ORIENTED CIRCLES IN SPACE*

BY

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INTRODUCTION

A wheel mounted on a spindle may be rotated in either of two directions. Thus rotation is a property which may be adjoined to a wheel in either of two ways. Analogously, *orientation* may be thought of as an arbitrary property which may be adjoined to any not null circle in two ways, thus giving rise to two *oriented circles*. In the mathematical development of this idea, two kinds of tangency present themselves, called proper and improper tangency. The study of oriented circles in two dimensions has been interestingly developed, especially by Laguerre, who discovered a one-to-one transformation of oriented circles that preserves one kind of tangency but not the other. In this paper I have considered oriented circles in space of three dimensions, and found a few theorems about invariants and linear systems. In one respect the results are disappointing, because all attempts to generalize the transformation of Laguerre were unsuccessful.

1. PENTASPHERICAL CONTINUUM†

All configurations considered in the following pages are supposed to lie in what may be called a *perfect pentaspherical continuum*, that is, a set of objects called *points* in one-to-one correspondence with all sets of five homogeneous coördinates, not all zero, $x_0 : x_1 : x_2 : x_3 : x_4$, which satisfy the relation

$$\sum_{i=0}^{4} x_i^2 \equiv (xx) = 0. \tag{1}$$

To have something definite in mind, let us observe that the points of finite cartesian space can be made into a perfect pentaspherical continuum by adjoining to them an infinite number of ideal points forming an ideal null sphere. Points that satisfy a linear relation

$$\sum_{i=0}^{4} y_i x_i \equiv (y\,x) = 0 \tag{2}$$

form a sphere. The five coefficients (y) are regarded as the coördinates of

* Presented to the Society, January 2, 1915.

† Darboux, *Sur une classe remarquable de courbes et de surfaces algébriques.*

the sphere. If (y) satisfies (1) the sphere is *null* and the point (y) is its *vertex*. In the extended cartesian space above alluded to, the isotropic planes appear as null spheres whose vertices are the ideal points. We use the same notation for point and sphere coördinates, but no confusion will arise from it.

The angle of two not null spheres (x) and (y) is found to be

$$\theta = \cos^{-1}\frac{(xy)}{\sqrt{(xx)}\sqrt{(yy)}}. \tag{3}$$

We shall study, in general, properties that are invariant under the group of conformal transformations of our space, which will be given by the orthogonal substitutions in the coördinates (x).

2. Plücker coördinates*

Two spheres, (x) and (y), will intersect in a circle. Consider the quantities

$$\lambda p_{ij} = (x_i y_j - x_j y_i) \tag{4}$$

where λ is a proportionality factor, not zero. If any other sphere of the pencil through this circle be used, e. g.,

$$lx_i + my_i \qquad (i = 0, 1, 2, 3, 4),$$

the effect will be merely to multiply every p_{ij} by the same factor. These quantities are called the plücker coördinates of the circle. They are connected by the following relations:

$$p_{ij} = -p_{ji}, \qquad p_{ii} = 0 \qquad (i, j = 0, 1, 2, 3, 4); \tag{5}$$

$$\begin{aligned}
\tfrac{1}{2}\Omega_0(pp) &\equiv p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0,\\
\tfrac{1}{2}\Omega_1(pp) &\equiv p_{02}p_{43} + p_{04}p_{32} + p_{03}p_{24} = 0,\\
\tfrac{1}{2}\Omega_2(pp) &\equiv p_{01}p_{34} + p_{03}p_{41} + p_{04}p_{13} = 0,\\
\tfrac{1}{2}\Omega_3(pp) &\equiv p_{01}p_{42} + p_{04}p_{21} + p_{02}p_{14} = 0,\\
\tfrac{1}{2}\Omega_4(pp) &\equiv p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} = 0.
\end{aligned} \tag{6}$$

Conversely, every set of quantities p_{ij} not all zero, that satisfy (5) and (6), represent a circle. Of the twenty coördinates p_{ij} $(i \neq j)$, ten may be selected in several ways that are *essentially different*, that is, no two of them are connected by (5), e. g.,

$$p_{01}, p_{02}, p_{03}, p_{04}, p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}.$$

The other ten are merely the negatives of these.

If a set of numbers p_{ij} and q_{ij} satisfy (5) alone, then the following identities

* Koenigs, *Contributions à la théorie des cercles dans l'espace*, Annales de la Faculté des Sciences de Toulouse, vol. 2 (1888).

will hold,

$$\sum_j p_{ij}\,\Omega_j(pp) \equiv 0 \qquad (i = 0, 1, 2, 3, 4), \tag{7}$$

$$\sum_j p_{ij}\,\Omega_j(qp) \equiv -\tfrac{1}{2}\sum_j q_{ij}\,\Omega_j(pp), \tag{8}$$

where by $\Omega_j(qp)$ is meant the polarized form of $\frac{1}{2}\Omega_j(pp)$. From (7) we see that only three independent conditions are imposed by (6). For suppose for example that $p_{34} \neq 0$. Then if

$$\Omega_0(pp) = \Omega_1(pp) = \Omega_2(pp) = 0,$$

it follows that

$$\Omega_3(pp) = \Omega_4(pp) = 0.$$

The condition that two circles be in involution, that is, that each be orthogonal to a sphere through the other, is

$$\sum_{ij} p_{ij}\,q_{ij} = 0. \tag{9}$$

The condition that two circles be cospherical is

$$\Omega_i(pq) = 0, \qquad (i = 0, 1, 2, 3, 4), \tag{10}$$

only two of which are independent conditions.

The condition that a circle be null* is

$$\sum p_{ij}^2 = 0. \tag{11}$$

The common orthogonal sphere of two non-cospherical circles is

$$\lambda x_i = \Omega_i(pq). \tag{12}$$

This is null and the two circles have a common point if

$$\sum_j [\Omega_j(pq)]^2 = 0. \tag{13}$$

Two circles are in bi-involution, that is every sphere through either is orthogonal to the other, if

$$\sum_n p_{in}\,q_{jn} = 0 \qquad (i, j = 0, 1, 2, 3, 4), \tag{14}$$

where only four of the equations are independent.

3. Orientation of circles

Let us introduce a redundant coördinate, p, defined by the relation

$$\sum p_{ij}^2 + p^2 = 0, \tag{15}$$

in which the summation is to be taken over only ten *essentially different* plücker coördinates, that is, no two of which are connected by (5). Two

* In the extended cartesian space to which we have alluded, in § 1, the isotropic lines appear as a special kind of null circle. They satisfy (11) but cannot be transformed into other null circles by any conformal transformation.

values of p, differing only in sign, go with every set of plücker coördinates except in case the circle is null when the redundant coördinate is zero on account of (11). Now let us say that in place of every not null circle, we have two *oriented circles that differ only in orientation.* In the real domain, it is customary to think of the two orientations of a circle as the two directions in which a moving point may describe it. In general, however, we may think of the orientation merely as an arbitrary property that may be adjoined to any not null circle in two ways. In the case of a null circle the two orientations are said to coincide. Thus we have a one-to-one correspondence between the oriented circles and the quantities p_{ij} and p, satisfying (5), (6), and (15), which are accordingly called the *coördinates* of the oriented circles.

But we can give a much more satisfactory meaning to the orientation of a circle. In the pencil of spheres through a not null circle are two null spheres whose vertices are called the *foci* of the circle. Let (x) and (y) denote the coördinates of the foci of a not null circle. The two null spheres through the circle have also these same coördinates (x) and (y). Let the *orientation* of a circle be a discrimination in favor of one of the foci, by saying that the circle whose *first focus* is (x) and *second focus* is (y) differs in orientation from the circle whose *first focus* is (y) and *second focus* is (x). In the case of proper null circles the foci coincide and the two orientations coincide as they ought. If the circle is an improper null circle, consisting of two coincident isotropic lines, every sphere in the pencil through it is null, and the orientation becomes illusory; but we shall define the two orientations as coincident also in this case.

