TRANSFORMATIONS T OF CONJUGATE SYSTEMS OF CURVES ON A SURFACE*

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When a surface $S$ is referred to a conjugate system of lines, its point coördinates are solutions of a partial differential equation of the Laplace type, called the point equation for the given conjugate system. Throughout this paper we consider surfaces referred to conjugate systems, and hence we will use the symbol $S$ to denote either the surface or the parametric system upon it, as the case may be. When the developables of a linear congruence $G$ meet $S$ in the parametric conjugate system, we say that $G$ and $S$ are conjugate to one another. A second surface $S_1$ conjugate to $G$ is said to be in the relation of a transformation $T$ to $S$, or to be a $T$ transform of $S$. Darboux† has shown that each solution $\phi$ of the adjoint equation of the point equation of $S$ determines a congruence $G$ conjugate to $S$, and that each solution of this point equation determines for any of these congruences $G$ a conjugate surface $S_1$. Hence each pair of function $\phi$ and $\theta$ determines a transformation $T$, and every such transformation is so determined. It is the purpose of this paper to develop the theory of these transformations.

It is shown that if $S_1$ and $S_2$ are two $T$ transforms of $S$, there exist $\infty^2$ surfaces $S_{12}$ each of which is in the relation of transformations $T$ with $S_1$ and $S_2$, and the determination of these surfaces requires only quadratures. We have thus proved the existence of a theorem of permutability of transformations $T$, which includes a similar theorem for the transformations $K$ of conjugate systems with equal invariants‡ (see § 9), just as the latter embraces as a particular case the theorem established by Bianchi§ for transformations $D_m$ of isothermic surfaces. In § 12 we extend the theorem of permutability so as to be concerned with eight surfaces.

When the function $\theta$ determining a transformation is a constant and the point coördinates are in the cartesian form, the corresponding tangent planes to $S$ and $S_1$ are parallel, in which case we say that we have a parallel trans-

‡ Cf. Eisenhart, These Transactions, vol. 15 (1914), pp. 404–8. Hereafter this memoir will be referred to as $M_1$.
formation. This result and the consideration of the relation between two transformations $T$ determined by the same $\phi$ but different functions $\theta$ lead to results formerly found by the author* for certain types of transformations $T$ and later by Jonas,† and enable us to put the equations of a general transformation $T$ in another convenient form.

In § 9 we consider in particular the case where the point equation of $S$ has equal invariants and its transforms possess the same property. The resulting transformations are the transformations‡ $K$ previously studied by us in their relation to the transformations of Moutard of differential equations. As there shown, these transformations $K$ include the transformations $D_m$ of isothermic surfaces discovered by Darboux.§

Transformations $T$ can be treated analytically also in terms of the tangential coördinates of the surface. This is done in § 10, and the relations between the two sets of equations are determined. In particular, the case where the tangential equation has equal invariants is studied, with the result that we are led to the transformations $\Omega$ previously discovered by the author.||

If $x, y, z$ are the cartesian coördinates of a surface $S$ and $\omega$ is any solution of the point equation of $S$, the surface $\bar{S}$ whose cartesian coördinates are $x/\omega, y/\omega, z/\omega$ is referred to a conjugate system. We say that $\bar{S}$ is a radial transform of $S$. Combinations of radial and $T$ transformations are studied in §13 in relation to the theorem of permutability of transformations $T$. In particular, it is shown that this theorem can be applied when a radial transformation is treated as a special type of transformation $T$.

1. Transformations $T$ in homogeneous point coördinates

The necessary and sufficient condition that four functions, $x, y, z, w$, be the homogeneous point coördinates of a surface $S$, referred to a conjugate system of lines of parameters $u$ and $v$, is that these functions satisfy an equation of the form

$$\frac{\partial^2 \theta}{\partial u \partial v} + a \frac{\partial \theta}{\partial u} + b \frac{\partial \theta}{\partial v} + c \theta = 0,$$

where $a, b, c$ are functions of $u$ and $v$. We refer to this equation as the point equation of the conjugate system.

If the developables of a rectilinear congruence meet $S$ in the parametric

‡ Jf, pp. 397–430.
|| M_2.
curves, the congruence and the parametric system are said to be conjugate to one another. Darboux* has shown that when a solution \( \phi \) of the adjoint equation of (1), namely

\[
\frac{\partial^2 \phi}{\partial u \partial v} - a \frac{\partial \phi}{\partial u} - b \frac{\partial \phi}{\partial v} + \left( c - \frac{\partial a}{\partial u} - \frac{\partial b}{\partial v} \right) \phi = 0,
\]

is known, a congruence \( G_1 \) conjugate to the parametric system is given by quadratures. In fact, the point coördinates, \( x', y', z', w'; x'', y'', z'', w'' \) of the focal points \( F' \) and \( F'' \) of the congruence are given by expressions of the form

\[
x' = \int \phi_1 \left( \frac{\partial x}{\partial u} + bx \right) du + x \left( \frac{\partial \phi_1}{\partial v} - a \phi_1 \right) dv,
\]

\[
x'' = \int x \left( \frac{\partial \phi_1}{\partial u} - b \phi_1 \right) du + \phi_1 \left( \frac{\partial x}{\partial v} + ax \right) dv.
\]

Furthermore, each solution of (1) leads by quadratures to another conjugate system conjugate to the above congruence. For, if \( \theta_1 \) is a solution of (1), the function \( \sigma_1 \) given by

\[
\sigma_1 = \int \phi_1 \left( \frac{\partial \theta_1}{\partial u} + b \theta_1 \right) du + \theta_1 \left( \frac{\partial \phi_1}{\partial v} - a \phi_1 \right) dv,
\]

is a solution of the point equation of the surface \( (F'_1) \), the locus of \( F'_1 \). Hence by the theorem of Levy† the surface \( S_1 \), whose coördinates are given by equations of the form

\[
x_1 = x_1' - \frac{\sigma_1}{\sigma_1} \frac{\partial x_1'}{\partial v} = x_1 - \frac{\sigma_1}{\theta_1} x,
\]

is conjugate to the congruence \( G_1 \) whose focal surfaces are \( (F'_1) \) and \( (F''_1) \). We say that \( S_1 \) is obtained from \( S \) by a transformation \( T \).

The equations of \( S_1 \) can be given another form, if we look upon the lines of the congruence \( G_1 \) as tangent to the focal surface \( (F''_1) \) also. Evidently the function \( \tau_1 \), given by

\[
\tau_1 = \int \theta_1 \left( \frac{\partial \phi_1}{\partial u} - b \phi_1 \right) du + \phi_1 \left( \frac{\partial \theta_1}{\partial v} + a \theta_1 \right) dv,
\]

is a solution of the point equation of \( (F''_1) \). Hence the equations of \( S_1 \) can be given the form

\[
x_1 = -x_1'' + \frac{\tau_1}{\theta_1} x.
\]

From (4) and (6) we find the relation

\[ \phi_1 \theta_1 = \sigma_1 + \tau_1. \]  

From (5) or (7) we get by differentiation

\[ \frac{\partial x_1}{\partial u} = \tau_1 \frac{\partial}{\partial u} \left( \frac{x}{\theta_1} \right), \quad \frac{\partial x_1}{\partial v} = -\sigma_1 \frac{\partial}{\partial v} \left( \frac{x}{\theta_1} \right). \]  

