CONFORMAL CLASSIFICATION OF ANALYTIC ARCS OR ELEMENTS: POINCARÉ'S LOCAL PROBLEM OF CONFORMAL GEOMETRY*

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§ 1. Statement of the Problem

In the geometry based on the infinite group of conformal transformations of the plane (or on the equivalent theory of analytic functions of one complex variable), two types of problems must be carefully distinguished: those relating to *regions* and those relating to *curves* or arcs.

Two *regions* of the plane are equivalent when there exists a conformal representation of the one on the other, the representation to be regular at every interior point. The classic Riemann theory shows that all simply connected regions are equivalent, any one being convertible into say the unit circle. The difficulties connected with the behavior of the boundary (which may be a Jordan curve or a more general point set) have been cleared up in the recent papers of Osgood, Study, and Caratheodory.

Logically simpler problems relating to *curves* or *arcs* have received very scant attention. Two arcs are equivalent provided the one can be converted into the other by a conformal transformation, the transformation to be regular at the points of the arcs, and therefore in some (unspecified) regions including the arcs in their interiors.

The main problem hitherto discussed by the writer in his papers on conformal geometry is the invariant theory of *curvilinear angles*.† Such a configuration (which may be designated also as an analytic angle) consists of two arcs through a common point, both arcs being real, analytic, and regular at the vertex.‡ In this theory it is necessary to distinguish rational and irrational angles. If θ denotes the magnitude of the angle (invariant of first order), then when θ/π is rational there exists a unique conformal invariant

* Presented to the Society, October 25, 1913.

† See *Conformal geometry*, Proceedings of the fifth international congress, Cambridge (1912), vol. 2, pp. 81–87. See also G. A. Pfeiffer's Columbia dissertation, to be published in the American Journal of Mathematics, October, 1915.

‡ The author has also carried out the theory for analytic angles in the complex domain, the sides being regular arcs, real or imaginary. The new feature which then arises is that certain imaginary angles have an *infinite* number of conformal invariants.

of higher order, involving the curvatures and a certain number of higher derivatives of the two curved sides of the angle. On the other hand, if θ/π is irrational, no such higher invariant exists. The transformation is of course assumed to be regular in the neighborhood of the vertex.

The object of the present paper is to study an even simpler and more fundamental problem: the equivalence theory of a *single curve* or arc. When can one analytic arc be converted into another analytic arc by a conformal transformation of the plane? It is apparently implied, in the current literature, that there is no problem here; for any curve (it is implied) can be converted into any other—in particular, into the axis of reals. But this is based on the assumption (not usually stated) that the arcs are real and regular. If we give up either or both of these assumptions we have actual problems which certainly seem worthy of treatment. Our subject (roughly) is *the invariant theory of a single general analytic arc.*

More exactly, the configuration we shall discuss is not an analytic arc but rather that arc together with a specific point of the arc. This (compound) configuration we shall term an *analytic element.* It consists of a point (called base point, which we shall throughout this paper take as origin) and an analytic arc through the point. It may be described also as a *differential element of infinite order.**

Our problem is then precisely what Poincaré has called the LOCAL PROBLEM† of conformal geometry: Given in the first plane (the plane of $z = x + iy$) a point o and an analytic arc l passing through o, and in the second plane (the plane of $Z = X + iY$) a point O and an analytic arc L passing through O; is it possible to find a conformal transformation, that is, is it possible to find Z as an analytic function of z, so as to convert o into O and l in L, the function to be regular in the neighborhood of $z = 0$? This means that we are to find the integral power series

$$Z = c_1 z + c_2 z^2 + c_3 z^3 + \cdots$$

with its first coefficient c_1 different from zero.

Poincaré dismisses this local problem with the remark that there exist

* The writer has introduced elsewhere the concept of *divergent* differential element of infinite order: this corresponds to a divergent power series and may be represented by a non-analytic arc having specified values for all the successive derivatives. Thus to every power series corresponds a geometric entity which may be real or imaginary, regular or irregular, convergent or divergent. This entity is the most general differential element. If it is convergent we call it an analytic element, or, more loosely, an analytic arc or curve.

† As distinguished from the Riemann problem which Poincaré calls the *problème étendue.* See Palermo Rendiconti, vol. 22 (1907), pp. 185–220. Poincaré's object is here to extend both problems to the theory of analytic functions of two complex variables (four-dimensional space).

always an infinitude of solutions.* Obviously he was assuming not merely the *reality* of the arcs considered, which is of course natural since conformal geometry ordinarily means the geometry of the real Gauss plane,—but also the *regularity* of the arcs in the neighborhood of the given points, a very restrictive assumption.

The most general analytic element, if we take the given point o as origin, is represented by writing x and y as integral power series in a parameter t, without absolute terms (that is terms of degree zero). If we eliminate t we obtain y as a series in x which may proceed according to integral or fractional powers of x. If the coefficients are real the element is called *real*; otherwise it is called *imaginary*. If fractional exponents enter and can not be avoided by interchanging x and y (this will then necessarily be the case for *any* choice of rectangular axes), we call the element *irregular*; otherwise the element is *regular*.†

What Poincaré had in mind was the familiar fact that all real regular elements are equivalent: any such element can be reduced conformally to the canonical form $y = 0$ (that is the axis of reals, together, of course, with the origin as base point), and this in an infinitude of ways.