We shall establish definite connections between this definition of orientation and the choice of the sign of the redundant coördinate by the following formula which gives the coördinates in terms of the first focus (x) and the second focus (y):

$$\rho p_{ij} = x_i y_j - x_j y_i, \qquad \rho p = (xy). \tag{16}$$

We must, of course, show that this value of p satisfies (15); but first observe that if the rôles of (x) and (y) are interchanged, the sign of every p_{ij} is reversed, but p is unaltered, which, by the homogeneity, amounts to changing the sign of p alone. Equation (15) may be multiplied by 2 and written in the form

$$\sum_{i,j=0}^{4} p_{ij}^2 + 2p^2 = 0,$$

where the summation is taken over all coördinates and not merely over ten essentially different ones. Substituting the values given by (16) we obtain

$$\sum_{i,j=0}^{4} [x_i^2 y_j^2 - 2x_i x_j y_i y_j + x_j^2 y_i^2] + 2(xy)^2$$
$$= (xx)(yy) - 2(xy)^2 + (xx)(yy) + 2(xy)^2 \equiv 0,$$

because $(xx) = (yy) = 0$, according to (1).

It would be perfectly logical to take $p = -(xy)$ instead of as in (16). It would only have associated the opposite orientation or choice of foci with a given choice of the sign of p. However we shall cling to the association established by formula (16).

If the circle is an improper null circle we may take the (x) and (y) as any two of the null spheres passing through it, and (16) holds true because $(xy) = 0$, and hence $p = 0$ as it ought. But in the case of proper null circles, formula (16) becomes illusory by giving zero for every coördinate, since $(x) \equiv (y)$. We can generalize (16) in a manner to cover this case and also prove useful later on. Let (x) be the null sphere whose vertex is the first focus of a circle, and (z) *any* other sphere through the circle, and let (y) be the second focus. Then

$$y_i = \lambda x_i + \mu z_i \qquad (i = 0, 1, 2, 3, 4).$$

Substituting for (y) in (16), and setting $\rho/\mu = \sigma$, we obtain

$$\begin{aligned} \sigma p_{ij} &= x_i z_j - x_j z_i, \\ \sigma p &= (xz), \end{aligned} \tag{17}$$

where (x) is the first focus and (z) is any sphere through the circle other than (x). This formula is equivalent to (16) if the circle is not null, and evidently holds in all cases. For null circles we have

$$(xz) = 0 \quad \text{but} \quad (x) \not\equiv (z).$$

Conversely, if we are given a set of coördinates, p_{ij}, p that satisfy (5), (6), and (15), we can calculate the first focus (x) by the formula

$$\lambda x_i = p \sum_j p_{ij}\, \xi_j - \sum_j \sum_n p_{in}\, p_{jn}\, \xi_j \qquad (i = 0, 1, 2, 3, 4), \tag{18}$$

where (ξ) is any sphere not orthogonal to the circle. In order to prove this formula, let us assume that (x') is the first focus of the circle in question and (z) any other sphere through the circle, and seek to prove that

$$x_i = \mu x_i' \qquad (i = 0, 1, 2, 3, 4).$$

Now formula (17) gives the coördinates in terms of (x') and (z), and these we substitute in (18), recollecting $(x'x') = 0$. The result is

$$\begin{aligned} \lambda x_i &= (x'z) \textstyle\sum_j (x_i' z_j - x_j' z_i)\, \xi_j - \sum_j \sum_n (x_i' z_n - x_n' z_i)(x_j' z_n - x_n' z_j)\, \xi_j \\ &= (x'z)[x_i'(z\xi) - z_i(x'\xi)] - x_i'(zz)(x'\xi) + x_i'(x'z)(z\xi) \\ &\qquad + z_i(x'z)(x'\xi) = [2(x'z)(z\xi) - (zz)(x'\xi)]\, x_i'. \end{aligned}$$

Q. E. D.

To obtain the second focus of the circle, we need only change the sign of the redundant coördinate in formula (18).

An improper null circle is in bi-involution with itself, that is, each sphere through it is orthogonal to every sphere through it. The conditions for this are formed from (14) by making $(p) \equiv (q)$, giving

$$\sum_n p_{in}\, p_{jn} = 0 \qquad (i, j = 0, 1, 2, 3, 4),$$

and furthermore p zero. Hence formula (18) is indeterminate, giving zeros on the right, as we should expect since there are no determined foci for an improper null circle.

4. Angle of two oriented circles

Consider two not null circles (p) and (q), both lying on a not null sphere (z), and for the present let us think of them as not oriented. They meet at some angle, or rather a pair of supplementary angles which we wish to calculate. Let (u) and (v) be the coördinates of two spheres orthogonal to (z) and meeting it in the circles (p) and (q) respectively. Then (u) and (v) meet at the same angles as (p) and (q). Forming the plücker coördinates of the two circles from (u), (z) and (v), (z), as in formula (4), we have, since $(uz) = (vz) = 0$,

$$\frac{\sum p_{ij}\, q_{ij}}{\sqrt{\sum p_{ij}^2}\sqrt{\sum q_{ij}^2}} = \begin{vmatrix} (uv) & (uz) \\ (zv) & (zz) \end{vmatrix} \Big/ \sqrt{\begin{vmatrix} (uu) & (uz) \\ (zu) & (zz) \end{vmatrix}} \sqrt{\begin{vmatrix} (vv) & (vz) \\ (zv) & (zz) \end{vmatrix}} = \frac{(uv)}{\sqrt{(uu)}\,\sqrt{(vv)}},$$

where the summations are to be extended over only ten essentially different terms. By (3) this is the cosine of the angle made by (u) and (v), and so it is also the cosine of the angle made by (p) and (q), the choice of the sign of the denominator determining which of the two supplementary angles we have. Suppose we now orient the circles (p) and (q) by giving the redundant coördinates p and q, and define their angle θ to be

$$\theta = \cos^{-1} \frac{\sum p_{ij}\, q_{ij}}{-pq}.$$

What we have here done is to give a law by which one of the two angles is determined when the orientations are given; and we choose the negative sign in the denominator so that an oriented circle will make a null angle with itself, as is seen by making (q) identical with (p) and attending to (15). Hence

$$2 \sin^2 \theta/2 = 1 - \cos \theta = \frac{\sum p_{ij}\, q_{ij} + pq}{pq}.$$

The circles are tangent if the angle is zero or π. We shall define them as *properly tangent* when their angle is zero, and *improperly tangent* when it is π. For proper tangency then

$$\sum p_{ij}\, q_{ij} + pq = 0.$$

The expression

$$\frac{\sum p_{ij}\, q_{ij}}{\sqrt{\sum p_{ij}^2}\sqrt{\sum q_{ij}^2}}$$

formed for *any* two not null circles, is an absolute invariant under the conformal transformations of our space. It was first studied by Koenigs,* and again by Coolidge.†

We wish to examine a similar absolute invariant, namely

$$\vartheta(pq) \equiv \frac{\sum p_{ij}\, q_{ij} + pq}{pq},$$

the summation extending over only ten terms. This is defined uniquely for any two not null circles. Let us generalize the angle, θ, of any two not null oriented circles (p) and (q) by the definition

$$\tfrac{1}{2}\sin^2\theta/2 = \vartheta(pq). \tag{19}$$

As we saw, when the two circles lie upon a not null sphere, θ is a particular one of the two supplementary angles at which they intersect. In the general case the following construction gives an angle equal to θ.

Pass a sphere through (p) tangent to (q), and pass another sphere through (q) orthogonal to the first sphere. These two spheres meet in a circle, (r), tangent to (q) and cospherical with (p). Let (r) be oriented so as to be properly tangent to (q); then the angle made by (r) and (p) is equal to θ.