2. The inverse of a transformation \( T \)

It is readily found from (9) that \( x_1, y_1, z_1, w_1 \) satisfy the equation

\[ \frac{\partial^2 \theta'_1}{\partial u \partial v} - \frac{\sigma_1}{\tau_1} \left( a + \frac{\partial \log \theta_1}{\partial v} \right) \frac{\partial \theta'_1}{\partial u} - \frac{\tau_1}{\sigma_1} \left( b + \frac{\partial \log \theta_1}{\partial u} \right) \frac{\partial \theta'_1}{\partial v} = 0. \]  

Since the relation between \( S \) and \( S_1 \) is reciprocal, there exist functions \( \phi_1^{-1} \) and \( \theta_1^{-1} \) by means of which \( S \) is the transform of \( S_1 \). We shall show that

\[ \theta_1^{-1} = 1, \quad \phi_1^{-1} = \frac{\phi_1 \theta_1}{\sigma_1 \tau_1}. \]  

We remark that as the congruence \( G_1 \) is the same, on the assumption that \( \theta_1^{-1} = 1 \), we must have analogously to (3)

\[ \frac{\partial}{\partial u} (\rho x'_1) = \phi_1^{-1} \left[ \frac{\partial x_1}{\partial u} - \frac{\tau_1}{\sigma_1} \left( b + \frac{\partial \log \theta_1}{\partial u} \right) x_1 \right], \]

\[ \frac{\partial}{\partial v} (\rho x'_1) = x_1 \left[ \frac{\partial \phi_1^{-1}}{\partial v} + \frac{\sigma_1}{\tau_1} \left( a + \frac{\partial \log \theta_1}{\partial v} \right) \phi_1^{-1} \right], \]  

where \( \rho \) is to be determined. If the value of \( x'_1 \) from (5) be substituted in the first of these equations, the result is reducible to the form

\[ A \frac{\partial x}{\partial u} + B x = 0, \]

where \( A \) and \( B \) are determinate expressions. Since similar equations hold in \( y, z, \) and \( w \), \( A \) and \( B \) must be equal to zero. From these equations we find that \( \rho = 1/\sigma \) and that \( \phi_1^{-1} \) is of the form (11). It is readily shown that these values satisfy the second of (12).

From (4) and (6) we have

\[ \frac{\partial^2 \sigma_1}{\partial u \partial v} = \frac{1}{\theta_1 \phi_1} \frac{\partial \sigma_1}{\partial u} \frac{\partial \sigma_1}{\partial v} + \theta_1 \phi_1 k, \]

\[ \frac{\partial^2 \tau_1}{\partial u \partial v} = \frac{1}{\theta_1 \phi_1} \frac{\partial \tau_1}{\partial u} \frac{\partial \tau_1}{\partial v} + \theta_1 \phi_1 h, \]  

where \( h \) and \( k \) are the invariants of (1) and are given by

\[ h = \frac{\partial a}{\partial u} + ab - c, \quad k = \frac{\partial b}{\partial v} + ab - c. \]
By means of (4) and (6) equation (10) may be given the form

\[
\frac{\partial^2 \theta'}{\partial u \partial v} - \frac{\tau_1}{\tau_1 \phi_1} \frac{\partial \tau_1}{\partial u} \frac{\partial \theta'}{\partial v} - \frac{\tau_1}{\tau_1 \phi_1} \frac{\partial \sigma_1}{\partial u} \frac{\partial \theta'}{\partial v} = 0.
\]

In consequence of (13) the adjoint of this equation is reducible to

\[
\frac{\partial^2 \phi'}{\partial u \partial v} + \frac{\sigma_1}{\tau_1 \phi_1} \frac{\partial \tau_1}{\partial v} \frac{\partial \phi'}{\partial u} + \frac{\tau_1}{\sigma_1 \phi_1} \frac{\partial \sigma_1}{\partial v} \frac{\partial \phi'}{\partial u}
\]

\[
+ \phi' \left[ \frac{\tau_1}{\tau_1} + \frac{1}{\sigma_1^2} \left( \frac{\partial \sigma_1}{\partial u} \frac{\partial \tau_1}{\partial v} + \frac{\partial \sigma_1}{\partial v} \frac{\partial \tau_1}{\partial u} \right) \right] = 0.
\]

From (11) we find

\[
\frac{\partial^2 \phi_{1^{-1}}}{\partial u \partial v} = \frac{1}{\tau_1} \frac{\partial \tau_1}{\partial u} \frac{\partial \tau_1}{\partial v} + \frac{1}{\sigma_1^2} \frac{\partial \sigma_1}{\partial u} \frac{\partial \sigma_1}{\partial v} - \frac{\tau_1}{\sigma_1} \phi_1 \left( \frac{h}{\tau_1^2} + \frac{k}{\sigma_1^2} \right).
\]

Making use of this result, we verify readily that $\phi_{1^{-1}}$ is a solution of (16).

If $\phi'$ is any solution of (16), then

\[
\frac{\partial^2}{\partial u \partial v} \left( \frac{\phi'}{\phi_{1^{-1}}} \right) = \frac{\tau_1}{\phi_1} \frac{\partial \sigma_1}{\partial u} \frac{\partial \phi'}{\partial v} \left( \frac{\phi'}{\phi_{1^{-1}}} \right) + \frac{\sigma_1}{\phi_1} \frac{\partial \tau_1}{\partial u} \frac{\partial \phi'}{\partial v} \left( \frac{\phi'}{\phi_{1^{-1}}} \right).
\]

Hence if $\phi_1$ and $\phi_2$ are two solutions of (2), the equations

\[
\frac{\partial}{\partial u} \left( \frac{\phi_{12}}{\phi_{1^{-1}}} \right) = \sigma_1 \frac{\partial}{\partial u} \left( \frac{\phi_2}{\phi_1} \right), \quad \frac{\partial}{\partial v} \left( \frac{\phi_{12}}{\phi_{1^{-1}}} \right) = -\tau_1 \frac{\partial}{\partial v} \left( \frac{\phi_2}{\phi_1} \right)
\]

are consistent, and the function $\phi_{12}$ so defined is a solution of (16).

3. Transformations $T$ in cartesian coördinates.

Parallel transformations $T$

We consider now the case when the point coördinates are non-homogeneous and rectangular. The point equation is of the form

\[
\frac{\partial^2 \theta}{\partial u \partial v} + a \frac{\partial \theta}{\partial u} + b \frac{\partial \theta}{\partial v} = 0.
\]

When we take $w = 1$, we have from the corresponding equation (9)

\[
\frac{\partial w_1}{\partial u} = \tau_1 \frac{\partial}{\partial u} \left( \frac{1}{\theta_1} \right), \quad \frac{\partial w_1}{\partial v} = -\sigma_1 \frac{\partial}{\partial v} \left( \frac{1}{\theta_1} \right).
\]

Moreover, the cartesian coördinates $x_1$, $y_1$, $z_1$ of $S_1$ are given by equations of the form

\[
\frac{\partial}{\partial u} \left( x_1 w_1 \right) = \tau_1 \frac{\partial}{\partial u} \left( \frac{x}{\theta_1} \right), \quad \frac{\partial}{\partial v} \left( x_1 w_1 \right) = -\sigma_1 \frac{\partial}{\partial v} \left( \frac{x}{\theta_1} \right).
\]
By means of (19) these equations are reducible to

\[ \frac{\partial x_1}{\partial u} = \frac{\tau_1}{w_1 \theta_1^2} \left( \theta_1 \frac{\partial x}{\partial u} + (x_1 - x) \frac{\partial \theta_1}{\partial u} \right), \]

(21)

\[ \frac{\partial x_1}{\partial v} = -\frac{\sigma_1}{w_1 \theta_1^2} \left( \theta_1 \frac{\partial x}{\partial v} + (x_1 - x) \frac{\partial \theta_1}{\partial v} \right). \]

Now the point equation of \( S_1 \) is

\[ \frac{d^2 l}{du dv} + \left[ \frac{\partial \log w_1}{\partial v} - \left( a + \frac{\partial \log \theta_1}{\partial v} \right) \frac{\sigma_1}{\tau_1} \right] \frac{\partial \theta'_1}{\partial u} \]

(22)

\[ + \left[ \frac{\partial \log w_1}{\partial u} - \left( b + \frac{\partial \log \theta_1}{\partial u} \right) \frac{\tau_1}{\sigma_1} \right] \frac{\partial \theta'_1}{\partial v} = 0. \]

The adjoint of this equation is obtained by replacing \( \phi'_1 \) in (16) by \( \phi_1/w_1 \).