Our new problem is to classify, with respect to the general conformal group, all analytic elements, real and imaginary, regular and irregular.

That distinctions arise in the imaginary domain is obvious, since minimal lines cannot be converted into other lines. For imaginary regular elements the problem is very simple, since it is necessary simply to consider the order of contact of the given element (o, l) with the minimal lines through the given point. It may be discussed synthetically, though for uniformity of treatment we give below (§ 7) the analytic discussion.

But for irregular elements, even in the real plane, the results we find are fairly complicated. It is clear, for example, that the cuspidal element $y = x^{\frac{3}{2}}$ cannot be converted into the regular element $y = 0$, nor into the irregular element $y = x^{\frac{4}{3}}$, for these curves differ qualitatively in an obvious way (in the nature of the singular point at the origin). But suppose the two proposed

* Poincaré shows that the analogous problem in four-dimensional space (in connection with functions of two variables) has in general no solution, but may in special cases have either a finite or an infinite number of solutions.

† See the systematic definitions in Study's *Vorlesungen über Geometrie*, Heft 1, §§ 5, 12. Study however is dealing with analytic curves, not analytic elements; so he speaks of the regular and irregular points (Stellen) of the curve, while we apply the adjectives to the elements (or sometimes to the arcs or curves belonging to the elements). There is no actual ambiguity however. An ordinary node, it should be noticed, is *not* an example of an irregular element, but comes rather under the concept of an analytic angle: we have in fact merely two regular arcs with a common point, that is, two distinct regular elements. An ordinary cusp is the most familiar instance of an irregular element. Any algebraic singularity may be resolved into a number of regular and irregular elements.

elements have the same kind of irregularity (in a sense to be later defined, depending on agreement of certain exponents, certain arithmetic invariants), will they necessarily be equivalent? If not, certain combinations of the coefficients will be invariant, that is, there will be absolute or differential invariants. For example, it turns out that every differential element of the form

$$y = x^{\frac{3}{2}} + \gamma_4 x^{\frac{4}{2}} + \gamma_5 x^{\frac{5}{2}} + \cdots$$

can be (formally) reduced to $y = x^{\frac{3}{2}}$; on the other hand, not every element of form

$$y = x^{\frac{4}{3}} + \gamma_5 x^{\frac{5}{2}} + \gamma_6 x^{\frac{6}{2}} + \cdots$$

can be reduced to $y = x^{\frac{4}{3}}$. Hence in the first type there are no invariants; in the second type there exist invariants—in fact an infinitude of them.

In general, irregular types of elements have absolute invariants; certain exceptions exist, namely, those in which the corresponding series in x proceeds according to powers of the square root of x. The exact statement of the results will be found italicized on pages 338, 339, 347, 349.

In carrying out the discussion, for both the real and the imaginary cases, we find it convenient to represent our curves, not in cartesian coördinates x, y, but in minimal coördinates u, v, where

$$u = x + iy, \qquad v = x - iy.$$

For a real point x and y are both real, while u and v are conjugate complex quantities. The general analytic element is then represented by writing v as a series which may proceed according to integral powers of either u or some root of u, say $\sqrt[p]{u}$. The integer p is then an obvious arithmetic invariant. When $p = 1$, the element is regular; when $p > 1$ the element is irregular.

We shall throughout this paper write our element in the form

$$v = \alpha_q u^{q/p} + \alpha_{q+1} u^{(q+1)/p} + \alpha_{q+2} u^{(q+2)/p} + \cdots,$$

where we assume $q \geqq p$. This is fair since, if $q < p$, we could interchange the coördinates u and v, which would render $q > p$. We always assume that the leading coefficient α_q does not vanish.

The integer q is a second arithmetic invariant. All elements obtained by taking arbitrary values of the coefficients in the above equation, but fixing the values of both p and q, we shall define as forming a single *species*, the species (p, q).

Just as projective geometry may be discussed either for the real plane or the complex plane, so we may have conformal geometry either for the real or the complex domain. In the real plane we have ∞^2 points defined by two real variables x and y or one complex combination $z = x + iy$: this is the usual gaussian plane. In the complex plane we have ∞^4 points defined by two independent complex coördinates x and y, or by the two linear combinations

$$u = x + iy, \qquad v = x - iy,$$

which are no longer necessarily conjugate but are completely independent complex numbers.

In passing from real to complex projective geometry (of the plane) we also extend our projective group, so that it contains 16 instead of 8 parameters. So in complex conformal geometry we have a larger group of transformations than the usual conformal group. The general conformal transformation (real or imaginary) is obtained by writing U as a power series in u, and V as a power series in v, the coefficients in the series being independent complex quantities; only when these coefficients are conjugate will the transformation be real. In cartesian coördinates our larger group is found in the form

$$X = \phi(x, y), \qquad Y = \psi(x, y),$$

where ϕ and ψ are any power series in two variables with real or imaginary coefficients satisfying the Cauchy-Riemann equations.

