AN EXISTENCE THEOREM FOR A CERTAIN DIFFERENTIAL EQUATION OF THE nTH ORDER*

BY

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1. The usual method for investigating the solutions of a differential equation of the nth order in the neighborhood of a singular point has been to reduce the problem to a system of n equations of the first order. This procedure was adopted by Poincaré† and others for the treatment of equations of a general form, and the results obtained have been summarized and simplified by Dulac.‡ No explicit statement has been made concerning an equation of the form

$$\frac{d^n y}{dx^n} = \frac{N}{D} = \frac{ax + by + \cdots}{\alpha x + \beta y + \cdots}, \tag{1}$$

where N and D are convergent power series in x and y vanishing for $x = 0$, $y = 0$, but an application of the general theory yields the following information concerning the integrals in the neighborhood of the values $x = 0$, $y = 0$.

Equation (1) may be reduced to the system

$$\frac{dy}{dx} = c + y_1, \qquad \frac{dy_1}{dx} = c_2 + y_2, \qquad \cdots, \qquad \frac{dy_{n-1}}{dx} = \frac{N}{D}, \tag{2}$$

where c, c_2, $\cdots$, c_{n-1} are arbitrary finite constants and the initial values of the variables are all zero. If $\lambda = \alpha + c\beta$ is not zero then the variables can be expressed as power series in terms of a single quantity z; that is,

$$x = x(z), \qquad y = y(z), \qquad \cdots, \qquad y_{n-1} = y_{n-1}(z), \tag{3}$$

such that if z satisfies an equation

$$\frac{dz}{dt} = \lambda z + z^2 P, \tag{4}$$

then the functions (3) will satisfy the system (2). The quantity P is a power

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† Journal de mathématiques, ser. 4, vol. 1 (1885), p. 167, and vol. 2 (1886), p. 151.

‡ Bulletin de la société mathématique de France, vol. 40 (1912), p. 324.

series in z whose coefficients depend upon the system (2). The solution z of equation (4) can be expressed as a power series in $\theta = ke^{\lambda t}$, k being an arbitrary constant. After substitution of this expression in the system (3), θ can be found in terms of x from the first equation and, when this value is put in the second, y is obtained as a function of x which satisfies equation (1). This solution involves the $n - 1$ finite constants $c, c_2, \cdots, c_{n-1}$, which are arbitrary except for the condition $\alpha + c\beta \neq 0$. The exact form of the solution is not given and its construction by this general method would involve a number of transformations and considerable calculation.

The object of the present paper is to obtain this result directly in explicit form and to derive results in some cases for which $\alpha + c\beta = 0$.

2. By setting $y = x(c + v)$, where v is to be determined as a function of x vanishing with x, and c is a constant, equation (1) becomes

$$x\frac{d^n v}{dx^n} + n\frac{d^{n-1} v}{dx^{n-1}} = \frac{a + b(c + v) + xN'}{\alpha + \beta(c + v) + xD'}.$$

If $\alpha + c\beta \neq 0$ the second member can be expanded as a power series in x and v, and the equation takes the form

$$(5) \qquad x\frac{d^n v}{dx^n} + n\frac{d^{n-1} v}{dx^{n-1}} = a_{00} + a_{10}x + a_{01}v + \cdots + a_{ij}x^i v^j + \cdots.$$

Suppose a solution is assumed in the form

$$(6) \qquad v = c_2 x + c_3 x^2 + \cdots + c_{n-1}x^{n-2} + c_n x^{n-1} + \cdots + c_{m+1}x^m + \cdots.$$

On substituting this value of v in equation (5) and equating the coefficients of corresponding powers of x, it follows that $c_2, c_3, \cdots, c_{n-1}$ may be chosen arbitrarily, while the remaining coefficients of the series (6) are uniquely determined by

$$\begin{aligned} &\underline{|n}\, c_n = a_{00}, \\ &\cdots\cdots\cdots\cdots, \\ (7) \qquad &\{m(m-1)\cdots(m-n+1) \\ &\qquad + nm(m-1)\cdots(m-n+2)\}c_{m+1} \\ &\qquad\qquad = F_{m+1}(a_{ij}, c_2, \cdots, c_{n-1}), \end{aligned}$$

where F_{m+1} is a rational integral function of $c_2, \cdots, c_{n-1}$ and the coefficients $a_{ij}\,(i + j < m - n)$.