We consider the transformations for which \( \phi_1 \) is any solution of (2) and \( \theta_1 = 1 \). The corresponding functions \( \sigma_1 \) and \( \tau_1 \) are given by

\[ \sigma'_1 = \int b \phi_1 du + \left( \frac{\partial \phi_1}{\partial v} - a \phi_1 \right) dv, \]

(23)

\[ \tau'_1 = \int \left( \frac{\partial \phi_1}{\partial u} - b \phi_1 \right) du + a \phi_1 dv. \]

It is readily shown that these particular values are in the following relations with the functions \( \sigma_1 \) and \( \tau_1 \) as given by (4) and (6):

(24)

\[ \tau'_1 = \frac{\tau_1}{\theta_1} - w_1, \quad \sigma'_1 = \frac{\sigma_1}{\theta_1} + w_1. \]

If \( S^{(1)} \) denotes the corresponding transform of \( S \), and its cartesian coördinates are denoted by \( x^{(1)}, y^{(1)}, z^{(1)} \), we have from (19) and (20)

(25)

\[ \frac{\partial x^{(1)}}{\partial u} = \tau'_1 \frac{\partial x}{\partial u}, \quad \frac{\partial x^{(1)}}{\partial v} = -\sigma'_1 \frac{\partial x}{\partial v}. \]

From the form of these equations it is evident that the tangent planes to \( S \) and \( S^{(1)} \) at corresponding points are parallel. Hence, when \( \theta_1 = 1 \), we have the parallel transformations \( T \).

4. **Theorem of permutability of transformations** \( T \)

Suppose that we have two transforms \( S_1 \) and \( S_2 \) of \( S \) determined by the respective sets of functions \( \sigma_1, \tau_1 \) and \( \sigma_2, \tau_2 \), where \( \sigma_2 \) and \( \tau_2 \) are given by (4) and (6) when \( \theta_1 \) and \( \phi_1 \) are replaced by \( \theta_2 \) and \( \phi_2 \). A solution \( \theta_{12} \) of equation (22) is given by the quadratures

(26)

\[ \frac{\partial}{\partial u} (w_1 \theta_{12}) = \tau_1 \frac{\partial}{\partial u} \left( \frac{\theta_2}{\theta_1} \right), \quad \frac{\partial}{\partial v} (w_1 \theta_{12}) = -\sigma_1 \frac{\partial}{\partial v} \left( \frac{\theta_2}{\theta_1} \right), \]

which are of the same form as (20).
We consider the surface $S_{12}$ obtained from $S_1$ by the transformation $T$ determined by $\theta_{12}$ and $\phi_{12}$, as given by (17), where now

$$\phi_{12}^{-1} = \frac{w_1 \theta_1 \phi_1}{\tau_1 \sigma_1}. \tag{27}$$

The function $w_{12}$ of this transformation is given by

$$\frac{\partial w_{12}}{\partial u} = \tau_{12} \frac{\partial}{\partial u} \left( \frac{1}{\theta_{12}} \right), \quad \frac{\partial w_{12}}{\partial v} = -\sigma_{12} \frac{\partial}{\partial v} \left( \frac{1}{\theta_{12}} \right), \tag{28}$$

where $\sigma_{12}$ and $\tau_{12}$ are defined by equations analogous to (4) and (6), namely

$$\sigma_{12} = \int \phi_{12} \left[ \frac{\partial \theta_{12}}{\partial u} + \theta_{12} \left( \frac{\partial \log w_1}{\partial u} - \left( b + \frac{\partial \log \theta_1}{\partial u} \right) \tau_1 \right) \right] du$$

$$+ \theta_{12} \left[ \frac{\partial \phi_{12}}{\partial v} - \phi_{12} \left( \frac{\partial \log w_1}{\partial v} - \left( a + \frac{\partial \log \theta_1}{\partial v} \right) \sigma_1 \right) \right] dv, \tag{29}$$

$$\tau_{12} = \int \theta_{12} \left[ \frac{\partial \phi_{12}}{\partial u} - \phi_{12} \left( \frac{\partial \log w_1}{\partial u} - \left( b + \frac{\partial \log \theta_1}{\partial u} \right) \tau_1 \right) \right] du$$

$$+ \phi_{12} \left[ \frac{\partial \theta_{12}}{\partial v} + \theta_{12} \left( \frac{\partial \log w_1}{\partial v} - \left( a + \frac{\partial \log \theta_1}{\partial v} \right) \sigma_1 \right) \right] dv. \tag{30}$$

By means of (17), (26), and (27) these expressions are reducible to

$$\sigma_{12} = \int \phi_{12} \left[ -\frac{\tau_1}{w_1} \frac{\partial}{\partial u} \left( \frac{\theta_2}{\theta_1} \right) - \left( b + \frac{\partial \log \theta_1}{\partial u} \right) \frac{\theta_{12} \tau_1}{\sigma_1} \right] du$$

$$+ \theta_{12} \left[ \frac{\phi_{12} \tau_1}{\sigma_1} \left( a \phi_1 - \frac{\partial \phi_1}{\partial v} \right) + \theta_1 \phi_1 w_1 \frac{\partial}{\partial v} \left( \frac{\phi_2}{\phi_1} \right) \right] dv, \tag{31}$$

$$\tau_{12} = \int \theta_{12} \left[ \frac{\phi_{12} \sigma_1}{\tau_1} \left( b \phi_1 - \frac{\partial \phi_1}{\partial u} \right) - \frac{\theta_1 \phi_1 w_1}{\tau_1} \frac{\partial}{\partial u} \left( \frac{\phi_2}{\phi_1} \right) \right] du$$

$$+ \phi_{12} \left[ \frac{\sigma_1}{w_1} \frac{\partial}{\partial v} \left( \frac{\theta_2}{\theta_1} \right) - \left( a + \frac{\partial \log \theta_1}{\partial v} \right) \frac{\theta_{12} \sigma_1}{\tau_1} \right] dv.$$
Proceeding in like manner with \(S_2\), we obtain a transform \(S_{21}\) by means of functions \(\theta_{21}, \phi_{21}, \sigma_{21}, \tau_{21}\). The equations similar to (31) are obtained by interchanging the subscripts 1 and 2 in (31). From these two sets of equations it follows that \(S_{12}\) and \(S_{21}\) are the same surface, if

\[\tau_{12} \tau_1 \theta_2 \theta_{21} w_1 w_{21} = \tau_{21} \tau_2 \theta_1 \theta_{12} w_1 w_{12},\]

(32)

and

\[\sigma_{12} \sigma_1 \theta_2 \theta_{21} w_1 w_{21} = \sigma_{21} \sigma_2 \theta_1 \theta_{12} w_1 w_{12},\]