Now two spheres may be passed through (p) tangent to (q), and furthermore the rôles of (p) and (q) may be interchanged, so that this construction can be made in four ways. All four angles are proved equal to θ as follows: Calling ϕ the angle of (r) and (p), and recollecting that (r) and (p) are cospherical, we have $\frac{1}{2}\sin^2\phi/2 = \vartheta(pr)$. Hence we need only prove that $\vartheta(pr) = \vartheta(pq)$. Both $\vartheta(pq)$ and the angle above constructed are invariant under the conformal transformations, and so we may begin by moving the circles into a convenient position relative to the coördinate system. Accordingly, let the sphere through (p) tangent to (q) be $(0, 0, 0, 0, 1)$, and that through (q) orthogonal to it $(0, 0, 0, 1, 0)$. Then by means of (4) we see that all plücker coördinates of (p) are zero except those with the subscript 4, and all of (q) are zero except those with the subscript 3. Hence

$$\vartheta(pq) = \frac{p_{34}\, q_{34} + pq}{pq}.$$

The oriented circle (r), being the intersection of the two spheres given above,

* Loc. cit., p. 79.

† *A Study of the circle cross*, these Transactions, vol. 14 (1913), p. 149.

has but one plücker coördinate not zero, r_{34}. By (15),

$$r_{34}^2 + r^2 = 0.$$

Since (r) and (q) are properly tangent,

$$q_{34}\, r_{34} + qr = 0.$$

Multiplying by r and substituting $-r_{34}^2$ for r^2, we obtain

$$q_{34}\, rr_{34} - qr_{34}^2 = 0, \quad \text{or} \quad q_{34} = r_{34}/r \cdot q.$$

Hence

$$\vartheta(pq) = \frac{p_{34}\, q_{34} + pq}{pq} = \frac{p_{34}\, r_{34}/r \cdot q + pq}{pq} = \frac{p_{34}\, r_{34} + pr}{pr} = \vartheta(pr). \quad \text{Q. E. D.}$$

Now we fixed upon no special one of the four constructions. The same proof holds for all four. This establishes the uniqueness of the construction of the angle, and completes the proof of the theorem.

The construction for the angle is seen to yield, in the case of cospherical circles on a not null sphere, the angle at which they intersect, as it ought. In this case, a sphere through (p) tangent to (q) is merely the sphere on which (p) and (q) lie, and the constructed circle (r) coincides with (q). As to circles lying upon the same null sphere, which case has not been covered yet, recourse to formula (16) shows that two not null circles with the same first focus, or the same second focus, have a null angle; and if the first focus of one coincides with the second focus of the other, their angle is π. This is according to the definition by formula (19). But two circles, in fact, any two curves upon a null sphere considered in cartesian space meet at a null or a straight angle. For their angle is that made by their two tangent lines at their point of intersection. These two lines lie in an isotropic plane tangent to the null sphere, and so make a null or straight angle. This lends reasonableness to the definition by formula (19). Finally, if the circles are in bi-involution, the construction for their angle falls through; for all spheres through either are orthogonal to the other. Formula (19) gives their angle as a right angle, as we can prove by use of (14).

Our definition of proper and improper tangency was given only in case of circles on a not null sphere. We can now extend this definition:

If two not null circles are tangent in the sense of meeting in two coincident points, then they are *properly tangent* if their angle is null, and *improperly tangent* if their angle is π.

5. Some miscellaneous theorems

Formula (19) confronted by (14) yields

THEOREM 1. *Two not null circles in involution make a right angle.*

Every linear combination of two oriented circles

$$r_{ij} = \lambda p_{ij} + \mu q_{ij}, \qquad r = \lambda p + \mu q,$$

will be the coördinates of an oriented circle, i. e. satisfy (5), (6), and (15), if and only if

$$\Omega_i(pq) = 0 \quad (i = 0, 1, 2, 3, 4), \qquad \sum p_{ij}\, q_{ij} + pq = 0.$$

Thus (p) and (q) must be cospherical and make a null angle if their angle be defined. If they lie upon a not null sphere, they are properly tangent and (r) sweeps out the properly tangent pencil determined by (p) and (q). But if they have the same first focus, or same second focus, they satisfy the conditions, and (r) sweeps out the pencil, not necessarily tangent, determined by (p) and (q) upon the null sphere which is the common focus. In fact if we take four oriented circles all having the same first focus (or same second focus), each two will satisfy the above conditions, and so the values (r), linear combinations of the four, will be the coördinates of oriented circles; and (r) will sweep out every oriented circle having that first (or second) focus.

An invariant of four oriented circles which should receive mention is

$$\frac{[\sum p_{ij}\, q_{ij} + pq]\,[\sum r_{ij}\, s_{ij} + rs]}{[\sum p_{ij}\, s_{ij} + ps]\,[\sum r_{ij}\, q_{ij} + rq]} = \frac{\sin^2 \frac{1}{2}\theta_{pq} \sin^2 \frac{1}{2}\theta_{rs}}{\sin^2 \frac{1}{2}\theta_{ps} \sin^2 \frac{1}{2}\theta_{rq}}.$$

If any of the circles is null the invariant still exists but cannot be expressed in terms of the angles which are not determined for null circles.

THEOREM 2. *The locus of the first foci of the oriented circles on a given not null sphere that cut a given oriented circle thereof at a fixed angle, is a sphere through the given circle.*

Let (z) be the given sphere and (α) the first focus of the given oriented circle (q), and (x) the first focus of an oriented circle (p) on (z) cutting (q) at the fixed angle $\theta = \cos^{-1} k$. Then

$$\cos\theta = \frac{\sum p_{ij}\, q_{ij}}{-pq} = \frac{(x\alpha)(zz) - (z\alpha)(xz)}{-(xz)(\alpha z)} = k,$$

or

$$(zz)(x\alpha) + (k-1)(z\alpha)(xz) = 0.$$

Hence (x) satisfies a linear relation and generates a sphere.

THEOREM 3. *Given six independent homogeneous equations in the coördinates p_{ij} and p, of degrees $n_1, n_2, n_3, n_4, n_5, n_6$ respectively; in general there will be just $10 n_1 n_2 n_3 n_4 n_5 n_6$ oriented circles whose coördinates satisfy the conditions.*

Now equations (6) constitute only three independent conditions. If we solve the first three equations of (6) together with the given equations and (5) and (15), we obtain $16 n_1 n_2 n_3 n_4 n_5 n_6$ solutions. All of these for which $p_{34} \neq 0$ will satisfy the last two equations of (6) as we saw by (7). But if any of these solutions have $p_{34} = 0$ they must be rejected for they will not in general satisfy the last two equations of (6), since these are independent of

the first three when $p_{34} = 0$. Solve

$$\Omega_0(pp) = \Omega_1(pp) = p_{34} = 0$$

together with (5), (15) and the six given equations, finding $8n_1 n_2 n_3 n_4 n_5 n_6$ solutions. By (7) these are seen to satisfy

$$p_{32}\,\Omega_2(pp) = 0, \qquad p_{42}\,\Omega_2(pp) = 0,$$

so they are divided into two categories: those that satisfy $\Omega_2(pp) = 0$ and those that satisfy $p_{32} = p_{42} = 0$. Finally solve

$$p_{32} = p_{42} = p_{34} = 0$$

with (6), (15), and the given equations, and get $2n_1 n_2 n_3 n_4 n_5 n_6$ solutions. These evidently make $\Omega_0(pp) = \Omega_1(pp) = 0$; $\Omega_2(pp) \neq 0$. Thus there are $6n_1 n_2 n_3 n_4 n_5 n_6$ solutions for which

$$\Omega_0(pp) = \Omega_1(pp) = \Omega_2(pp) = p_{34} = 0$$

and for these

$$\Omega_3(pp)\,\Omega_4(pp) \neq 0. \qquad \text{Q. E. D.}$$

Suppose we select p and any ten essentially different plücker coördinates, and regard them as the homogeneous coördinates of a point in ten-dimensional space. Then Theorem 3 shows that oriented circles correspond to a six-dimensional spread of tenth order lying in the ten-dimensional space. We shall denote this ten-dimensional space by S_{10}; and the six-dimensional spread of tenth order by S_6^{10}.

