s(t)$, $\sigma(t)$, $\tau(t)$ so determined are of class $C'$ near a value of $t$ defining an interior point of $S_3$ unless $t$ is one of the exceptional values at which $V$ has a corner point. By substituting $\sigma(t)$, $\tau(t)$ in the equations (11), (13), and (14), the three families $E_{04}$, $E_{45}$, $E_{56}$ are determined with properties similar to those of the auxiliary theorems of §3. It follows readily that the derivative of $J(E_{06})$ has the value given in the theorem.

The argument is the same for a point $6$ in $S_1$ but not in $S_3$, except that the equations whose solutions determine the family $E_{06}$ are then

$$x(t) = \phi(s, u, v, w), \quad y(t) = \psi(s, u, v, w), \quad z(t) = \chi(s, u, v, w),$$

$$u^2 + v^2 + w^2 = 1.$$
If the stronger set of eight properties described in the preceding sections is satisfied by the arc $E_03$, then

$$J(E_03) < J(V_{03})$$

for any curve $V_{03}$ distinct from $E_03$ in the field $F_{13}$ and on the same side of $S$ with the arc $E_03$. It is understood that $V_{03}$ consists of a finite number of non-singular arcs of class $C'$.

For consider the continuous function

$$W(t) = J(E_{03}) + J(V_{03}),$$

where $6$ is the point of $V$ determined by the parameter value $t$. With the help of the last theorem but one, the derivative of $W(t)$ exists except at a finite number of values of $t$, and has itself the value

$$W'(t) = -E(x, y, z; x', y', z'; x', y', z').$$

On account of IV and the properties of the function $E$ this is always negative except when the positive directions along $V_{03}$ and $E_{03}$ coincide at the point 6. Furthermore

$$W(t_0) = J(V_{03}), \quad W(t_1) = J(E_{03}).$$

The inequality (33) must therefore be satisfied unless the positive directions along $E_{03}$ and $V_{03}$ coincide at every point 6 of $V$ where the derivative (34) is well defined.

If the positive directions along $V$ coincide with those of $E_{03}$ at every point of $V$, then $V_{03}$ and $E_{03}$ must be coincident. The proof is somewhat complicated but may be made with the help of the following lemma:

**Lemma 3.** Any arc of $V$ defined by an interval $t' \leq t \leq t''$ and containing no points of $S$ or $B$, is an extremal of one of the families (6), (14) along which $s$ increases from $s'$ to $s''$ as $t$ traverses the interval $t' t''$. An arc of $V$ lying entirely on the surface $S$ must be an arc of one of the surface extremals (13) with similar properties.

An arc of $V$ with which the lemma is concerned must lie in the interior of $F_1$ or $F_2$, or else on the surface $S$. The proofs in the three cases are quite similar, so that it will be sufficient to suppose the arc $V$ to lie entirely on $S$. The values $s = \sigma, \sigma, \tau$ are then continuous functions $\sigma(t), \tau(t)$ defined over the interval $t' \leq t \leq t''$ by equations (V) and (29), and they satisfy the equations

$$x(t) = \xi [\sigma(t), \tau(t)], \quad y(t) = \eta [\sigma(t), \tau(t)],$$

$$z(t) = \zeta [\sigma(t), \tau(t)].$$

At least one of the determinants of the matrix

$$\begin{vmatrix} \xi_\sigma & \eta_\sigma & \zeta_\sigma \\ \xi_\tau & \eta_\tau & \zeta_\tau \end{vmatrix}$$
is different from zero at every value of $t$ on account of $V^\prime$ and the relations (12). Hence the theory of implicit functions applied to a suitably chosen pair of the equations (35) shows that $\sigma(t)$ and $\tau(t)$ are of class $C^\alpha$. If the direction of $V$ coincides with that of the surface extremal at each point of the arc in question, then

$$x'(t) = \kappa \xi_\sigma, \quad y'(t) = \kappa \eta_\sigma, \quad z'(t) = \kappa \xi_\sigma \quad (\kappa > 0),$$

and by differentiating equations (35) the relations

$$\sigma'(t) = \kappa, \quad \tau'(t) = 0$$

are readily deduced. Hence the lemma is true for an arc on $S$, and for the other two cases by similar arguments.

**Lemma 4.** If $t_4$ is a lower bound of values of $t$ such that for $t_4 \leq t \leq t_3$ the curve $V_{03}$ coincides with $E_{03}$, then no value $t' < t_4$ can define a point of $V$ interior to $S_2$, or a point of $V$ on the surface $S$ and distinct from $B$.

There must be a lower bound $t_4$ of the kind described since near the point 3 the curve $V_{03}$ coincides with an extremal of the field passing through the point 3, according to Lemma 3, and $E_{03}$ is the only such extremal.

If $t'$ defined a point interior to $S_2$, then there would be a first value $t'' > t'$ such that the arc $t' t''$ of $V$ would be entirely interior to $S_2$ and distinct from $E_{03}$ except at the value $t''$. The value $t''$ would necessarily define a point either of $B$, or $S$, or $E_{23}$. But the arc $t' t''$ of $V$ would in that case be an extremal arc along which $s$ increases with $t$ from a value $s'$ to a value $s''$. This is, however, impossible, since any such arc which is distinct from $B$, $S$, $E_{23}$ for $s = s'$ retains this property for all values $s > s'$.

With the help of this result and by similar arguments it follows that $t'$ could not define a point of $S$ distinct from $B$.

The identity of $V_{03}$ and $E_{03}$ now follows without difficulty. The value $t = t_4$ must in fact define a point 4 on the arc $E_{01}$, since in case this is not true there would certainly be values $t' < t_4$ satisfying the conditions of Lemma 4. The arc $t_0 t_1$ of $V$ then lies entirely in the field $S_1$, and no point of it can be distinct from $E_{01}$. Otherwise there would be an extremal arc of the field $S_1$ distinct from $E_{01}$ whose end point, as $s$ increased toward the parameter value defining it, would lie on $E_{01}$. There is, however, no such arc.