(33)

where

\[\frac{1}{\Theta_{12}} = \theta_2 \theta_{21} + \theta_1 \theta_{12} - \theta_{12} \theta_{21}.\]

When we express the condition that this value of \(x_{12}\) shall satisfy equations (31), we get

\[
\left[ \tau_{12} \tau_1 - \theta_1 \theta_{12} w_1 w_{12} \Theta_{12} \left( \frac{\theta_2 \theta_{21} \tau_1}{\theta_1 w_1} + \frac{\theta_1 \theta_{12} \tau_2}{\theta_2 w_2} - \theta_{12} \theta_{21} \right) \right] \left[ x_1 \left( - \theta_{21} \frac{\partial \theta_2}{\partial u} \right) - \theta_2 \frac{\partial \theta_1}{\partial u} + \theta_1 \frac{\partial \theta_2}{\partial u} \right] + x_2 \left( \theta_{12} \frac{\partial \theta_1}{\partial u} + \theta_2 \frac{\partial \theta_1}{\partial u} - \theta_1 \frac{\partial \theta_2}{\partial u} \right) - x \left( \theta_{21} \frac{\partial \theta_2}{\partial u} + \theta_{12} \frac{\partial \theta_1}{\partial u} + \frac{1}{\Theta_{12}} \frac{\partial x}{\partial u} \right) = 0,
\]

(34)

\[
\left[ \sigma_{12} \sigma_1 + \theta_1 \theta_{12} w_1 w_{12} \Theta_{12} \left( \frac{\theta_2 \theta_{21} \sigma_1}{\theta_1 w_1} + \frac{\theta_1 \theta_{12} \sigma_2}{\theta_2 w_2} + \theta_{12} \theta_{21} \right) \right] \left[ x_1 \left( - \theta_{21} \frac{\partial \theta_2}{\partial v} \right) - \theta_2 \frac{\partial \theta_1}{\partial v} + \theta_1 \frac{\partial \theta_2}{\partial v} \right] + x_2 \left( \theta_{12} \frac{\partial \theta_1}{\partial v} + \theta_2 \frac{\partial \theta_1}{\partial v} - \theta_1 \frac{\partial \theta_2}{\partial v} \right) - x \left( \theta_{21} \frac{\partial \theta_2}{\partial v} + \theta_{12} \frac{\partial \theta_1}{\partial v} + \frac{1}{\Theta_{12}} \frac{\partial x}{\partial v} \right) = 0.
\]

On the assumption that \(\theta_1\) and \(\theta_2\) are independent, these are equivalent to

\[\sigma_1 \sigma_{12} = - \theta_1 \theta_{12} w_1 w_{12} \Theta_{12} \left( \frac{\theta_2 \theta_{21} \sigma_1}{\theta_1 w_1} + \frac{\theta_1 \theta_{12} \sigma_2}{\theta_2 w_2} + \theta_{12} \theta_{21} \right),\]

\[\tau_1 \tau_{12} = \theta_1 \theta_{12} w_1 w_{12} \Theta_{12} \left( \frac{\theta_2 \theta_{21} \tau_1}{\theta_1 w_1} + \frac{\theta_1 \theta_{12} \tau_2}{\theta_2 w_2} - \theta_{12} \theta_{21} \right).
\]

From these equations and one analogous to (8) we obtain

\[\phi_{12} = \frac{\theta_2 \theta_{12} w_1 w_{12} \Theta_{12} \phi_{21}}{\sigma_1 \tau_1} \left( - \phi_{12} + \phi_{21} + (\sigma_1 \tau_2 - \sigma_2 \tau_1) \frac{1}{\theta_2 w_2} \right).
\]

When this value is substituted in (17), we find that the transformation func-
tions must have the expressions

\[ w_{12} = \frac{w_2}{\theta_1 \theta_{12} \theta_{12}}, \]

\[ \phi_{12} = \frac{w_1 \theta_1 \phi_1}{\tau_1 \sigma_1} \left( -w_2 \theta_{21} + \frac{1}{\theta_2 \phi_1} \left( \tau_2 \sigma_1 - \tau_1 \sigma_2 \right) \right), \]

\( \sigma_{12} = -\frac{w_1 w_2}{\sigma_1} \left( \frac{\theta_2 \theta_{21} \sigma_1}{\theta_1 w_1} + \frac{\theta_1 \theta_{12} \sigma_2}{\theta_2 w_2} + \theta_{12} \theta_{21} \right), \)

\[ \tau_{12} = \frac{w_1 w_2}{\tau_1} \left( \frac{\theta_2 \theta_{21} \tau_1}{\theta_1 w_1} + \frac{\theta_1 \theta_{12} \tau_2}{\theta_2 w_2} - \theta_{12} \theta_{21} \right). \]

It is readily verified that these values for \( \sigma_{12} \) and \( \tau_{12} \) satisfy equations (30).

In consequence of (35) equation (33) can be written

\[ (36) \quad \theta_1 \theta_{12} w_{12} x_{12} = w_2 \left( \theta_2 \theta_{21} x_1 + \theta_1 \theta_{12} x_2 - \theta_{12} \theta_{21} x \right). \]

Thus we have established a theorem of permutability of general transformations \( T \). There are two arbitrary constants involved, namely in the determination of \( \theta_{12} \) by (26) and of \( \theta_{21} \) by

\[ (37) \quad \frac{\partial}{\partial u} (w_2 \theta_{21}) = \tau_2 \frac{\partial}{\partial u} \left( \frac{\theta_1}{\theta_2} \right), \quad \frac{\partial}{\partial v} (w_2 \theta_{21}) = -\sigma_2 \frac{\partial}{\partial v} \left( \frac{\theta_1}{\theta_2} \right). \]

Accordingly we formulate

**Theorem 1.** If \( S_1 \) and \( S_2 \) are two transforms of \( S \), there exist \( \infty^2 \) surfaces \( S_{12} \), each of which is a transform of both \( S_1 \) and \( S_2 \); and their complete determination requires two quadratures.

We say that four such surfaces \( S, S_1, S_2, S_{12} \) form a quatern.

We consider, in particular, the case where \( S_2 \) is parallel to \( S \). If we take \( \theta_2 = 1 \), in accordance with (19) and (26) we have \( \theta_{12} = 1 \) as one solution. Now (33) becomes

\[ (38) \quad (x_{12} - x_2) \theta_1 = (x_1 - x) \theta_{21}. \]

Hence we have

**Theorem 2.** If \( S_2 \) is parallel to \( S \) and \( S_1 \) is any transform of \( S \), one of the surfaces \( S_{12} \) is parallel to \( S_1 \); moreover the lines joining corresponding points on \( S_{12} \) and \( S_2 \) and on \( S \) and \( S_1 \) are parallel.

If both \( S_1 \) and \( S_2 \) are parallel to \( S \), the functions \( \theta_1 \) and \( \theta_2 \) are constants. Hence from (33) it follows that \( x_{12} \) is a linear function of \( x, x_1, x_2 \), with constant coefficients, and consequently \( S_{12} \) also is parallel to \( S \).

5. **Envelope of the planes of a quatern**

If \( M, M_1, M_2, M_{12} \) are corresponding points of four surfaces of a quatern, it follows from (35) and (36) that these four points lie in a plane \( \pi \). Since
this plane contains the lines $MM_1$ and $MM_2$ which generate congruences conjugate to the parametric conjugate system on $S$, it envelopes a surface $\Sigma$ upon which the parametric curves form a conjugate system, as follows from the general theory of congruences.* Moreover, if $\Pi$ is the point of the envelope corresponding to $M$ on $S$, the tangent at $\Pi$ to one of these curves passes through the focal points $F'_1$ and $F'_2$ of the lines $MM_1$ and $MM_2$ respectively, and the tangent to the other curve passes through the focal points $F''_1$ and $F''_2$. We will now find the coordinates of $\Pi$.