6. The linear hypercomplex

All oriented circles whose coördinates satisfy a single linear homogeneous relation shall be said to form a *linear hypercomplex*. The general linear equation contains twenty-one terms, but by aid of (5) twenty of these combine in pairs and yield

$$\sum a_{ij}\, p_{ij} + ap = 0, \tag{20}$$

where the summation extends over only ten essentially different terms. The hypercomplex is called *reduced* in case $a = 0$, and is called *special* if $\Omega_i(aa) = 0$ $(i = 0, 1, 2, 3, 4)$. If the hypercomplex is neither reduced nor special, and if furthermore $\sum [\Omega_i(aa)]^2 \neq 0$, then it is said to be *general*.

The linear hypercomplex corresponds in S_{10} to the points of S_6^{10} in a hyperplane. A hyperplane is determined by ten of its points, hence

Theorem 4. *Ten arbitrary oriented circles determine in general one linear hypercomplex of oriented circles whose coördinates are linearly dependent on those of the ten.*

In a special linear hypercomplex the coefficients a_{ij} satisfy the conditions

for being the coördinates of a non-oriented circle. If we orient this circle in both ways by adjoining the proper redundant coördinates $\pm\sqrt{-\sum a_{ij}^2}$, we obtain two circles that differ only in orientation, and are called the *central circles* of the special hypercomplex. If the central circles are not null, we may write the special hypercomplex in the form

$$\frac{\sum a_{ij}\, p_{ij} \pm p\sqrt{-\sum a_{ij}^2}}{\pm p\sqrt{-\sum a_{ij}^2}} = 1 - \frac{a}{\pm\sqrt{-\sum a_{ij}^2}}.$$

THEOREM 5. *The oriented circles of a special linear hypercomplex whose central circles are not null make fixed supplementary angles with two oriented circles that differ only in orientation.*

No two circles that differ only in orientation can belong to the same non-reduced linear hypercomplex. The not null oriented circles that belong to a reduced hypercomplex fall into pairs that differ only in orientation.

By aid of (17), (20) may be written

$$\sum a_{ij}\,(x_i\, z_j - x_j\, z_i) + a\,(xz) = 0,$$

summed over only ten terms. For convenience let us introduce ten new numbers a_{ji} defined by $a_{ji} = -a_{ij}$ and the above equation becomes

$$\sum_{i,j=0}^{4} a_{ij}\, x_i\, z_j + a\,(xz) = 0,$$

summed over all terms. Set

$$\xi_i = \sum_j a_{ij}\, z_j \qquad (i = 0, 1, 2, 3, 4).$$

Then $(\xi z) = 0$ and $\sum_n \xi_n\, \Omega_n\,(aa) = 0$.

If (z) is held fixed, the focus (x) traces the sphere $\xi_i + az_i$ and by theorem 2, we have

THEOREM 6. *The oriented circles of a general linear hypercomplex upon an arbitrary not null sphere cut an oriented circle thereof at a certain angle which may differ for different spheres but is never a right angle.*

If the angle were ever a right angle we should be able to find two oriented circles of the hypercomplex differing only in orientation, which can only happen when the complex is reduced. If the hypercomplex is reduced, its circles that lie on an arbitrary sphere cut a circle thereof at right angles.

Suppose in the preceding work that (z) is null. Then (x) and (z) are the first and second foci of the circle (p). If we give (z) a fixed value (α), then (x) traces the sphere

$$u_i = \sum_j a_{ij}\, \alpha_j + a\alpha_i.$$

If, on the other hand, we give (x) a fixed value (α), then (z) traces the

sphere

$$v_j = \sum_i a_{ij}\,\alpha_i + a\alpha_j = -\sum_i a_{ji}\,\alpha_i + a\alpha_j .$$

Now (u) and (v) are tangent at the point (α). Hence

THEOREM 7. *The oriented circles of a linear non-reduced hypercomplex upon a null sphere fall into two classes: if the vertex of the null sphere be the first focus the second focus traces a sphere; if it be the second focus, the first focus traces a sphere. These two spheres are tangent to each other at the vertex of the null sphere.*

*The central sphere** of the hypercomplex shall be that sphere whose coördinates are the five quantities $\Omega_i\,(a\alpha)$.

The reasoning leading to Theorem 6 fails if $\xi_i \equiv 0$, or $\xi_i \equiv \lambda z_i$. The former case occurs only if (z) is the central sphere. The latter case occurs if (z) is any one of four particular null spheres. In both cases the circles of the hypercomplex on (z) are all the null circles on (z). But in the case of the reduced hypercomplex every circle on the central sphere belongs to the hypercomplex.

The circles orthogonal to two spheres will pass through two points, namely the vertices of the circle in which these spheres meet. To investigate the circles of the hypercomplex that pass through two points we may examine those that are orthogonal to the two spheres $(0, 0, 0, 0, 1)$, and $(0, 0, 0, 1, 0)$ which need have no special relation to the hypercomplex. The conditions imposed on the circles are that every coördinate whose subscript contains either 3 or 4 shall be zero. These conditions carry with them the satisfaction of (6); so that only (15) and (20) remain, and we have

$$p_{01}^2 + p_{02}^2 + p_{12}^2 + p^2 = 0, \qquad a_{01}\,p_{01} + a_{02}\,p_{02} + a_{12}\,p_{12} + ap = 0 .$$

These circles generate a surface. Now equation (13) is of the second order, and if solved with these yields four solutions for (p), showing that four of these circles meet an arbitrary circle (q). The surface in question meets a circle as often as it meets a line, hence

THEOREM 8. *The oriented circles of a general linear hypercomplex through two arbitrary points generate a two horned cyclide.*

The two horns or conical points are, of course, the two fixed points through which the circles pass.

Let θ and ϕ denote the angles that a variable oriented circle makes with two fixed not null oriented circles (α) and (β); and consider the oriented circles that satisfy

$$A \cos\theta + B \cos\phi + C = 0,$$

* G. Koenigs, loc. cit., page F. 12; also G. Castelnuovo, Atti della R. Istituto Veneto., ser. 7, vol. 2, part 1, page 861. This article is on line geometry in four dimensions. But this subject is equivalent to the study of non-oriented circles in three-way space; for a sphere may be regarded as a point of a four-dimensional space, and the pencil of spheres through a circle will correspond to a line in four dimensions.

where A, B, C, are constants. The oriented circles (p) that satisfy this generate a linear hypercomplex, for it may be written

$$A\frac{\sum \alpha_{ij}\, p_{ij}}{-\alpha p} + B\frac{\sum \beta_{ij}\, p_{ij}}{-\beta p} + C = 0,$$

$$\sum [A\beta\alpha_{ij} + B\alpha\beta_{ij}]\, p_{ij} - C\alpha\beta p = 0.$$

The central sphere of this hypercomplex is

$$z_i = 2AB\alpha\beta\, \Omega_i(\alpha\beta) \qquad (i = 0, 1, 2, 3, 4).$$

This sphere is orthogonal to the circles (α) and (β), for, by the identities (8) we obtain

$$\sum_n z_n\, \alpha_{in} = 2AB\alpha\beta \sum_n \alpha_{in}\, \Omega_n(\alpha\beta) \equiv -AB\alpha\beta \sum_n \beta_{in}\, \Omega_n(\alpha\alpha) = 0 \qquad (i = 0, 1, 2, 3, 4);$$

and similarly for (β).