The convergence of the series (6) can be demonstrated by comparing it with the solution of

$$(8) \qquad \frac{d^{n-1} u}{dx^{n-1}} = a'_{00} + a'_{10}x + a'_{01}u + \cdots + a'_{ij}x^i u^j + \cdots,$$

the coefficients a'_{ij} being positive constants equal to the absolute values of the a_{ij}. By Cauchy's fundamental theorem it is known that this equation has a solution for u as a power series in x which is convergent for values of x whose moduli are sufficiently small. The solution is

$$u = \alpha_2 x + \cdots + \alpha_{n-1} x^{n-2} + \alpha_n x^{n-1} + \cdots + \alpha_{m+1} x^m + \cdots, \tag{9}$$

where $\alpha_2, \cdots, \alpha_{n-1}$ are arbitrary. The remaining coefficients are determined by

$$\begin{aligned} &\underline{|n-1}\, \alpha_n = a'_{00}, \\ &\cdots\cdots\cdots\cdots\cdots, \\ &m(m-1)\cdots(m-n+2)\,\alpha_{m+1} = F'_{m+1}(a'_{ij}, \alpha_2, \cdots, \alpha_{n-1}). \end{aligned} \tag{10}$$

The function F'_{m+1} in equations (10) is a rational integral function of the same form as F_{m+1} in equations (7), the coefficients being all positive and greater than the absolute values of the corresponding coefficients in F_{m+1}. It follows then that the absolute value of c_{m+1} is less than α_{m+1} for every m provided $\alpha_2 = c_2, \cdots, \alpha_{n-1} = c_{n-1}$. Hence the series (6) converges for at least the same range of values as the series (9).

The solution of equation (1) is now given in the explicit form

$$y = cx + c_2 x^2 + \cdots + c_{n-1} x^{n-1} + \cdots + c_m x^m + \cdots,$$

where $c, c_2, \cdots, c_{n-1}$ are finite constants, arbitrary except for the condition $\alpha + c\beta \neq 0$. The remaining coefficients are uniquely determined in terms of the arbitrary constants and the coefficients in the differential equation. This statement contains the information which may be derived from the general theory as summarized by Dulac.

3. The preceding method may be applied without alteration to the case when in equation (1) the lowest terms in N are of degree p, and the lowest terms in D are of degree q if $p \geqq q$. If the terms of lowest degree in D are

$$D_0 x^q + D_1 x^{q-1} y + \cdots + D_q y^q,$$

the assumption concerning the constant c is

$$D_0 + D_1 c + \cdots + D_q c^q \neq 0.$$

Geometrically this assumption is expressed by saying that the integral curve is not tangent at the origin to a branch of the curve $D = 0$.

The existence theorem may now be stated in the following form·

THEOREM. *Given the differential equation*

$$\frac{d^n y}{dx^n} = \frac{N}{D} = \frac{N_0 x^p + N_1 x^{p-1} y + \cdots + N_p y^p + \cdots}{D_0 x^q + D_1 x^{q-1} y + \cdots + D_q y^q + \cdots} \tag{E}$$

in which N and D are convergent series in positive integral powers of x and y

and $p \geqq q$. *If*

$$(C) \qquad D_0 + D_1 c + \cdots + D_q c^q \neq 0,$$

then there exists a solution expressible in the form

$$(S) \qquad y = cx + \sum_{i=2}^{\infty} c_i x^i,$$

which is convergent for values of x *whose moduli are sufficiently small, and the coefficients of the first* $n - 1$ *powers of* x *are arbitrary except for the condition* (C).

The fundamental theorem of Cauchy applies to an equation of the form

$$(11) \qquad \frac{d^n y}{dx^n} = P,$$

in which P is a series in positive integral powers of x and y, and asserts the existence of a solution of the form (S) in which the coefficients of the first $n - 1$ powers of x are entirely arbitrary. The difference in the results may be stated geometrically by saying that for equation (11) the regular integral curves through the origin may go in *any* direction, while for the equation (E) they may go in any direction not tangent to a branch of the curve $D = 0$ (excluding always the direction of the y-axis).

4. We now return to equation (1) in which the numerator and denominator contain terms of the first degree and inquire for integral curves tangent at the origin to the curve $D = 0$. We ask if the equation can be satisfied by expressing y as a series in positive integral powers of x^σ, where σ is a positive constant. Since $dy/dx = -\alpha/\beta = c$ for $x = 0$, we make the substitution $y = -\alpha x/\beta + v$, and v is to be determined as a series in x^σ containing x^k as a factor, where k is a positive number greater than unity, and satisfying the equation

$$(12) \qquad \frac{d^n v}{dx^n} = \frac{Ax + bv + P_2}{\beta v + Q_2}.$$

In this equation P_2 and Q_2 contain no first degree terms, and $\beta A = a\beta - b\alpha$. Let the term in Q_2 of lowest degree independent of v be x^r. Then r is an integer greater than unity. Suppose that A is not zero, which means that the curves $N = 0$ and $D = 0$ are not tangent at the origin.

Let a solution of equation (12) be assumed in the form

$$(13) \qquad v = lx^k + mx^{k+\sigma} + \cdots \qquad (l \neq 0).$$

After multiplying equation (12) by the denominator of the second member and substituting the value of v, the index of the lowest power of x must be the same on each side of the equation. Two cases are to be considered.

I. Suppose k is not an integer less than n. If k is not greater than r, the lowest term in the left member is of degree $2k - n$, and in the right member is of the first degree. Hence $k = (n+1)/2$, which is possible only if n is even. If k is greater than r, the index equation is $k - n + r = 1$, which is contrary to the assumption that k is not an integer less than n.