In cartesian coördinates equations (5) and (7) are of the form

\begin{equation}
    w_1 x_1 = x'_1 \sigma'_1 - \frac{\sigma'_1}{\theta'_1} x, \quad w_1 x_1 = -x''_1 \tau'_1 + \frac{\tau'_1}{\theta'_1} x,
\end{equation}

where now $x'_1 \sigma'_1$ and $x''_1 \tau'_1$ are respectively equal to the right-hand members of (3). Similar equations with subscripts 2 hold for the congruence of lines $MM_2$.

The cartesian coördinates $\xi, \eta, \zeta$ of $\Pi$ are given by equations of the form

\begin{equation}
    \xi = \frac{1}{\sigma_1} \left( x_1 w_1 + \frac{\sigma_1}{\theta_1} x \right) + t_1 \left[ \frac{1}{\sigma_1} \left( x_1 w_1 + \frac{\sigma_1}{\theta_1} x \right) - \frac{1}{\sigma_2} \left( x_2 w_2 + \frac{\sigma_2}{\theta_2} x \right) \right],
\end{equation}

\begin{equation}
    = \frac{1}{\tau_1} \left( -x_1 w_1 + \frac{\tau_1}{\theta_1} x \right) + t_2 \left[ \frac{1}{\tau_1} \left( -x_1 w_1 + \frac{\tau_1}{\theta_1} x \right) - \frac{1}{\tau_2} \left( -x_2 w_2 + \frac{\tau_2}{\theta_2} x \right) \right],
\end{equation}

where $t_1$ and $t_2$ are to be determined. When these two expressions for $\xi$ are equated, we get an equation of the form

\begin{equation}
    Ax + Bx_1 +Cx_2 = 0,
\end{equation}

where $A$, $B$, and $C$ are determinate functions. Since similar equations in the $y$'s and $z$'s also must hold, we must have $A = B = C = 0$. From the first two of these equations we get

\begin{align*}
    \phi_1 \left( w_2 + \frac{\sigma_2}{\theta_2} \right) + t_1 \left( w_2 \phi_1 - w_1 \phi_2 + \frac{\sigma_2 \tau_1 - \sigma_1 \tau_2}{\theta_1 \theta_2} \right) &= 0, \\
    \phi_1 \left( w_2 - \frac{\tau_2}{\theta_2} \right) + t_2 \left( w_2 \phi_1 - w_1 \phi_2 + \frac{\sigma_2 \tau_1 - \sigma_1 \tau_2}{\theta_1 \theta_2} \right) &= 0.
\end{align*}

These values satisfy $C = 0$, and when substituted in the above expressions

for $\xi$ we get

$$\xi \left( w_2 \phi_1 - w_1 \phi_2 + \frac{\sigma_2 \tau_1 - \sigma_1 \tau_2}{\theta_1 \theta_2} \right) = w_2 \phi_1 x_2 - w_1 \phi_2 x_1$$

(41)

$$- \frac{\sigma_1 \tau_2 - \sigma_2 \tau_1}{\theta_1 \theta_2} x.$$  

We shall find the functions of the theorem of permutability when homogeneous coordinates are used. Now the functions $\theta_{12}$ and $\theta_{21}$ are given by

$$\frac{\partial \theta_{ij}}{\partial u} = \tau_i \frac{\partial}{\partial u} \left( \frac{\theta_j}{\theta_i} \right), \quad \frac{\partial \theta_{ij}}{\partial v} = - \sigma_i \frac{\partial}{\partial v} \left( \frac{\theta_j}{\theta_i} \right) \quad (i = 1, 2, j = 1, 2, \ i \neq j),$$

(42) and the coordinates $x_{12}, \cdots, w_{12}$ of $S_{12}$ must satisfy the equations of the form

$$\frac{\partial x_{ij}}{\partial u} = \tau_{ij} \frac{\partial}{\partial u} \left( \frac{x_i}{\theta_{ij}} \right), \quad \frac{\partial x_{ij}}{\partial v} = - \sigma_{ij} \frac{\partial}{\partial v} \left( \frac{x_i}{\theta_{ij}} \right) \quad (i = 1, 2, j = 1, 2, \ i \neq j).$$

(43) From (35) it follows that the functions $\tau_{12}, \sigma_{12}, \phi_{12}$ are of the form

$$\tau_{12} \tau_{21} = \tau_1 \tau_2 \frac{\theta_2 \theta_{21} \tau_1}{\theta_1} + \theta_1 \theta_{12} \tau_2 - \theta_{12} \theta_{21},$$

(44) $$\sigma_{12} \sigma_{21} = \sigma_1 \sigma_2 \frac{\theta_2 \theta_{21} \sigma_1}{\theta_1} + \theta_1 \theta_{12} \sigma_2 + \theta_{12} \theta_{21},$$

$$\phi_{12} = \frac{\theta_1 \phi_1}{\sigma_1 \tau_1} \left( - \theta_{12} + (\sigma_1 \tau_2 - \sigma_2 \tau_1) \frac{1}{\theta_2 \phi_1} \right).$$

Moreover, the coordinate $x_{12}$ is expressed by

(45) $$\theta_1 \theta_{12} x_{12} = \theta_2 \theta_{21} x_1 + \theta_1 \theta_{12} x_2 - \theta_{12} \theta_{21} x.$$  

6. Transformations $T$ determined by the same function $\phi$

We consider now the relation of two transformations determined by $\theta_1$ and $\theta_2$ respectively but by the same function $\phi_1$. If we put

$$\left( \tau_1 \right)_2 = \int \theta_2 \left( \frac{\partial \phi_1}{\partial u} - b \phi_1 \right) du + \phi_1 \left( \frac{\partial \theta_2}{\partial v} + a \theta_2 \right) dv,$$

$$\left( \sigma_1 \right)_2 = \int \phi_1 \left( \frac{\partial \theta_2}{\partial u} + b \theta_2 \right) du + \theta_2 \left( \frac{\partial \phi_1}{\partial v} - a \phi_1 \right) dv,$$

we have in consequence of (26)

$$\left( \tau_1 \right)_2 = \frac{\theta_2}{\theta_1} \tau_1 - w_1 \theta_{12}, \quad \left( \sigma_1 \right)_2 = \frac{\theta_2}{\theta_1} \sigma_1 + w_1 \theta_{12}.$$  

(46) When these values are substituted in equations analogous to (20), namely

$$\frac{\partial}{\partial u} (x_2 w_2) = (\tau_1)_2 \frac{\partial}{\partial u} \left( \frac{x}{\theta_2} \right), \quad \frac{\partial}{\partial v} (x_2 w_2) = - (\sigma_1)_2 \frac{\partial}{\partial v} \left( \frac{x}{\theta_2} \right),$$

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the latter can be integrated in the form

\[(47) \quad x_2 w_2 = x_1 w_1 - x w_1^0 \frac{\theta_{12}}{\theta_2}.\]

In like manner from equations analogous to (19) we get

\[(48) \quad w_2 = \frac{w_1}{\theta_2} (\theta_2 - \theta_{12}) + c,\]

where \(c\) is a constant. By means of this result equation (47) can be given the form

\[(49) \quad (x_2 - x) w_2 = (x_1 - x) w_1 - cx.\]

From this it is seen that the congruence \(G_2\) of lines joining corresponding points on \(S\) and \(S_2\) is the same as the congruence \(G_1\) only in case \(c = 0\) in (48).