§ 2. General Method and Results

In order to find the conformal invariants of the general analytic element of species (p, q), namely

$$v = \alpha_q u^{q/p} + \alpha_{q+1} u^{(q+1)/p} + \alpha_{q+2} u^{(q+2)/p} + \cdots \qquad (\alpha_q \neq 0), \tag{1}$$

we inquire when this element is equivalent to some other element of the same species

$$V = A_q U^{q/p} + A_{q+1} U^{(q+1)/p} + A_{q+2} U^{(q+2)/p} + \cdots. \qquad (A_q \neq 0). \tag{2}$$

Equivalence means that the first equation can be converted into the second by a transformation of the form

$$\begin{aligned} U &= a_1' u + a_2' u^2 + \cdots, \\ V &= b_1' v + b_2' v^2 + \cdots, \end{aligned} \tag{3}$$

in which neither a_1' nor b_1' is to vanish since the transformation is to be regular at the origin. We shall find it convenient to write our transformation in the less symmetric form

$$\begin{aligned} u &= a_1 U + a_2 U^2 + \cdots \qquad (a_1 \neq 0), \\ V &= b_1 v + b_2 v^2 + \cdots \qquad (b_1 \neq 0), \end{aligned} \tag{4}$$

where the a's are of course the coefficients of the series obtained by reverting the first series in (3).

To express the fact that (4) converts (1) into (2) we may eliminate the three quantities u, v, V from the four equations (1), (2), (4). The result is an

equation in U which must hold identically, so that we may equate coefficients of like powers. In order to avoid fractional exponents we make the substitution

$$U = t^p; \tag{5}$$

our *fundamental condition of equivalence* then takes the form

$$A_q t^q + A_{q+1} t^{q+1} + \cdots = b_1 P + b_2 P^2 + \cdots, \tag{6}$$

where

$$\begin{aligned} P \equiv \alpha_q t^q (a_1 + a_2 t^p + a_3 t^{2p} + \cdots)^{q/p} \\ + \alpha_{q+1} t^{q+1} (a_1 + a_2 t^p + a_3 t^{2p} + \cdots)^{(q+1)/p} + \cdots, \end{aligned} \tag{6'}$$

which can obviously be developed as a series in integral powers of t, beginning with t^q, since a_1 does not vanish.

Equating coefficients of like powers of t in (6), we obtain an infinite set of equations involving the constants α and A of the two curves and the constants a and b of the transformation. Two given curves (1) and (2) will be equivalent provided this infinite set of equations can be solved for the a's and b's, subject of course to the essential restriction $a_1 \neq 0$, $b_1 \neq 0$.

If for all values of the α's and A's, the a's and b's can be found, then the curves are always equivalent and no invariants exist. Otherwise certain conditions must be imposed on the α's and A's in order to render the equations consistent, and this indicates the existence of invariants. Which of these possibilities actually occurs we shall find depends essentially on the values of integers p and q determining the species. Some species have invariants, others have not.

The integers p and q are obviously of invariant character under the conformal group. In some types other such *arithmetic invariants* exist. But our main question is to find *absolute* or *differential invariants*, that is, expressions depending on a finite number of the coefficients of the curve, say α_q up to α_{q+r}, which are unchanged by conformal transformations, and thus are converted into expressions of the same form in the new coefficients A_q up to A_{q+r}. The principal result obtained (proofs are given later) is as follows:

The regular species $(1, 1)$, $(1, 2)$, $(1, 3)$, $\cdots$, *have no differential invariants. But with the single exception of the species* $(2, 2)$, *all the irregular species have differential invariants.*

When $p = 1$, there are no invariants; also when $p = 2$ and $q = 2$ there are no invariants; but in all other cases the analytic element (1) has invariants.

We shall divide our discussion into three parts, namely, $p = 1$, $p = 2$, $p > 2$. The first and third cases are very simple; but the second is somewhat complicated, since we shall find that distinct discussions are required for three subcases, namely, $q = 2$, $q = 3$, $q > 3$.

We shall in each case either prove that no invariants exist, or else give the explicit expression of the first invariant (that is, invariant of lowest order, where order refers to the subscript $q + r$). A few specimens of additional invariants are given. The number of additional invariants is always infinite.

We restate our notation. Any analytic element may be written in minimal coördinates $u = x + iy$, $v = x - iy$, as follows

$$v = \alpha_q u^{q/p} + \alpha_{q+1} u^{(q+1)/p} + \alpha_{q+2} u^{(q+2)/p} + \cdots$$

where the coefficients α are any complex numbers (making the series convergent), the first coefficient α_q being different from zero; p and q are positive integers and $q \geqq p$. The integers p and q are then arithmetic invariants under the conformal group. All elements with the same p and q we speak of as forming the *species* (p, q). We shall call p the *index* of the element and q the *rank* of the element. If $p = 1$ the element is *regular*; if $p > 1$ it is *irregular*. If $q = p$ the tangent line (at the origin or base point) is not minimal; while if $q > p$ the tangent is minimal. Of course *all real elements are included in the former category* $q = p$.

Absolute or differential invariants, that is, functions of the coefficients unaltered by the conformal group, exist for all irregular species except in the case of species $(2, 2)$. *The order of the lowest invariant is* $q + 2$ *when* $p > 2$. *If however* $p = 2$ *the order is* $q + 3$ *when* $q > 3$, *and* $q + 5$ *when* $q = 3$. *Every species that has invariants has actually an infinite number of invariants.*

The following table exhibits some of the results in detail:

	1	2	3	4	5	6
1	*	*	*	*	*	*
2		†	8	7	8	9
3			5	6	7	8
4				6	7	8
5					7	8
6						8

Here the species is determined by the value of p in the left column and the value of q in the top row. In the body of the table we then find the order of the lowest absolute invariant. The asterisk indicates that the corresponding species has no absolute invariant and further that all members of that species are conformally equivalent. The dagger, in the case of the species $(2, 2)$, indicates that there are no absolute invariants but that the members of the species are not all conformally equivalent (not even in the formal sense): there exists a certain arithmetic invariant, hence there is a division of this species into an infinite number of conformally distinct sub-

species. See § 6 below. We recall that all *real* curves are included in the species $(1,1)$, $(2,2)$, $(3,3)$, etc.; all *real regular* curves are included in $(1,1)$.