II. Suppose k is an integer less than n. If σ is not an integer, the index equation is $2k + \sigma - n = 1$ or $k + \sigma + r - n = 1$, which contradicts the assumption that σ is not an integer. If σ is an integer it may be taken as unity without loss of generality. The lowest power of x in $d^n v/dx^n$ will be x^q, where q is zero or a positive integer, and the index equation becomes $k + q = 1$, which contradicts the assumption that k is greater than unity, or $r + q = 1$, which is impossible.

The preceding argument shows that equation (1) may admit a solution for y expressible as a positive power series in x^σ $(\sigma > 0)$ such that $dy/dx = -\alpha/\beta$ when $x = 0$, only if n is even. If such a series exists the first two terms will be

$$(N) \qquad y = -\frac{\alpha}{\beta}x + lx^{(n+1)/2} + \cdots.$$

5. We shall consider in detail the case when $n = 2$, and show that equation (1) admits a solution for y as a power series in $x^{\frac{1}{2}}$. On introducing ξ as the independent variable by the relation $x = \xi^2$, equation (1) becomes

$$(14) \qquad \frac{1}{4\xi^3}\left\{\xi\frac{d^2y}{d\xi^2} - \frac{dy}{d\xi}\right\} = \frac{a\xi^2 + by + \cdots}{\alpha\xi^2 + \beta y + \cdots}.$$

Following the condition (N) it is convenient to set

$$y = -\frac{\alpha}{\beta}\xi^2 + \xi^3(l + v),$$

where v is to be determined as a power series in ξ vanishing with ξ. Then equation (14) takes the form

$$(15) \qquad \xi^2\frac{d^2v}{d\xi^2} + 5\xi\frac{dv}{d\xi} + 3(l + v) = \frac{4A + \xi P_0}{\beta l + \beta v + \xi Q_0},$$

in which P_0 and Q_0 are series in ξ and v. The constant l is determined by setting $\xi = 0$, $v = 0$ which gives $3\beta l^2 = 4A$. The second member of equation (15) can be expanded as a power series in ξ and v, and the equation becomes

$$(16) \qquad \xi^2\frac{d^2v}{d\xi^2} + 5\xi\frac{dv}{d\xi} + 6v = A\xi + \cdots + A_{ij}\xi^i v^j + \cdots \qquad (i+j>1).$$

Let a solution be assumed in the form

$$(17) \qquad v = a_1\xi + a_2\xi^2 + \cdots + a_n\xi^n + \cdots.$$

The equations for the determination of the coefficients are

$$(18) \qquad (5+6)a_1 = A, \qquad \cdots, \qquad \{n(n-1)+5n+6\}a_n = F_n.$$

The quantities F_n are rational integral functions of $A_{ij}\,(i+j \leqq n)$, and the coefficients in equation (17) are uniquely determined.

The convergence of the series (17) can be demonstrated by comparing it with the solution of

$$(19) \qquad 5\xi\frac{du}{d\xi} + 6u = A'\,\xi + \cdots + A'_{ij}\,\xi^i u^j + \cdots \qquad (i+j>1),$$

the coefficients A'_{ij} being positive constants equal to the absolute values of the A_{ij}. If a solution is assumed in the form

$$(20) \qquad u = \alpha_1\,\xi + \cdots + \alpha_n\,\xi^n + \cdots,$$

the equations for the determination of the coefficients are

$$(21) \qquad (5+6)\alpha_1 = A', \qquad \cdots, \qquad (5n+6)\alpha_n = F'_n.$$

In equations (21), F'_n is a rational integral function of the same form as F_n in equations (18), the coefficients being all positive and greater than the absolute values of the corresponding coefficients in F_n. Evidently the absolute value of a_n is less than α_n for every n. Now equation (19) is the well-known equation of Briot and Bouquet* which admits a solution of the form (20) if the coefficient of u (in this case 6/5) is not a negative integer. The series (20) is therefore known to be convergent for values of x whose moduli are sufficiently small. Consequently the series (17) is convergent for at least the same range of values.

The conclusion is that if $a\beta - b\alpha \mp 0$, then equation (1), in which $n=2$, admits a solution of the form

$$y = -\frac{\alpha}{\beta}x + x^{6/5}\,P,$$

where P is a power series in $x^{1/5}$.

When this result is combined with that of the theorem above, we can make the following geometric statement concerning a differential equation of the form (1) and of the second order. If the origin is an ordinary point on each of the curves $N=0$ and $D=0$, and if these curves have distinct tangents at the origin, then, in every direction through the origin (except, perhaps, tangent to the y-axis) there passes one integral curve. These integral curves are all regular except the one tangent to the curve $D=0$. This one has a branch point at the origin.

It has been said above that for n even the equation (1) may possibly admit a

*Journal de l'école polytechnique, vol. 21 (1856), p. 161.

solution of the form (N). For n greater than 2 the search for such solutions involves the terms in D of order higher than the first and the investigation will not be undertaken in this paper.

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