When the above expressions for \((\sigma_1)_2\) and \((\tau_1)_2\) are substituted in (37), we find

\[(50) \quad w_2 \theta_{21} = - w_1 \theta_{12} \frac{\theta_1}{\theta_2} + k,\]

where \(k\) is an additive constant.

From (33) and (34) we obtain for the present case

\[(51) \quad \frac{w_2}{\Theta_{12}} = (\theta_2 - \theta_{12}) k + \theta_1 \theta_{12} c, \quad \frac{w_{12}}{\theta_1 \theta_{12}} = \frac{(\theta_2 - \theta_{12}) k}{\theta_1 \theta_{12}} + c, \quad x_{12} (k (\theta_2 - \theta_{12}) + c \theta_1 \theta_{12}) = k \frac{w_2}{w_1} \frac{\theta_2}{\theta_1} x_2.\]

Hence if \(k = 0\), the surface \(S_{12}\) reduces to a point; if \(c = 0\), it coincides with \(S_2\).

In the inverse transformation from \(S_1\) to \(S\) the function \(w_1^{-1}\) has the value \(1/\theta_1\), as is evident from (20). If we look upon \(S\) and \(S_{12}\) as transforms of \(S_1\), the analogue of equation (47) is

\[(52) \quad x_{12} w_{12} = w_1^{-1} \left( x - x_1 \frac{\theta_2}{\theta_{12}} \right).\]

This equation is satisfied by the value of \(x_{12}\), given by (51), provided \(k = -1\).

Incidentally we remark that the last of (51) can be written

\[(53) \quad x_{12} = x_2 / \left( 1 - \frac{c}{k} \theta_{21} \right).\]

The denominator of this equation is a solution of the point equation of \(S_2\). Moreover, corresponding points on \(S_2\) and \(S_{12}\) are on a line through the origin. This is a type of transformations which we will consider later (§ 13); we call them radial transformations.

Accordingly we have
Theorem 3. When a transform $S_1$ of $S$ is known, the determination of another transform $S_2$ with the same function $\phi$ requires a single quadrature; then the fourth surface of the quatern is a radial transform of $S_2$.

7. Another form of transformations $T$

Particular importance attaches to the results of the preceding section when we take a parallel surface for $S_2$. As in § 3, we call it $S^{(1)}$ and its coordinates $x^{(1)}, y^{(1)}, z^{(1)}$. We take $\theta_2 = 1$, then $\theta_1 = 1$. Also we take $c = 1$. Then (47) assumes the desired form

$$x_1 = x + \frac{x^{(1)}}{w_1}.$$  

In consequence of (24) equations (19) can be written

$$\frac{\partial}{\partial u} (w_1 \theta_1) = -\tau_1 \frac{\partial \theta_1}{\partial u}, \quad \frac{\partial}{\partial v} (w_1 \theta_1) = \sigma_1 \frac{\partial \theta_1}{\partial v}.$$  

Hence if we put

$$w_1 \theta_1 = -\theta^{(1)},$$  

we have

$$\frac{\partial \theta^{(1)}}{\partial u} = \tau_1 \frac{\partial \theta_1}{\partial u}, \quad \frac{\partial \theta^{(1)}}{\partial v} = -\sigma_1 \frac{\partial \theta_1}{\partial v},$$  

which are similar to (25). By means of (55) equation (54) is reducible to*

$$x_1 = x - \frac{\theta_1}{\theta^{(1)}} x^{(1)}.$$  

The significance of this result is that the problem of finding transformations $T$ is reduced to that of finding parallel transforms and the integration of equation (1).

Equations (57) enable us to show that when we take

$$\theta_1 = ax + by + cz, \quad \theta^{(1)} = ax^{(1)} + by^{(1)} + cz^{(1)},$$  

where $a, b, c$ are constants, then $S_1$ is the plane $ax_1 + by_1 + cz_1 = 0$.

In consequence of (24) and (54) equations (39) giving the coordinates of the focal points of the congruence $G_1$ are reducible to

$$x'_1 = x + \frac{x^{(1)}}{\sigma_1}, \quad x''_1 = x - \frac{x^{(1)}}{\tau_1}.$$  

As an application of these results we seek the condition that $S_1$ shall be normal to the lines of the congruence $G_1$. From (57) it is seen that $x^{(1)}, y^{(1)}, z^{(1)}$ are the direction-parameters of the lines of this congruence. Hence

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* Cf. Jonas, l. c., p. 102.
we must have 
\[ \sum x^{(i)} \frac{\partial x_1}{\partial u} = 0, \quad \sum x^{(i)} \frac{\partial x_1}{\partial v} = 0. \]

Substituting the values of \( x_1, y_1, z_1 \) from (57), we have to within a constant factor
\[
\theta_1^{(1)} = \sqrt{x^{(1)2} + y^{(1)2} + z^{(1)2}}.
\]

From (22) it follows that the point equation of \( S^{(1)} \) is
\[
\frac{\partial^2 \theta_1^{(1)}}{\partial u \partial v} - a \frac{\partial \theta_1^{(1)}}{\partial u} \frac{\partial \theta_1^{(1)}}{\partial v} = 0.
\]

Expressing the condition that the above value of \( \theta_1^{(1)} \) satisfies this equation, we get
\[
\sum \frac{\partial x^{(i)}}{\partial u} \frac{\partial x_1}{\partial v} - \frac{\partial \theta_1^{(1)}}{\partial u} \frac{\partial \theta_1^{(1)}}{\partial v} = 0,
\]
and consequently
\[
\sum \frac{\partial x \partial x_1}{\partial u \partial v} - \frac{\partial \theta_1 \partial \theta_1}{\partial u \partial v} = 0.
\]

But this is the condition that \( x^2 + y^2 + z^2 - \theta_1^2 \) also is a solution of the point equation of \( S \). Moreover, it follows from (59) and (57) that in this case \( \theta_1 \) is the distance from \( S \) to \( S_1 \). Hence from this point of view we have established the known

**Theorem 4.** When the developables of the congruence of normals to a surface \( S_1 \) meet a surface \( S \) in a conjugate system, the function \( t \) giving the distance between corresponding points on \( S \) and \( S_1 \) is a solution of the point equation of \( S \) as is also the function \( x^2 + y^2 + z^2 - t^2 \).

Returning to the general case, we have from (50) in consequence of (55)
\[
\theta_{21} = \theta_1^{(1)} - 1.
\]

We have taken \( k = -1 \) so that (52) shall hold. Then \( S_{12} \) is the parallel \( S_{1}^{(1)} \) of \( S_1 \) by means of which \( S \) is obtained from \( S_1 \). Consequently the present form of (53) is
\[
x_1^{(1)} = \frac{x^{(1)}}{\theta_1^{(1)}}.
\]

8. Another form of the equations of the theorem of permutability

Suppose now that we have two transforms \( S_1 \) and \( S_2 \) of \( S \); we wish to give the theorem of permutability a new form in view of the preceding results. Evidently the functions \( \theta_{12} \) and \( \theta_{21} \) are given by expressions analogous to (57), namely
\[
\theta_{12} = \theta_2 - \frac{\theta_1}{\theta_1^{(1)}} \theta_2^{(1)} \quad \theta_{21} = \theta_1 - \frac{\theta_2}{\theta_2^{(2)}} \theta_1^{(2)},
\]
where \( \theta_i^{(j)} \) is defined by
\[
\frac{\partial \theta_i^{(j)}}{\partial u} = \tau_i \frac{\partial \theta_i}{\partial u}, \quad \frac{\partial \theta_i^{(j)}}{\partial v} = -\sigma_i \frac{\partial \theta_i}{\partial v},
\]
\( \sigma_j \) and \( \tau_j \) being given by equations obtained from (23) on replacing 1 by \( j \).