THEOREM 9. *If an oriented circle so move that the cosines of its angles with two fixed oriented circles satisfy a general linear relation, it traces a linear hypercomplex whose central sphere is the common orthogonal sphere of the two circles.*

Conversely, any linear hypercomplex can be described in this way. The problem is, given the hypercomplex (a), to find two suitable oriented circles (α) and (β). They must be orthogonal to the central sphere, by Theorem 9. First suppose that the central sphere of the complex is not null, and take for this sphere $(0, 0, 0, 0, 1)$. Then by (7) we have

$$a_{04} = a_{14} = a_{24} = a_{34} = 0.$$

Also all coördinates of (α) and (β) with the subscript 4 are zero; and so all of equations (6) are satisfied automatically by them except the last one. Our problem reduces to finding (α) and (β) such that

$$\lambda\alpha_{ij} + \mu\beta_{ij} = a_{ij} \qquad (i, j = 0, 1, 2, 3),$$

subject to the further conditions

$$\Omega_4(\alpha\alpha) = \Omega_4(\beta\beta) = 0.$$

Eliminating (β) we obtain

$$1/\mu^2\, \Omega_4(aa) - 2(\lambda/\mu^2)\, \Omega_4(a\alpha) = 0, \qquad \lambda = \Omega_4(aa)/2\Omega_4(a\alpha),$$

$$\Omega_4(a\alpha) \neq 0.$$

Hence

$$\beta_{ij} = \Omega_4(aa)\, \alpha_{ij} - 2\Omega_4(a\alpha)\, a_{ij}.$$

Thus (α) may be chosen and (β) found to satisfy the demands.

Again, suppose that the central sphere of the given hypercomplex is null. We may take for its coördinates $(0, 0, 0, 1, \sqrt{-1})$. Then by (7)

$$a_{34} = 0, \qquad a_{i3} + \sqrt{-1}\, a_{14} = 0 \qquad (i = 0, 1, 2).$$

Similar relations will be imposed on the coördinates of (α) and (β); and in view of them, the first three equations of (6) are satisfied, and the last two reduce to the same condition. Our problem reduces to exactly what it was before, and can be solved as before. Again, if the hypercomplex is special, the coefficients are the plücker coördinates of some circle. In this case it is sufficient to take the (α) and (β) cospherical and such that the central circles of the hypercomplex belong to their (not tangent) pencil. Finally if the hypercomplex is $p = 0$, it consists of all the null circles, and this generation fails.

THEOREM 10. *Any linear hypercomplex except that of all null circles can be generated in an infinite number of ways by an oriented circle so moving that the cosines of its angles with two fixed oriented circles satisfy a general linear relation.*

Let us return to the general case where the central sphere is not null, and enquire whether the two circles can be in bi-involution. The conditions are

$$\beta_{01} = \rho\alpha_{23}, \qquad \beta_{02} = \rho\alpha_{31}, \qquad \beta_{03} = \rho\alpha_{12}, \qquad \beta_{23} = \rho\alpha_{01}, \qquad \beta_{31} = \rho\alpha_{02},$$
$$\beta_{12} = \rho\alpha_{03},$$

which may be extracted from formula (14). In short, we seek a single circle (α) such that $\lambda\alpha_{ij} + \mu'\alpha_{kl} = a_{ij}$. Hence

$$\lambda\alpha_{kl} + \mu'\alpha_{ij} = a_{kl}, \qquad \alpha_{ij} = 1/(\lambda^2 - \mu'^2)(\lambda a_{ij} - \mu' a_{kl}),$$

and we must have

$$0 = \Omega_4(\alpha\alpha) = 1/(\lambda^2 - \mu'^2)^2[\lambda^2\Omega_4(aa) - 2\lambda\mu'\Sigma a_{ij}^2 + \mu'^2\Omega_4(aa)].$$

Set

$$H = \frac{\sum a_{ij}^2}{\Omega_4(aa)}.$$

Then there are two values of $\lambda : \mu'$ and so two values for (α) namely

$$\alpha'_{ij} = [H + \sqrt{H^2 - 1}]\,a_{ij} - a_{kl}, \qquad \alpha''_{ij} = [H - \sqrt{H^2 - 1}]\,a_{ij} - a_{kl}.$$

These two circles form a *cross*.* If (α) is chosen as one of them (β) must be the other. This circle cross may be used to define the hypercomplex as in Theorem 9, and is uniquely determined in general except, of course, for the orientation.

By a process similar to that used to deduce Theorem 9 we have

THEOREM 11. *If an oriented circle move so that the ratio of its invariants with two fixed oriented circles is constant, it describes a linear hypercomplex the central sphere of which is the common orthogonal sphere of the fixed circles.*

Can the general hypercomplex be generated in this way? As in the work leading up to Theorem 10, we first take the central sphere of the hypercomplex

* Coolidge first defined the circle cross, these Transactions, vol. 14 (1913), p. 149.

(a) as $(0, 0, 0, 0, 1)$, and then seek two oriented circles (α) and (β) such that

$$\lambda\alpha_{ij} + \mu\beta_{ij} = a_{ij} \quad (i, j = 0, 1, 2, 3), \qquad \lambda\alpha + \mu\beta = a,$$

where

$$\Omega_4(\alpha\alpha) = \Omega_4(\beta\beta) = 0, \qquad \sum\alpha_{ij}^2 + \alpha^2 = \sum\beta_{ij}^2 + \beta^2 = 0.$$

We can eliminate (β) in two ways this time and obtain

$$\Omega_4(\beta\beta) = \frac{1}{\mu^2}\Omega_4(aa) - 2\frac{\lambda}{\mu^2}\Omega_4(a\alpha) = 0,$$

$$\sum\beta_{ij} + \beta^2 = \frac{1}{\mu^2}[\sum a_{ij} + a^2] - \frac{2\lambda}{\mu}[\sum a_{ij}\alpha_{ij} + a\alpha] = 0.$$

Eliminating λ from these two equations gives

$$[\sum a_{ij}^2 + a^2]\,\Omega_4(a\alpha) - \Omega_4(aa)\,[\sum a_{ij}\alpha_{ij} + a\alpha] = 0,$$

assuming

$$\Omega_4(a\alpha) \neq 0, \qquad \sum a_{ij}\alpha_{ij} + a\alpha \neq 0.$$

Hence if we take (α) as a not null circle orthogonal to the central sphere and satisfying these conditions, another circle (β) can be found as desired. We find small interest in pushing the theorem to cases other than the general one.

THEOREM 12. *A general linear hypercomplex can be generated in an infinite number of ways by an oriented circle so moving that its invariants with two oriented circles orthogonal to the central sphere bear a constant ratio.*

7. THE LINEAR HYPERCONGRUENCE

The totality of oriented circles whose coördinates satisfy two independent linear conditions is called a *linear hypercongruence,* namely

$$\sum a_{ij}p_{ij} + ap = 0, \qquad \sum b_{ij}p_{ij} + bp = 0. \tag{21}$$

The hypercongruence corresponds to the intersection of S_6^{10} and two hyperplanes of S_{10}. Hence

THEOREM 13. *Nine arbitrary oriented circles determine in general one linear hypercongruence of oriented circles whose coördinates are linearly dependent on those of the nine.*

The oriented circles of a hypercongruence all belong to the *pencil* of linear hypercomplexes

$$l_{ij} = \lambda a_{ij} + \mu b_{ij}, \qquad l = \lambda a + \mu b.$$

THEOREM 14. *The oriented circles of a linear hypercongruence belong to one reduced hypercomplex.*

If, however, a and b are both zero, then every hypercomplex in the pencil is reduced.

The central sphere* of (l) is

$$\Omega_i(ll) = \lambda^2 \Omega_i(aa) + 2\lambda\mu\Omega_i(ab) + \mu^2 \Omega_i(bb).$$

THEOREM 15. *If two hypercomplexes have the same central sphere this is also the central sphere of every hypercomplex of their pencil.*

The proof of this theorem is immediate if the common central sphere be taken as $(0, 0, 0, 0, 1)$.