When $p > 2$, the lowest invariant is a function of the first *three* coefficients α_q, α_{q+1}, α_{q+2}. When $p = 2$ and $q > 3$, the lowest invariant involves the first *four* coefficients. Finally when $p = 2$ and $q = 3$, the result involves *six* coefficients.

§ 3. Discussion for $p > 2$

This is extremely simple, for we find that the first three equations obtained by the process already described, equating coefficients in the fundamental equation (6), involve only two of the transformation constants. The equations are in fact

$$A_q = \alpha_q b_1 a^{q/2}, \qquad A_{q+1} = \alpha_{q+1} b_1 a^{(q+1)/2}, \qquad A_{q+2} = \alpha_{q+2} b_1 a^{(q+2)/2}.$$

Eliminating a_1 and b_1, we find

$$\frac{\alpha_q \alpha_{q+2}}{\alpha_{q+1}^2} = \frac{A_q A_{q+2}}{A_{q+1}^2}.$$

Hence when the index p of the analytic element (1) *is greater than two the invariant of lowest order is*

$$\frac{\alpha_q \alpha_{q+2}}{\alpha_{q+1}^2};$$

the order $= q + 2$, weight $= 2q + 2$, degree $= 2$.

§ 4. Discussion for $p = 2$, $q > 3$

In this case the first two equations

$$A_q = \alpha_q b_1 a^{q/2}, \qquad A_{q+1} = \alpha_{q+1} b_1 a^{(q+1)/2}$$

determine as before the values of a_1 and b_1. The next two equations

$$A_{q+2} = \alpha_{q+2} b_1 a_1^{(q+2)/2} + \frac{q}{2} \alpha_q b_1 a^{(q+2)/2} a_2,$$

$$A_{q+3} = \alpha_{q+3} b_1 a^{(q+3)/2} + \frac{q}{2} \alpha_{q+1} b_1 a^{(q-1)/2} a_2$$

bring in only a_2, not b_2. Eliminating a_1, b_1, and a_2 from our four equations and separating the coefficients α and A, we find

When p equals two and q is greater than three, the irregular element of species (p, q) has the following invariant of lowest order

$$\frac{\alpha_q (\alpha_q \alpha_{q+3} - \alpha_{q+1} \alpha_{q+2})}{\alpha_{q+1}^3}.$$

Order $= q + 3$, weight $= 3q + 3$, degree $= 3$.

§ 5. Discussion for $p = 2$, $q = 3$

We shall now show that an ordinary cusp with a minimal tangent line has a differential invariant. This is the species defined by $p = 2$, $q = 3$; that is

$$v = \alpha_3 u^{\frac{3}{2}} + \alpha_4 u^{\frac{4}{2}} + \alpha_5 u^{\frac{5}{2}} + \cdots .$$

We shall need the first six equations obtained in the usual way from the equivalence identity. They are

$$b_1 \alpha_3 = A_3 a_1^{3/2}, \qquad b_1 \alpha_4 = A_4 a_1^{4/2}, \qquad b_1 \alpha_5 = \tfrac{3}{2} A_3 a_1^{1/2} a_2 + A_5 a_1^{5/2},$$

$$b_1 \alpha_6 + b_2 \alpha_3^2 = 2A_4 a_1 a_2 + A_6 a_1^{6/2},$$

$$b_1 \alpha_7 + 2b_2 \alpha_3 \alpha_4 = A_3 (\tfrac{3}{8} a_1^{-1/2} a_2^2 + \tfrac{3}{2} a_1^{1/2} a_3) + \tfrac{5}{2} A_5 a_1^{3/2} a_2 + A_7 a_1^{7/2},$$

$$b_1 \alpha_8 + b_2 (\alpha_4^2 + 2\alpha_3 \alpha_5) = A_4 (a_2^2 + 2a_1 a_3) + 3A_6 a_1^2 a_2 + A_8 a_1^4 .$$

We are now trying to prove the existence, not the non-existence of invariants; hence it is necessary to have these equations in full, not merely in leading terms, as will be the case in § 6.

The first two equations involve two transformation coefficients a_1, b_1; the next two bring in two new coefficients a_2, b_2; but the following two bring in merely a_3, since b_3 does not as yet appear. Hence it is possible to eliminate the transformation coefficients, and obtain a relation between the coefficients $\alpha_3 \cdots \alpha_8$ of the old curve, and the coefficients $A_3 \cdots A_8$ of the new curve. Carrying out the elimination we find that the two sets of coefficients may be separated,* the result being the expression

$$\frac{\alpha_3^2 (6\alpha_3 \alpha_4 \alpha_7 - 3\alpha_4^2 \alpha_6 - 22\alpha_4 \alpha_5^2 - 9\alpha_3^2 \alpha_8 + 18\alpha_3 \alpha_5 \alpha_6) + 4\alpha_3 \alpha_4^3 \alpha_5}{\alpha_4^5}$$

equal to the same expression in the A's.