Now
\[
\frac{1}{\Theta_{12}} = \frac{\theta_1 \theta_2}{\theta_1^{(1)} \theta_2^{(1)}} \left( \theta_1^{(2)} \theta_2^{(2)} - \theta_1^{(1)} \theta_2^{(1)} \right),
\]
and from (33) we have
\[
\left( \theta_1^{(1)} \theta_2^{(2)} - \theta_2^{(1)} \theta_1^{(2)} \right) (x_{12} - x) = (\theta_1^{(2)} \theta_2^{(2)} - \theta_1^{(2)} \theta_2^{(1)}) x^{(1)} + (\theta_1^{(1)} \theta_1^{(2)} - \theta_1^{(1)} \theta_2^{(2)}) x^{(2)}.
\]

From this equation and (57) we obtain
\[
\left( \theta_1^{(1)} \theta_2^{(2)} - \theta_2^{(1)} \theta_1^{(2)} \right) (x_{12} - x_1) = (\theta_1^{(1)} \theta_2^{(2)} - \theta_1^{(2)} \theta_1^{(1)}) \left( \frac{\theta_1^{(2)}}{\theta_1^{(1)}} x^{(1)} - x^{(2)} \right).
\]

We note that the expression in the last parenthesis is similar in form to the right-hand member of (57). Hence if we put
\[
x_1^{(2)} = x^{(2)} - \frac{\theta_1^{(2)}}{\theta_1^{(1)}} x^{(1)},
\]
the surface \( S_1^{(2)} \) whose coordinates are \( x_1^{(2)}, y_1^{(2)}, z_1^{(2)} \) is a transform of \( S_1^{(2)} \).

If equation (66) be differentiated, the resulting equations are reducible to
\[
\frac{\partial x_1^{(2)}}{\partial u} = \left( \tau_1' - \theta_1^{(2)} \tau_1 \right) \theta_1 w_1 \frac{\partial x_1}{\partial u},
\]
\[
\frac{\partial x_1^{(2)}}{\partial v} = \left( \sigma_1' - \theta_1^{(2)} \sigma_1 \right) \theta_1 w_1 \frac{\partial x_1}{\partial v}.
\]

Hence \( S_1^{(2)} \) is parallel to \( S_1 \). We wish to show that it is the parallel surface whose coordinates enable the equations of the transformation from \( S_1 \) to \( S_{12} \) to be given a form similar to (57). The first derivatives of the coordinates of this desired parallel surface are equal to
\[
\tau_{12} \frac{\partial x_1}{\partial u}, \quad -\sigma_{12} \frac{\partial x_1}{\partial v},
\]
where in consequence of (24), (35), (57), and (55)
\[
\tau_{12}' = \frac{\tau_{12}}{\theta_{12}'} - w_{12} = \frac{w_1 \theta_1}{\tau_1} \left( \tau_1' - \theta_1^{(2)} \tau_1 \right),
\]
\[
\sigma_{12}' = \frac{\sigma_{12}}{\theta_{12}'} + w_{12} = \frac{w_1 \theta_1}{\sigma_1} \left( \sigma_1' - \theta_1^{(2)} \sigma_1 \right).
\]
Comparing these results with (67) we find that $S^{(2)}_1$ is the desired surface.

If we write (65) in the form

$$x_{12} = x_1 - \frac{\theta_2 - \theta_1}{\theta_2^{(1)} \theta_2^{(2)}} x_{1}^{(2)},$$

we note its similarity to (57).

9. Transformations $K$

We consider now the particular conjugate systems for which the invariants $h$ and $k$ of the point equation are equal. From (14) it is seen that in this case the point equation may be written

$$\frac{\partial^2 \theta}{\partial u \partial v} + \frac{\partial \log \sqrt{\theta}}{\partial u} + \frac{\partial \log \sqrt{n}}{\partial v} \frac{\partial \theta}{\partial v} = 0.$$

From (22) it is seen that the parametric system on $S_1$ will have equal invariants, if $\sigma_1$ and $\tau_1$ are equal. From (4) and (6) it follows that to within a constant factor we must have

$$\phi_1 = 2\theta_1 \rho.$$

It is readily verified that this value of $\phi_1$ satisfies the equation adjoint to (70). From (8) we have

$$\sigma_1 = \tau_1 = \theta_1^2 \rho.$$

Now equations (21) become

$$\frac{\partial x_1}{\partial u} = \frac{\rho}{w_1} \left( \theta_1 \frac{\partial x}{\partial u} + (x_1 - x) \frac{\partial \theta_1}{\partial u} \right),$$

$$\frac{\partial x_1}{\partial v} = -\frac{\rho}{w_1} \left( \theta_1 \frac{\partial x}{\partial v} + (x_1 - x) \frac{\partial \theta_1}{\partial v} \right),$$

where $w_1$ is given by

$$\frac{\partial w_1}{\partial u} = -\frac{\partial \theta_1}{\partial u}, \quad \frac{\partial w_1}{\partial v} = \frac{\partial \theta_1}{\partial v}.$$  

This transformation is the same which we have considered at length in a former paper, and called a transformation $K$.*

Equations (24) reduce to

$$\sigma'_1 = \rho \theta_1 + w_1, \quad \tau'_1 = \rho \theta_1 - w_1,$$

and consequently the expressions (39) for the coordinates of the focal points

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* M., p. 400.
of the congruence $G_1$ become

$$x_1' = \frac{\rho \theta_1 x + w_1 x_1}{\rho \theta_1 + w_1}, \quad x_2' = \frac{\rho \theta_1 x - w_1 x_1}{\rho \theta_1 - w_1}.$$  

The foregoing results are stated in

**Theorem 5.** When a surface $S$ is referred to a conjugate system with equal point invariants and $\theta_1$ is any solution of the point equation of $S$, the surface $S_1$ whose coordinates are given by quadratures of the form (73), is referred to a conjugate system with equal point invariants, and the developables of the lines joining corresponding points on $S$ and $S_1$ meet these surfaces in these parametric curves. Moreover, the focal points of the congruence are harmonic to the corresponding points on $S$ and $S_1$.

We assume that $S_1$ and $S_2$ are two surfaces in the relation of transformations $K$ with $S$, and apply the theorem of permutability. Equations (26) and (37) are now reducible to

$$\frac{\partial}{\partial u}(w_i \theta_{ij}) = \rho \left( \theta_i \frac{\partial \theta_j}{\partial u} - \theta_j \frac{\partial \theta_i}{\partial u} \right),$$

$$(i = 1, 2, \; i \neq j),$$

from which it follows that

$$w_2 \theta_{12} + w_1 \theta_{12} = \text{const.}$$

From (35) it is seen that the necessary and sufficient condition that $\sigma_{12} = \tau_{12}$ is

$$w_2 \theta_{21} + w_1 \theta_{12} = 0.$$  

Hence we have

**Theorem 6.** When $S_1$ and $S_2$ are two surfaces in the relation of transformations $K$ with a surface $S$, of the $\infty^2$ surfaces $S_{12}$ forming quaterns with them in accordance with the theorem of permutability of transformations $T$, $\infty^1$ are in the relation of transformations $K$ with $S_1$ and $S_2$.