THEOREM 16. *If two linear hypercomplexes are special, making fixed angles with two circles, their pencil has one common central sphere orthogonal to the two circles, unless these two circles be cospherical when all hypercomplexes of the pencil are special, and consist of oriented circles making fixed angles with the circles of the pencil determined by the two.*

To prove this theorem observe that if $\Omega_i(aa) = \Omega_i(bb) = 0$ $(i = 0, 1, 2, 3, 4)$, then $\Omega_i(ll) = 2\lambda\mu\Omega_i(ab)$. And if furthermore $\Omega_i(ab) = 0$, then $\Omega_i(ll) = 0$.

THEOREM 17. *If one hypercomplex be special and another not, then the central spheres of their pencil generate a coaxial system.*

THEOREM 18. *If two linear hypercomplexes have different central spheres, then their central spheres generate a one-parameter family whereof two are orthogonal to an arbitrary sphere, called a conic series.*

The envelope of the series is the cyclide

$$\sum \Omega_i(aa) x_i \sum \Omega_i(bb) x_i - [\sum \Omega_i(ab) x_i]^2 = 0.$$

If $\Omega_i(ab) = 0$, $(i = 0, 1, 2, 3, 4)$, then $\rho\Omega_i(aa) + \sigma\Omega_i(bb) = 0$. For by means of the identities (7) and (8)

$$\sum_n a_{in} \Omega_n(aa) = 0, \qquad \sum_n a_{in} \Omega_n(bb) = 0$$

$$\sum_n b_{in} \Omega_n(aa) = 0, \qquad \sum_n b_{in} \Omega_n(bb) = 0,$$

and $\Omega_i(aa)$ and $\Omega_i(bb)$ are determined by the same set of linear equations. Hence the pencil of hypercomplexes have the same central sphere, or else all of them are special.

By Theorem 6 we observe that the oriented circles of a linear hypercongruence upon an arbitrary sphere cut two oriented circles thereof at fixed angles. Hence they are tangent to two circles in the pencil of these two.

THEOREM 19. *The oriented circles of a linear hypercongruence upon an arbitrary not null sphere are properly tangent to two oriented circles thereof.*

By aid of Theorem 7, we obtain

THEOREM 20. *The oriented circles of a linear hypercongruence upon an arbitrary null sphere fall into two classes; if the vertex of the null sphere be the*

* The central spheres of the hypercongruenee are studied by G. Castelnuovo, loc. cit., p. 863 ff.

first focus, the second focus traces a circle; if it be the second focus, the first focus traces a circle. These two circles are tangent at the vertex of the null sphere.

Let us examine for exceptions to Theorem 19. By means of Theorem 14, we may regard the general hypercongruence as determined by one reduced and one non-reduced hypercomplex. But as we remarked, the reduced hypercomplex imposes no condition upon the circles of its central sphere. Hence

THEOREM 21. *The oriented circles of a general hypercongruence upon the central sphere of the reduced hypercomplex of the corresponding pencil are those cutting an oriented circle thereof at a fixed angle.*

We observe further that the oriented circles of a non-reduced hypercomplex upon its central sphere are all null.

THEOREM 22. *The oriented circles of the general hypercongruence upon the central sphere of a non-reduced hypercomplex of the pencil are all null and their centers trace a circle upon the sphere.*

This theorem may be regarded as the limiting case of Theorem 19 where the oriented circles are all tangent to two oriented circles that differ only in orientation. Thus Theorem 21 mentions the only not null sphere that has a two parameter family of the oriented circles of the hypercongruence on it. For in every other case the reduced hypercomplex requires the circles to cut a circle at a right angle, and the non-reduced hypercomplex requires them to cut an oriented circle at some other angle, and these two conditions must always be independent.

The locus of the centers of the null circles of the hypercongruence that lie upon the series of central spheres is easily found. In the proof preceding Theorem 6, we found that the first foci of the oriented circles of the hypercomplex (a), which lie upon an arbitrary sphere (z), trace a certain sphere (ξ) where

$$\xi_i = \sum_j a_{ij} z_j .$$

Now in our case we take for (z) the central sphere of the general hypercomplex of the pencil, (l). Then the null circles of this sphere that belong to the hypercomplex (a) must have their centers lying on the sphere

$$\xi_i = \sum_j a_{ij} \Omega_j (ll) = \sum_j [\lambda^2 \Omega_j (aa) + 2\lambda\mu\Omega_j (ab) + \mu^2 \Omega_j (bb)] a_{ij} .$$

By using identities (7) and (8) we obtain

$$\xi_i = \mu \sum_j [\mu a_{ij} \Omega_j (bb) - \lambda b_{ij} \Omega_j (aa)] .$$

What we wish to do is to find the locus of the intersection of (ξ) and the central sphere of (l) as $\lambda : \mu$ varies. We need only eliminate $\lambda : \mu$ between

$$(\xi x) = 0, \qquad \sum_i \Omega_i (ll) x_i = 0,$$

and we obtain the equation

$$[\sum_{ij} a_{ij}\,\Omega_j(bb)\,x_i]^2 \sum_j \Omega_i(aa)\,x_i + 2\,[\sum_{ij} a_{ij}\,\Omega_j(bb)\,x_i]\,[\sum_{ij} b_{ij}\,\Omega_j(aa)\,x_i] \sum_i \Omega_i(ab)\,x_i + [\sum_{ij} b_{ij}\,\Omega_j(aa)\,x_i]^2 \sum_i \Omega_i(bb)\,x_i = 0.$$

Consider the circles of the general linear hypercongruence orthogonal to $(0, 0, 0, 0, 1)$ and $(0, 0, 0, 1, 0)$, which spheres need bear no special relation to the hypercongruence. All coördinates with the subscript 3 or 4 are zero. Conditions (6) are all satisfied because of this. We have then equations (5), (15), and (21) to determine the remaining coördinates, which have two solutions. Hence

THEOREM 23. *Two circles of the general linear hypercongruence pass through an arbitrary pair of points.*

8. THE LINEAR COMPLEX

Oriented circles whose coördinates satisfy three independent linear relations generate a *linear complex,*

$$(22)\qquad \sum a_{ij}\,p_{ij} + ap = 0, \qquad \sum b_{ij}\,p_{ij} + bp = 0, \qquad \sum c_{ij}\,p_{ij} + cp = 0.$$

These oriented circles correspond, of course, to the points of S_6^{10} that lie in a linear spread of seven dimensions.

THEOREM 24. *Eight arbitrary oriented circles determine in general one linear complex of oriented circles whose coördinates depend linearly upon those of the eight.*

The oriented circles of a linear complex are common to a *net* of hypercomplexes

$$l_{ij} = \lambda a_{ij} + \mu b_{ij} + \nu c_{ij}, \qquad l = \lambda a + \mu b + \nu c.$$

THEOREM 25. *The oriented circles of a complex belong to a pencil of reduced hypercomplexes.*

If a, b, and c are all zero, then all hypercomplexes of the net are reduced.

The central sphere* of (l) is

$$\Omega_i(ll) = \lambda^2\,\Omega_i(aa) + \mu^2\,\Omega_i(bb) + \nu^2\,\Omega_i(cc) + 2\lambda\mu\Omega_i(ab) + 2\lambda\nu\Omega_i(ac) + 2\mu\nu\Omega_i(bc).$$

THEOREM 26. *The central spheres of a linear set of hypercomplexes generate in general a quartic congruence of spheres.*

The envelope of this congruence is

$$\begin{vmatrix} \sum\Omega_i(aa)\,x_i & \sum\Omega_i(ab)\,x_i & \sum\Omega_i(ac)\,x_i \\ \sum\Omega_i(ba)\,x_i & \sum\Omega_i(bb)\,x_i & \sum\Omega_i(bc)\,x_i \\ \sum\Omega_i(ca)\,x_i & \sum\Omega_i(cb)\,x_i & \sum\Omega_i(cc)\,x_i \end{vmatrix} = 0.$$

THEOREM 27. *If three hypercomplexes have a common central sphere, then this is the central sphere for every hypercomplex of their net.*

* Central spheres are discussed by Castelnuovo, loc. cit., p. 868 ff.