Hence this expression is an absolute invariant of our element of species $(p = 2, q = 3)$. It is of order 8, weight 20, and degree 5.

In cartesian coördinates such an irregularity is represented by

$$y = ix + c_3 x^{\frac{3}{2}} + c_4 x^{\frac{4}{2}} + \cdots .$$

It is not usually possible to reduce this conformally to the simple cubic cusp

$$y = ix + x^{\frac{3}{2}}$$

with minimal tangent line. A necessary condition is that the absolute invariant (a certain combination of c_3, c_5, c_6, c_7, c_8) shall have a special numerical value.†

* This calculation was carried out by Mr. J. A. Northcott in my seminar at Columbia University, 1913.

† The special case where α_4 vanishes of course defines an invariant subspecies under the given species, the first absolute invariant then becoming infinite. An invariant of higher order may then be found for this special case (see § 8).

§ 6. Discussion for $p = 2$, $q = 2$

We now prove that the species $p = 2$, $q = 2$, namely

$$v = \alpha_2 u^{\frac{2}{3}} + \alpha_3 u^{\frac{3}{2}} + \alpha_4 u^{\frac{4}{3}} + \cdots,$$

has no differential invariants, making first the assumption $\alpha_3 \neq 0$.

The equivalence identity takes the form

$$b_1 P + b_2 P^2 + \cdots = A_2 t^2 + A_3 t^3 + \cdots,$$

where

$$P \equiv P_2 t^2 + P_3 t^3 + \cdots,$$

the coefficients here being

$$P_2 = \alpha_2 a_1, \qquad P_3 = \alpha_3 a_1^{3/2}, \qquad P_4 = \alpha_2 a_2 + \cdots,$$

$$P_5 = \tfrac{3}{2}\alpha_3 a_1^{1/2} a_2 + \cdots, \qquad P_6 = \alpha_2 a_3 + \cdots, \qquad P_7 = \tfrac{3}{2}\alpha_3 a_1^{1/2} a_3 + \cdots,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Only the leading terms, that is terms involving the a of highest subscript, are written out since the other terms are unnecessary for the present purpose.

Equating coefficients of powers of t in the identity, we have an infinite system of equations involving the constants α and A of the two arbitrary curves, and the constants a and b of the transformation. We must show that these equations can always be solved for the a's and b's, with neither a_1 nor b_1 vanishing.

The first pair of equations, arising from t^2 and t^3, is

$$\alpha_2 a_1 b_1 = A_2,$$

$$\alpha_3 a_1^{3/2} b_1 = A_3,$$

giving the unique solution

$$a_1 = \left(\frac{A_3 \alpha_2}{A_2 \alpha_3}\right)^2, \qquad b_1 = \frac{A_2^3 \alpha_3^2}{A_3^3 \alpha_2^2},$$

finite and different from zero, since the first two terms in the series for the curves are assumed to be actually present.

The next pair of equations, arising from t^4 and t^5, are linear in two new unknowns, a_2 and b_2, the determinant of the left hand members being

$$\begin{vmatrix} \alpha_2 b_1 & \alpha_2 a_1 \\ \frac{3}{2}\alpha_3 a_1^{1/2} b_1 & 2\alpha_2 \alpha_3 a_1^{\frac{5}{2}} \end{vmatrix}.$$

To show that this does not vanish, we may cancel out the non-vanishing factors, α_2, α_3, a_1, b_1; the remaining purely numerical factor is found to be

$$\begin{vmatrix} 1 & 1 \\ \frac{3}{2} & 2 \end{vmatrix} = \tfrac{1}{2}.$$

Similarly the next pair of equations bring in linearly two new unknowns, a_3 and b_3, and the numerical factor in their determinant is

$$\begin{vmatrix} 1 & 1 \\ \frac{3}{2} & 3 \end{vmatrix} = \tfrac{3}{2}.$$

It is easily verified that the nth pair of equations, arising from the terms t^{2n} and t^{2n+1}, always bring in just two new unknowns, a_n and b_n, and that these equations admit a unique solution, since the determinant, factorable into powers of α_2, α_3, a_1, b_1, together with the number

$$\begin{vmatrix} 1 & 1 \\ \frac{3}{2} & n \end{vmatrix} = n - \tfrac{3}{2},$$

cannot vanish.

It follows that the coefficients of the transformation cannot be eliminated from the first $2n$ equations, no matter how great we take n. Hence there cannot be any invariant relations involving a finite number of coefficients of the two curves. It follows that our curve can be reduced formally to the normal form $v = u^{\frac{3}{2}}$, and that this reduction is unique.

So far we have used the assumption $\alpha_3 \neq 0$, that is, that the term $u^{\frac{3}{2}}$ is actually present. Since we are discussing the class $p = 2$, $q = 2$ the term $u^{\frac{3}{2}}$ must of course be present; that is $\alpha_2 \neq 0$. Let us now consider what happens if any number of successive terms after the first are absent.

