ADDENDA AND ERRATA, VOLUMES, 11, 14

VOLUME 11

Page 25. ERNEST B. LYTLE.* *Proper multiple integrals over iterable fields.*

1. Consider the fundamental relation

$$(A) \qquad \underline{\int}_{\mathfrak{A}} f \leqq \underline{\int} \underline{\int}_{\mathfrak{C}} f \leqq \overline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} f \leqq \overline{\int}_{\mathfrak{A}} f$$

where f is any limited function defined over the limited field $\mathfrak{A}$, and where the integrals are Pierpont integrals and hence are applicable to fields which are not metric (defined in these Transactions, vol. 11 (1910), p. 26). The problem under consideration here is to find the most general conditions upon the field $\mathfrak{A}$ under which relation (A) is true.

Pierpont has shown† that relation (A) is true when the field $\mathfrak{A}$ is metric. In a former paper‡ the author found a more general class of fields called iterable for which relation (A) holds true. This class of iterable fields includes all metric fields and some non-metric fields.

In this note I show that iterability of the field $\mathfrak{A}$, that is,

$$\int \overline{\mathfrak{C}} = \overline{\mathfrak{A}},$$

is also a necessary condition that the class of limited functions over $\mathfrak{A}$ simultaneously satisfy relation (A).

2. THEOREM 1. *If all limited functions defined over $\mathfrak{A}$ satisfy relation (A), then $\mathfrak{A}$ is iterable.*

Consider the contraposite, if $\mathfrak{A}$ is not iterable then not all limited functions over $\mathfrak{A}$ satisfy (A); and we see that to prove Theorem 1 it is only necessary to show there exists a limited function over a non-iterable field which does not satisfy (A). Such a function is as follows:

Example 1. In $\mathfrak{A} = \{x, y\}$, let $0 \leqq x \leqq 1$; for rational x let $0 \leqq y \leqq 1$, and for irrational x let $0 \leqq y \leqq \frac{1}{2}$. Let $f(x, y) = 1$ over this $\mathfrak{A}$.

This $\mathfrak{A}$ is not iterable, for

$$\underline{\int}_{\mathfrak{B}} \overline{\mathfrak{C}} = \tfrac{1}{2}, \qquad \overline{\int}_{\mathfrak{B}} \overline{\mathfrak{C}} = 1,$$

* Presented to the Society (Madison), September 8, 1913.

† These Transactions, vol. 7 (1906), p. 167.

‡ These Transactions, vol. 11 (1910), p. 25.

and therefore $\int_{\mathfrak{B}} \mathfrak{C}$ does not even exist. Further, $f(x, y)$ here does not satisfy relation (A), for

$$\underline{\int}_{\mathfrak{A}} f = 1, \qquad \text{and} \qquad \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f = \tfrac{1}{2}.$$

By combining Theorem 1 above and Theorem 14, p. 35 of my former paper cited above, we get the following important theorem:

THEOREM 2. *In order that all limited functions over $\mathfrak{A}$ simultaneously satisfy relation (A), it is necessary and sufficient that $\mathfrak{A}$ be iterable relative to x.*

3. It is to be noticed that iterability of the field is not necessary in order that a single particular limited function satisfy (A), as the following example shows.

Example 2. Let $\mathfrak{A}$ be defined as in Example 1 above. Let $f(x, y) = 1$ when x is rational and $f(x, y) = 2$ when x is irrational.

Here again $\mathfrak{A}$ is not iterable, but relation (A) is satisfied, for

$$\underline{\int}_{\mathfrak{A}} f = 1, \qquad \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f = 1, \qquad \overline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} f = 1, \qquad \overline{\int}_{\mathfrak{A}} f = \tfrac{3}{2}.$$

4. *Correction.* I wish to call attention to an error in Theorem 8, page 31 of my former paper cited above. This theorem as there stated will not hold for certain fields involving a Pringsheim aggregate* which has a two-dimensional content greater than zero while each linear section parallel to the x-axis (or y) has a linear content equal to zero.

The proof as there given will hold if we restate the theorem as follows:

THEOREM 3. *Let $f(x, y)$ be limited over the limited field $\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{A}_2$ where $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are so defined that on each linear section parallel to the x-axis (or y) the points of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are each everywhere dense with respect to the other. Let $f(x, y) \geqq 0$ over $\mathfrak{A}_1$, and $f(x, y) \leqq 0$ over $\mathfrak{A}_2$. Then relation (A) is true.*

Theorem 8 was used in no other place, so this error affects no other part of that paper.

The author is indebted to Prof. R. G. D. Richardson for calling his attention to the above error; also for suggesting the truth and method of attack of Theorem 1 of this paper.

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* Pierpont's Lectures on the Theory of Functions of Real Variables, vol. 1, p. 546.

Volume 14

Page 65. Oswald Veblen. *Decomposition of an n-Space by a Polyhedron.*

Page 66, line 4 of §4, for "consisting" read "whose interior consists."

Page 66, line 6 of §4, for "The" read "A."

Page 66, line 9 of §4, before the word "According" insert the sentence "It is easily seen that a region has at most one boundary."

Page 66, line 4 of §5, between the words "two-dimensional" and "*polygonal*" insert "*polyhedral or a.*"

A polyhedron is defined in §5 as a set of n-dimensional polyhedral regions, together with their boundaries, subject to certain restrictions. These restrictions are not properly stated. Instead of modifying the statements as they stand, I propose to make a change which simplifies the whole matter without any loss of generality.

It is easy to prove, by an argument like that in §7, that the set of points on any polyhedral region together with its boundary can be regarded as composed of a finite number of convex polyhedral regions together with their boundaries. Hence any set of points consisting of a finite number of polyhedral regions and their boundaries may be regarded as consisting of a finite set of convex polyhedral regions and their boundaries. In view of this observation, we replace the definition at the bottom of page 66 and the top of page 67 by the following:

An *n-dimensional polyhedron* is a set of points $[P]$ which satisfies the following conditions: (1) $[P]$ consists of the points in and on the boundaries of a finite number of n-dimensional convex polyhedral regions, $F_1, F_2, \cdots, F_k$, no two of which have a point in common, and which are such that whenever there is an $(n-1)$-dimensional convex region contained in the boundary of each of q F's and containing no point in or on the boundary of any other F, q is an even number; (2) there is no subset of $[P]$ which has the property (1).

This change in the definition renders superfluous the corollary (1) of §16 on page 72, but makes no change in the rest of the paper.

It is perhaps desirable to call attention to the fact that, according to the definition employed in this paper, the boundary of a general region need not be a closed set of points. It does, however, require the boundary of a convex or polyhedral region in a number-plane to be closed.