The proof is immediate on taking for the common central sphere $(0, 0, 0, 0, 1)$.

By Theorem 25 we may regard the complex as determined by two reduced and one not reduced linear hypercomplex. Thus the oriented circles thereof that lie upon an arbitrary sphere must cut two circles thereof at right angles, and make a certain other angle with another oriented circle. Two circles do this, hence

THEOREM 28. *Two oriented circles of a complex lie upon an arbitrary sphere.*

If, however, we take the central sphere of one of the reduced hypercomplexes, its circles are subject to only the two conditions imposed by the other two hypercomplexes, and as we proved in Theorem 19, we obtain

THEOREM 29. *The oriented circles of the complex upon the central sphere of one of the reduced hypercomplexes are those properly tangent to two oriented circles.*

THEOREM 30. *Every null circle of the complex lies on three central spheres of the net that are mutually tangent, and when three central spheres are mutually tangent at a point, their common null circle belongs to the complex.*

The proof of this theorem comes from the fact that the null circles of the general complex are the same as the null circles of the complex reduced by setting a, b, and c all zero, in which every circle of the complex lies on three central spheres.*

Consider the circles of the general complex orthogonal to $(0, 0, 0, 0, 1)$, which sphere need bear no special relation to the complex. All coördinates with subscript 4 are zero. The remaining coördinates are subject to conditions (21), (15), (5), and the last equation of (6). Let us enquire how many of them meet an arbitrary circle (q). The condition as given by (13) is quadratic. Hence we find eight solutions.

THEOREM 31. *The surface swept out by the oriented circles of the complex orthogonal to an arbitrary sphere is of the eighth order, having in euclidean space the circle at infinity as a quadruple curve.*

9. THE LINEAR CONGRUENCE

Oriented circles whose coördinates satisfy four independent linear equations form a *linear congruence*

$$\begin{aligned} &\sum a_{ij}\, p_{ij} + ap = 0, \qquad \sum b_{ij}\, p_{ij} + bp = 0, \\ &\sum c_{ij}\, p_{ij} + cp = 0, \qquad \sum d_{ij}\, p_{ij} + dp = 0. \end{aligned} \tag{23}$$

These oriented circles correspond to the points of S_6^{10} lying in a linear subspace of six dimensions, hence

THEOREM 32. *Seven arbitrary oriented circles determine in general one congruence of oriented circles whose coördinates are linearly dependent upon those of the seven.*

* For proof of this see G. Castelnuovo, loc. cit., p. 870.

The oriented circles of the congruence are common to the three-parameter family of hypercomplexes linearly dependent upon four

$$l_{ij} = \lambda a_{ij} + \mu b_{ij} + \nu c_{ij} + \rho d_{ij}, \qquad l = \lambda a + \mu b + \nu c + \rho d.$$

By a method used to prove Theorem 3, we find that five sets of the quantities l_{ij} satisfy equations (6). These five sets of quantities are consequently the plücker coördinates of five non-oriented circles. This is the *pentacycle* of Stephanos.* Each of these circles may be given an orientation in two ways, whence result the theorems

THEOREM 33. *The general linear congruence consists in thirty-two ways of oriented circles making a (different) fixed angle with each circle of a pentacycle.*

THEOREM 34. *Oriented circles making fixed angles with four arbitrary oriented circles make a fixed angle with a fifth which completes the pentacycle. If each of the first four angles is a right angle, the fifth is also a right angle.*

We observe that a two-parameter net of the hypercomplexes of (l) are reduced.

THEOREM 35. *The oriented circles of a congruence are common to a net of reduced hypercomplexes.*

Now no circle of the congruence lies on a general sphere, but the oriented circles upon the central sphere of one of the reduced hypercomplexes are, as we have seen, subject to only three conditions and two circles of the congruence lie upon each.

THEOREM 36. *Two oriented circles of the congruence lie upon the central sphere of each of the reduced hypercomplexes to which the congruence belongs.*

These are all the oriented circles of the congruence. For our congruence belongs to a reduced linear complex determined by the net of reduced hypercomplexes mentioned in Theorem 35; and Castelnuovo has shown (loc. cit., p. 870) that the circles of such a complex all lie upon the net of central spheres.

The sphere $(0, 0, 0, 0, 1)$ will bear no special relation to the general linear congruence, and we see that four oriented circles of the congruence are orthogonal to it.

THEOREM 37. *Four oriented circles of a linear congruence are orthogonal to an arbitrary sphere.*

To find the locus of the first foci of the oriented circles of the congruence, we must use the formula for determining the plücker coördinates of a circle by means of three spheres, (r), (s), and (t), orthogonal to it, namely

$$\alpha_{ij} = \begin{vmatrix} r_k & r_l & r_m \\ s_k & s_l & s_m \\ t_k & t_l & t_m \end{vmatrix}.$$

Now by Theorem 35, we may use in place of the hypercomplex (d) of

* C. Stephanos, Comptes Rendus, vol. 93.

formula (23) a reduced hypercomplex to determine our congruence. And by Theorem 10 this reduced hypercomplex may be written

$$\sum \alpha_{ij}\, p_{ij} + \sum \beta_{ij}\, p_{ij} = 0,$$

where (α) and (β) are the plücker coördinates of two circles. Let these two circles be determined by the two triads of spheres, (r), (s), and (t); and (r'), (s'), and (t') orthogonal to them. Furthermore let the variable circle (p) be given in terms of its first and second foci, (x) and (y). The above equation then becomes

$$\begin{vmatrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ y_0 & \cdot & \cdot & \cdot & y_4 \\ r_0 & \cdot & \cdot & \cdot & r_4 \\ s_0 & \cdot & \cdot & \cdot & s_4 \\ t_0 & \cdot & \cdot & \cdot & t_4 \end{vmatrix} + \begin{vmatrix} x_0 & \cdot & \cdot & \cdot & x_4 \\ y_0 & \cdot & \cdot & \cdot & y_4 \\ r_0' & \cdot & \cdot & \cdot & r_4' \\ s_0' & \cdot & \cdot & \cdot & s_4' \\ t_0' & \cdot & \cdot & \cdot & t_4' \end{vmatrix} = 0.$$

Denote the first determinant by Δ and the second by Δ'. Besides this equation we have the first three equations of (23), which in terms of (x) and (y) are

$$\sum_{i,j=0}^{4} a_{ij}\, x_i\, y_j + a\,(xy) = 0,$$
$$\sum_{i,j=0}^{4} b_{ij}\, x_i\, y_j + b\,(xy) = 0,$$
$$\sum_{i,j=0}^{4} c_{ij}\, x_i\, y_j + c\,(xy) = 0,$$

where the summation is to extend over all terms. Finally, because (x) and (y) are null spheres, we have

$$(xx) = (yy) = 0.$$

We wish to eliminate the second focus (y) and obtain the equation of the surface in (x). The first step is the pseudo-elimination of (y) by equating to zero the following determinant:

$$\begin{vmatrix} \dfrac{\partial\Delta}{\partial y_0} + \dfrac{\partial\Delta'}{\partial y_0} & \dfrac{\partial\Delta}{\partial y_1} + \dfrac{\partial\Delta'}{\partial y_1} & \cdots & \dfrac{\partial\Delta}{\partial y_4} + \dfrac{\partial\Delta'}{\partial y_4} \\ y_0 & y_1 & \cdots & y_4 \\ \sum_i a_{i0}\, x_i + a x_0 & \sum_i a_{i1}\, x_i + a x_1 & \cdots & \sum_i a_{i4}\, x_i + a x_4 \\ \sum_i b_{i0}\, x_i + b x_0 & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot & \sum_i b_{i4}\, x_i + b x_4 \\ \sum_i c_{i0}\, x_i + c x_0 & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot & \sum_i c_{i4}\, x_i + c x_4 \end{vmatrix} = 0.$$