Let $k = 2m + 1$ denote the subscript of the first term with a fractional exponent actually appearing in the series for the curve. The equation of the curve is thus of the form

$$v = \alpha_2 u + \alpha_4 u^2 + \cdots + \alpha_{2m} u^m + \alpha_{2m+1} u^{m+\frac{1}{2}} + \cdots.$$

The polynomial of mth degree in the first part of this development can always be transformed into a linear term. Hence we may, without loss of generality, take our curve in the reduced form

$$v = \alpha_2 u^{\frac{3}{2}} + \alpha_k u^{k/2} + \alpha_{k+1} u^{(k+1)/2} + \cdots,$$

where $k = 2m + 1$ is odd, and $\alpha_2 \neq 0$, $\alpha_k \neq 0$.

We must show that such a curve can always be transformed into an arbitrary curve of the same type

$$V = A_2 U^{\frac{3}{2}} + A_k U^{k/2} + A_{k+1} U^{(k+1)/2} + \cdots.$$

The equivalence identity is

$$b_1 P + b_2 P^2 + \cdots = A_2 t^2 + A_k t^k + \cdots,$$

where

$$P \equiv \alpha_2 a_1 t^2 + (\alpha_2 a_2 + \cdots) t^4 + (\alpha_2 a_3 + \cdots) t^6 + \cdots + \alpha_k a_1^{k/2} t^k$$
$$+ \left(\frac{k}{2} \alpha_k a^{(k-2)/2} a_2 + \cdots\right) t^{k+2} + \left(\frac{k}{2} \alpha_k a_1^{(k-2)/2} a_3 + \cdots\right) \cdot t^{k+4} + \cdots.$$

Here again only the leading terms with respect to the coefficients a are needed.

Writing out the system of equations obtained by equating coefficients of like powers of t, we find that the first two unknowns a_1, b_1, are not, as in the former case, given by the terms in t^2 and t^4, but rather by those in t^2 and t^k. These equations are not linear but give non-vanishing values for a_1 and b_1.

The next pair of equations, linear in a_2 and b_2, are given by the terms t^4 and t^{k+2}. In general the unknowns a_n and b_n are determined by a pair of linear equations, arising from the terms in t^{2n} and $t^{k+2(n-1)}$. The determinant involves as a factor, in addition to powers of a_1, b_1, α_2, α_k which do not vanish, only the number

$$\begin{vmatrix} 1 & 1 \\ \frac{k}{2} & n \end{vmatrix} = n - \frac{k}{2}$$

which can never vanish since k is odd.

Therefore the type considered has no absolute invariants. The only distinctions are those that arise from the variation of the integer $k = 2m + 1$. Any curve can be converted into any other having the same value of m.

Every irregular element of species $p = 2$, $q = 2$ can be reduced to the normal form*

$$v = u + u^{m+\frac{1}{2}}$$

where m is a positive integer $1, 2, 3, \cdots$. There are no differential invariants. The only arithmetic invariant is the integer m.

The transformation converting the curve into the normal form is determinate, but not always uniquely determinate. This is due to the fact that the first pair of equations, that is those giving a_1 and b_1, are non-linear, at least with respect to a_1. The equations are

$$\alpha_2 a_1 b_1 = A_2, \qquad \alpha_k a_1^{m+\frac{1}{2}} b_1 = A_k,$$

and have $2m - 1$ solutions

$$a_1 = \left(\frac{A_k \alpha_2}{A_2 \alpha_k}\right)^{\frac{2}{2m-1}}, \qquad b_1 = \frac{A_2}{\alpha_2} a_1.$$

Only when $m = 1$ is the required reducing transformation unique.

In particular the number of transformations converting one of our curves into itself is $2m - 1$. The only transformation of $v = u + u^{\frac{3}{2}}$ into itself is identity.†

* The question of convergency is left open for future discussion. This does not, of course, affect results as to the existence or non-existence of invariants (that is, differential invariants of finite order). See paper on *Conformal geometry* cited in § 1.

† We refer throughout only to direct conformal transformations. It may be shown that there is also a unique reverse conformal transformation which converts this curve into itself. The general reverse (or improper) conformal transformation is represented in minimal coordinates by writing U as a series in v, and V as a series in u.

If we pass from the notation of minimal coördinates u, v to that of cartesian coördinates x, y we may express our results as follows:

The class of curves (analytic elements) defined by $p = 2$, $q = 2$, that is curves having a cartesian equation of the form*

$$y = c_3 x^{\frac{3}{2}} + c_4 x^{\frac{4}{2}} + c_5 x^{\frac{5}{2}} + \cdots,$$

has no differential invariants under the conformal group. Such a curve can be reduced† conformally to the normal form

$$y = x^{m+\frac{1}{2}} \qquad (m = 1, 2, 3, \cdots).$$

The only conformal distinctions are with respect to the arithmetic invariant m. The number of direct conformal transformations of this normal form into itself is $2m - 1$.

Geometrically, the type of irregularity defined by $p = 2$, $q = 2$ is that of an ordinary cusp, with a non-minimal tangent. Variation of m signifies a difference in the (fractional) order of contact between the curve and its tangent line. The case $m = 1$, leads to the simplest cusp, represented by the semi-cubic parabola

$$y = x^{\frac{3}{2}}.$$

If we apply a conformal transformation to this, we obtain an analytic curve with a cusp of the same kind

$$y = c_3 x^{\frac{3}{2}} + c_4 x^{\frac{4}{2}} + \cdots \qquad (c_3 \neq 0).$$

Since $m = 1$, the transformation is uniquely determined by the curve. Hence have this peculiar result:

If we know the effect of a conformal transformation (of course assumed regular in the neighborhood of the cusp) on the points of the semi-cubic parabola (or any conformally equivalent curve), that is if we know the equation of the new or transformed curve, the transformation is uniquely determined.‡

This is apparently the simplest possible way of specifying (completely determining) a conformal transformation by means of curves.