This equation is true, but not yet free from (y). Let this determinant be expressed as the sum of two determinants, the first row of one having the elements $\partial\Delta/\partial y_i$, and the first row of the other $\partial\Delta'/\partial y_i$; the last four rows being as they stand above. Each of these determinants may be regarded as the

product of two matrices; for the elements of the first rows are themselves determinants taken from a certain matrix of four rows and five columns. Hence

$$\begin{vmatrix} (xy) & (ry) & (sy) & (ty) \\ 0 & \sum_{ij} a_{ij} x_i r_j + a(xr) & \cdots & \sum_{ij} a_{ij} x_i t_j + a(xt) \\ 0 & \sum_{ij} b_{ij} x_i r_j + b(xr) & \cdots & \sum_{ij} b_{ij} x_i t_j + b(xt) \\ 0 & \sum_{ij} c_{ij} x_i r_j + c(xr) & \cdots & \sum_{ij} c_{ij} x_i t_j + c(xt) \end{vmatrix}$$

$$+ \begin{vmatrix} (xy) & (r'y) & (s'y) & (t'y) \\ 0 & \sum_{ij} a_{ij} x_i t'_j + a(xr') & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \sum_{ij} c_{ij} x_i t'_j + c(xt') \end{vmatrix} = 0.$$

Factoring out (xy) we have the equation. Hence

THEOREM 38. *The locus of the first foci of the oriented circles of a linear congruence is a surface of the sixth order having in euclidean space the circle at infinity as a triple curve. The second foci trace another such locus, and the two surfaces fall together in case the hypercomplexes determining the congruence are all reduced.*

10. THE LINEAR SERIES

Oriented circles whose coördinates satisfy five independent linear relations form a *linear series,*

$$\begin{gathered} \sum a_{ij} p_{ij} + ap = 0, \qquad \sum b_{ij} p_{ij} + bp = 0, \qquad \sum c_{ij} p_{ij} + cp = 0, \\ \sum d_{ij} p_{ij} + dp = 0, \qquad \sum e_{ij} p_{ij} + ep = 0. \end{gathered} \tag{24}$$

THEOREM 39. *Six arbitrary oriented circles determine in general one linear series of oriented circles whose coördinates are linearly dependent on those of the six.*

The oriented circles of a linear series are common to a four-parameter family of hypercomplexes linearly dependent on five,

$$\begin{gathered} l_{ij} = \lambda a_{ij} + \mu b_{ij} + \nu c_{ij} + \rho d_{ij} + \sigma e_{ij}, \\ l = \lambda a + \mu b + \nu c + \rho d + \sigma e. \end{gathered}$$

If the given five hypercomplexes are reduced, every one of this system will be reduced, and the series will naturally be called *reduced.*

THEOREM 40. *The oriented circles of a non-reduced series belong to a three-parameter system of reduced hypercomplexes linearly dependent on four.*

By methods like those used to prove Theorem 3 we find that five members of this three parameter family are special hypercomplexes. Their central circles form a pentacycle, hence

THEOREM 41. *The oriented circles of a series make a right angle with each circle of a pentacycle, besides satisfying another independent condition.*

This pentacycle is in general unique.

There are an infinite number of special hypercomplexes in the four-parameter system (l). The plücker coördinates (but not the redundant coördinates) of the central circles of these are linearly dependent upon the five sets of coefficients in (24), and so they satisfy five reduced linear equations and generate a reduced series. By aid of Theorem 5, we obtain

THEOREM 42. *The oriented circles of a general linear series make fixed angles with the circles of a reduced series. This series contains every pentacycle of which it contains four circles.*

Five oriented circles determine in general one and only one reduced series.

The hypercomplexes (l), being linearly dependent upon five, will represent the points of a flat four-way spread in ten dimensions. This meets S_6^{10} in ten points by Theorem 3. In general, the ten points in which an arbitrary four-way spread meets S_6^{10} correspond to ten oriented circles which shall be called a *dekacycle*. The general dekacycle is determined by any five of its oriented circles, because five points determine a four-way spread; and the other five oriented circles are linearly dependent upon the chosen five. A special case of the dekacycle is obtained from the pentacycle by giving to each of its circles both orientations. The five sets of coefficients in (24) represent five points in S_{10}, and the four-way spread linearly dependent on them meets S_6^{10} in a dekacycle, hence

THEOREM 43. *The oriented circles of a general series are those that make a null angle with each oriented circle of a dekacycle.*

By Theorem 40 we may choose as reduced four of the five equations that determine a general series. In other words, (24) is not less general if we set

$$b = c = d = e = 0.$$

Having done this, set

$$a = a' - a''$$

and (24) may be written

$$\sum a_{ij}\, p_{ij} + a'\, p = a''\, p, \qquad \sum b_{ij}\, p_{ij} = 0, \qquad \cdots, \quad \sum e_{ij}\, p_{ij} = 0.$$

Now there is a dekacycle dependent on a_{ij}, a'; b_{ij}; c_{ij}; d_{ij}; e_{ij}, which we may indicate thus

$$\alpha_{ij}^{(k)} = \lambda^{(k)}\, a_{ij} + \mu^{(k)}\, b_{ij} + \cdots + \sigma^{(k)}\, e_{ij}, \qquad \alpha^{(k)} = \lambda^{(k)}\, a'.$$

Hence

$$\sum \alpha_{ij}^{(k)}\, p_{ij} + \alpha^{(k)}\, p = \lambda^{(k)}\, a''\, p.$$

Dividing this equation by $\alpha^{(k)}\, p$ we obtain

$$\theta\,(\alpha^{(k)}\, p) = a''/a' \qquad (k = 1, 2, 3, \cdots, 10).$$

Now evidently a''/a' can be made to have any value we choose. Hence

THEOREM 44. *A dekacycle can be found each of whose circles makes the*

same arbitrary angle with the oriented circles of an arbitrary series. The infinite number of dekacycles thus determined by a given series generate a reduced series.

If the arbitrary angle is chosen as $\pi/2$, the dekacycle reduces to the pentacycle of Theorem 41, each of its circles having both orientations.

THEOREM 45. *A linear series contains every dekacycle of which it contains five oriented circles.* For a five-way spread contains every four-way spread of which it contains five points.

THEOREM 46. *Every dekacycle belongs in general to one and only one reduced series.*

An exception occurs if the dekacycle falls into two pentacycles that differ only in orientation, in which case an infinite number of reduced series contain the dekacycle.

The condition that two circles intersect as given by (13) is quadratic in the coördinates of either. Hence, by Theorem 3, twenty circles of the linear series will meet an arbitrary circle.

THEOREM 47. *The oriented circles of a linear series sweep out a surface of the twentieth order which has, in euclidean space, the circle at infinity as a tenfold curve.*

If the series be reduced, the surface doubles upon itself and becomes one of the tenth order, for, while twenty circles still meet an arbitrary circle, yet these are evidently divided into ten pairs that differ only in orientation.

The condition that an oriented circle be null is linear, $p = 0$. Hence by Theorem 3

THEOREM 48. *There are ten null circles in a linear series.*

Let (ξ) and (α) be two arbitrary spheres, and consider the equation

$$p\sum_{ij} p_{ij}\,\alpha_i\,\xi_j - \sum_{ijn} p_{in}\,p_{jn}\,\alpha_i\,\xi_j = 0.$$

On consulting (18) we observe that this equation is satisfied not only by all oriented circles which have the first focus orthogonal to (α) but also by all those which have the second focus orthogonal to (ξ). Moreover this equation is quadratic in (p), and so by Theorem 3 there are twenty oriented circles of a general linear series that satisfy it. By symmetry ten of these lie with the first focus orthogonal to (α), and the other ten with the second foci orthogonal to (ξ). Hence

THEOREM 49. *The first foci of the oriented circles of a linear series trace a curve of the tenth order, and so do the second foci. These two curves meet in ten points, the centers of the null circles of the series. If the series is reduced the two curves coincide.*

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