7. Discussion for $p = 1$. Regular Elements

In this simple case there are no differential invariants. This is well-known for the species $(1, 1)$, that is, when the initial tangent is non-minimal. It is true also when the tangent is minimal, that is when q is greater than unity, as we shall show below.

* We take the tangent as axis of x and the normal as axis of y. This is legitimate since the tangent is non-minimal.

† At least in the formal sense.

‡ Contrast this with the familiar theorem: If we know the effect of a conformal transformation on the axis of reals, $y = 0$, the transformation is not determined but involves an infinitude of arbitrary constants.

Regular elements have no differential invariants under the conformal group. All the elements of the same species $(p = 1, q = k)$, *that is,*

$$v = \alpha_k u^k + \alpha_{k+1} u^{k+1} + \cdots,$$

are equivalent to each other, being reducible to the normal form

$$V = U^k;$$

so that arithmetic invariant k *is the only invariant.*

To prove this we may of course use the general identity (6). But we find it more convenient here to use the following simpler method, which settles at the same time all convergency questions.

To show that the given curve can be reduced to the normal form stated, it is sufficient to assume that v is unaltered, and to seek the proper transformation of u. Using

$$U = g(u), \qquad V = v,$$

we find that the function g must satisfy the relation

$$g(u) = u \sqrt[k]{\alpha_k + \alpha_{k+1} u + \alpha_{k+2} u^2 + \cdots}.$$

Since $\alpha_k \neq 0$, this formula defines an integral power series and the coefficient of u in the result does not vanish. Hence our conformal transformation exists.

The number of possible transformations is infinite, depending on an infinite set of arbitrary coefficients. To show this, it is sufficient to seek the most general transformation

$$U = a_1 u + a_2 u^2 + \cdots,$$

$$V = b_1 v + b_2 v^2 + \cdots,$$

which converts the canonical curve $V = U^k$ into itself. The requisite condition is

$$a_1 + a_2 u + a_3 u^2 + \cdots \equiv \sqrt[k]{b_1 + b_2 u^k + b_3 u^{2k} + \cdots},$$

which can be satisfied by taking the b's arbitrarily with $b_1 \neq 0$; then finding the a's, which are (up to a k-th root of unity) determined.*

It is thus seen that any curve of species $(1, k)$ *can be converted conformally into any curve of the same species in an infinity* (∞^∞) *of ways. If we take not only the two curves at random but also an arbitrary regular analytic correspondence between their points (the origin going into the origin) the conformal transformation exists and is uniquely determined.*

Of course in the real conformal plane the only regular elements are those

* On the other hand it is *not* possible, when k exceeds unity, to take the a's arbitrarily and then find the b's.

of species $(1, 1)$, that is to say k is then unity. The results are then well-known.

The natural conformal classification of regular analytic elements in the complex plane, expressed in the usual cartesian apparatus, is as follows:

Every REGULAR *analytic element in the complex plane can be reduced to one of these three forms:*

$$\text{(I)} \qquad y = 0,$$

$$\text{(II)} \qquad y = ix + x^k \qquad (k = 2, 3, 4, \cdots),$$

$$\text{(III)} \qquad y = ix.$$

The first is of course the most general form: it contains all elements whose tangent line is non-minimal (including as a special case all real elements.) The second includes all curved elements whose tangent line is minimal, the order of contact with the minimal line being $k - 1$. The third is the most special form; it includes merely the minimal straight lines of the plane.

In a previous paper,* the author showed that regular elements may be completely characterized, with respect to conformal equivalence, by the value of the limit of the ratio of the arc to the chord running from the given base point to a neighboring point. For ordinary curves, that is, class (I), this limit is of course unity. But for class (II) the result is

$$L = \text{limit}\,\frac{\text{arc}}{\text{chord}} = \frac{2\sqrt{k}}{k+1},$$

which is, for example, correct to two decimal places, .94 for $k = 2$, .86 for $k = 3$, .80 for $k = 4$, .74 for $k = 5$. Hence the value of L determines the value of the integer k, so that two elements having the same L will be conformally equivalent. Finally, for class (III) both arc and chord vanish identically, hence the limit L does not exist (is indeterminate).

§ 8. Some Additional Invariants

For each species (p, q) we have either shown that no invariants exist or we have obtained the explicit expression of the first absolute invariant (differential invariant of lowest order). Every species that has an invariant (that is every irregular species except $(2, 2)$) has in fact an infinite number of invariants. We shall here give some examples of the invariants of higher order.

Consider first the species $(3, 3)$, that is

$$(7) \qquad v = \alpha_3 u^{\frac{3}{3}} + \alpha_4 u^{\frac{4}{3}} + \alpha_5 u^{\frac{5}{3}} + \cdots \qquad (\alpha_3 \neq 0).$$

Here the first invariant, obtained by elimination from three equations, is, in accordance with the discussion for $p > 2$ given in § 3,

* *The ratio of the arc to the chord of an analytic curve need not approach unity.* Bulletin of the American Mathematical Society, vol. 20 (1914), pp. 524–531.

$$\frac{\alpha_3\,\alpha_5}{\alpha_4}.$$

Assuming that α_4 does not vanish, we find that we can also eliminate the constants of the transformation from the next three equations, the result being an invariant of order 8. Continuing in this way by grouping the equivalence equations into sets of three we find this result:

The general case of species (3, 3), *that is form* (7) *with* $\alpha_4 \neq 0$, *has absolute invariants of order* 5, 8, 11, 14, *etc.*

In the more special case defined by $\alpha_4 = 0$, $\alpha_5 \neq 0$, we find that there still exist an infinite number of invariants, the orders now being 7, 10, 13, 16, etc.

For larger values of p, the structure of a certain first succession of $p - 2$ invariants is very simple. Thus for $p = 5$, the first three invariants are

$$\frac{\alpha_q\,\alpha_{q+2}}{\alpha_{q+1}^2}, \qquad \frac{\alpha_q\,\alpha_{q+3}}{\alpha_{q+1}\,\alpha_{q+2}}, \qquad \frac{\alpha_q\,\alpha_{q+4}}{\alpha_{q+1}\,\alpha_{q+3}}.$$

The next higher invariants are more complicated.

For the species (3, 4), we have first a simple invariant of order 6, and second a complicated invariant of order 9. The expressions are

$$\frac{\alpha_4\,\alpha_6}{\alpha_5}, \qquad \frac{\alpha_4^2\,(5\alpha_5^2\,\alpha_7 - 4\alpha_4\,\alpha_5\,\alpha_8 + 2\alpha_4^2\,\alpha_9 - 3\alpha_4\,\alpha_6\,\alpha_7)}{\alpha_5^5}.$$

For the species (4, 5) the first three invariants, of orders 7, 8, and 11, are

$$\frac{\alpha_5\,\alpha_7}{\alpha_6^2}, \qquad \frac{\alpha_6\,\alpha_8}{\alpha_7^2}, \qquad \frac{12\alpha_5^3\,\alpha_6^2\,\alpha_9 - 7\alpha_5^4\,\alpha_7\,\alpha_9 - 10\alpha_5^4\,\alpha_6\,\alpha_{10} + 5\alpha_5^5\,\alpha_{11}}{\alpha_6^6}.$$

In our discussion of lowest invariants the most complicated species was (2, 3), since here it was necessary to use six equations before elimination was possible. Grouping the next equations of equivalence into sets of six, we find that new eliminations are always possible. Hence

The species (2, 3) *has an infinite number of invariants, the orders being* 8, 14, 20, 26, *etc.*

This is of course in the general case of that species. It is assumed that not only the first term $\alpha_3\,u^{\frac{3}{2}}$ is present, but also the second term $\alpha_4\,u^{\frac{4}{2}}$. Separate discussions are necessary if certain terms are absent.

For the arbitrary species (p, q) with a certain number of terms, after the first, absent, we state only this result. If the curve is of the form

$$v = \alpha_q\,u^{q/p} + \alpha_{q+k}\,u^{(q+k)/p} + \alpha_{q+k+1}\,u^{(q+k+1)/p} + \cdots,$$

where $p > 2$ and $k < p - 1$, the first invariant is

$$\frac{\alpha_q\,\alpha_{q+k+1}^k}{\alpha_{q+k}^{k+1}}.$$

For the real species (p, p) it is worth while to give the following more detailed results.

The case represented by

$$v = \alpha_q u^{p/p} + \alpha_{q+k} u^{(p+k)/p} + \cdots,$$

when $k < p - 1$ has a first invariant of order $p + k + 1$ and also, provided $k < p - 2$, a second invariant of order $2p - 1$. When $k = p - 1$, the first invariant is of order $2p + 1$.

If still more terms in the power series are absent, say if $\alpha_{p+1} = 0$, $\alpha_{p+2} = 0$, and so on, until $\alpha_{lp+k} \neq 0$, so that the element (p, p) is of the special form

$$v = \alpha_q u^{p/p} + \alpha_{lq+k} u^{(lp+k)/k} + \cdots,$$

then when $k < p - 1$ the first order is $lp + k + 1$; and when $k = p - 1$, assuming $p > 2$, the first order is $(l + 1)p + 1$.

§ 9. Real Elements

For real elements under the group of real conformal transformations we may, in conclusion, state our principal results in the usual cartesian coördinates as follows, taking the base point as origin and the initial tangent of the element as axis of x.

Regular real elements are conformally reducible to $y = 0$.

Irregular real elements of index two, that is

$$y = c_3 x^{\frac{3}{2}} + c_4 x^{\frac{4}{2}} + \cdots$$

have no differential invariants, and may be reduced (formally) to

$$y = x^{m+\frac{1}{2}}, \qquad (m = 1, 2, 3, \cdots).$$

Irregular real elements of index greater than two always have an infinite number of differential invariants under the real conformal group.

A direct discussion of the real theory in cartesian coördinates will appear in another paper. It should be observed that our classification into species (p, q) is based on the minimal notation u, v. If we pass to the cartesian notation x, y, the fractional exponents in the leading term are usually different from q/p. The index p remains as denominator, but the new numerator has no intrinsic significance. A new arithmetic invariant is then obtained which has such significance.

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