From (35) we have

$$\theta_1 w_{12} = w_1 (\theta_{12} - \theta_2) + w_2 \theta_1, \quad \theta_2 w_{21} = w_2 (\theta_{21} - \theta_1) + w_1 \theta_2,$$

and from (36)

$$\theta_1 w_{12} x_{12} = -w_1 \theta_2 x_1 + w_2 \theta_1 x_2 + w_1 \theta_{12} x.$$  

The coördinates $\xi, \eta, \zeta$, of $\Pi$, the point of contact of the plane $\pi$ with its envelope, as given by (41), are expressible in the form

$$\xi = -\frac{w_2 \theta_1 x_2 - w_1 \theta_2 x_1}{w_2 \theta_1 - w_1 \theta_2} = -\frac{\theta_1 w_{12} x_{12} - w_1 \theta_{12} x}{\theta_1 w_{12} - w_1 \theta_{12}}.$$
Hence II is the intersection of the lines $MM_{12}$ and $M_1M_2$; consequently the points $M_{12}$ of the $\infty^1$ surfaces $S_{12}$ lie on the line $MM_{12}$. Therefore we have

Theorem 7. If $S$, $S_1$, $S_2$, $S_{12}$ are four surfaces of a quatern for transformations $K$, the plane $\pi$ of the four corresponding points $M$, $M_1$, $M_2$, $M_{12}$ touches its envelope in the intersection II of the lines $MM_{12}$ and $M_1M_2$; the parametric lines on the envelope form a conjugate system whose tangents are harmonic to the lines $MM_{12}$ and $M_1M_2$, and contain the focal points of the lines $MM_1$, $MM_2$, $M_{12}M_1$, $M_{12}M_2$ for the congruences generated by them.*

10. Transformations $T$ in tangential coördinates

When a surface $S$ is referred to a conjugate system, if $x$, $y$, $z$, $w$ and $X$, $Y$, $Z$, $W$ are the point and tangential coördinates respectively of $S$ so that

$$(80) \quad Xx + Yy + Zz + Ww = 0,$$

then $X$, $Y$, $Z$, $W$ satisfy an equation of the form

$$(81) \quad \frac{\partial^2 \lambda}{\partial w \partial v} + \alpha \frac{\partial \lambda}{\partial u} + \beta \frac{\partial \lambda}{\partial v} + \gamma \lambda = 0.$$ 

Evidently the analytical theory of § 1 is independent of the geometrical interpretation there given, and has a meaning when applied to equation (81). This we will give and study the relation between the two sets of equations.

The adjoint of (81) is

$$\frac{\partial^2 \mu}{\partial u \partial v} - \alpha \frac{\partial \mu}{\partial u} - \beta \frac{\partial \mu}{\partial v} + \left( \gamma - \frac{\partial \alpha}{\partial u} - \frac{\partial \beta}{\partial v} \right) \mu = 0. \quad (82)$$

If $\lambda_1$ and $\mu_1$ are solutions of these equations, the following integrals have a meaning:

$$\bar{s}_1 = \int \mu_1 \left( \frac{\partial \lambda_1}{\partial u} + \beta \lambda_1 \right) du + \lambda_1 \left( \frac{\partial \mu_1}{\partial v} - \alpha \mu_1 \right) dv, \quad (83)$$

$$\bar{r}_1 = \int \lambda_1 \left( \frac{\partial \mu_1}{\partial u} - \beta \mu_1 \right) du + \mu_1 \left( \frac{\partial \lambda_1}{\partial v} + \alpha \lambda_1 \right) dv.$$

The functions $X_1$, $Y_1$, $Z_1$, $W_1$ defined by equations of the form

$$\frac{\partial X_1}{\partial u} = \bar{s}_1 \frac{\partial}{\partial u} \left( \frac{X}{X_1} \right), \quad \frac{\partial X_1}{\partial v} = -\bar{s}_1 \frac{\partial}{\partial v} \left( \frac{X}{X_1} \right), \quad (84)$$

are the tangential coördinates of a second surface, upon which the parametric curves form a conjugate system.

* Cf. M_1, p. 409.
In addition to (80) we have the equations of condition
\[ X \frac{\partial x}{\partial u} + Y \frac{\partial y}{\partial u} + Z \frac{\partial z}{\partial u} + W \frac{\partial w}{\partial u} = 0, \]
(85)
\[ X \frac{\partial x}{\partial v} + Y \frac{\partial y}{\partial v} + Z \frac{\partial z}{\partial v} + W \frac{\partial w}{\partial v} = 0. \]

Consider the function
\[ \theta_1 = X' x + Y' y + Z' z + W' w, \]
where \( X', Y', Z', W' \) are given by integrals of the form
(86)
\[ X' = \int \mu_1 \left( \frac{\partial X}{\partial u} + \beta X \right) du + X \left( \frac{\partial \mu_1}{\partial v} - \alpha \mu_1 \right) dv. \]

In consequence of (81) and (85) we have
\[ \frac{\partial \theta_1}{\partial u} = \sum \frac{\partial x}{\partial u} X'_1 + \frac{\partial w}{\partial u} W'_1, \quad \frac{\partial \theta_1}{\partial v} = \sum \frac{\partial x}{\partial v} X'_1 + \frac{\partial w}{\partial v} W'_1, \]
(88)
\[ \frac{\partial^2 \theta_1}{\partial u \partial v} = \sum \frac{\partial^2 x}{\partial u \partial v} X'_1 + \frac{\partial^2 w}{\partial u \partial v} W'_1, \]
where as usual the symbol \( \sum \) signifies the sum for three terms in \( x, y, z \).
Hence \( \theta_1 \) is a solution of equation (1).*

In like manner it can be shown that \( \lambda_1 \) given by
(89)
\[ \lambda_1 = X x'_1 + Y y'_1 + Z z'_1 + W w'_1, \]
where \( x'_1, y'_1, z'_1, w'_1 \) are given by equations of the form (3), is a solution of (81).
It is our purpose to show that equations (9) and (84) define the same transformation of \( S \), when \( \theta_1 \) and \( \lambda_1 \) are given by (86) and (89).

The analogue of equation (5) is
(90)
\[ X_1 = X'_1 - \frac{\sigma_1}{\lambda_1} X. \]

From (9) it follows that the points \( T_1 \) and \( T_2 \), whose coordinates \( x_1, \eta_1, \zeta_1, \omega_1; \zeta_2, \eta_2, \zeta_2, \omega_2 \) are of the form
(91)
\[ \xi_1 = \frac{\partial x}{\partial \theta_1}, \quad \xi_2 = \frac{\partial x}{\partial \theta_1}, \]
are the intersections of corresponding tangents to the parametric curves on \( S \) and \( S_1 \). Since
(92)
\[ \frac{\partial \xi_2}{\partial v} = \frac{\partial \xi_1}{\partial u} = - \left( \frac{\partial \log \theta_1}{\partial v} + a \right) \xi_1 - \left( \frac{\partial \log \theta_1}{\partial u} + b \right) \xi_2, \]

* Cf. Darboux, Leçons, vol. 2, p. 188.
the points $T_1$ and $T_2$ are the focal points of the congruence of lines $T_1T_2$. These lines are the intersections of the tangent planes to $S$ and $S_1$.

We shall show that $X'_1, Y'_1, Z'_1, W'_1$ are the tangential coordinates of the locus of $T_1$. In fact, it follows from (80), (85), (86), (87) and (88) that

$$\sum \xi\, X'_1 + \omega_1 W'_1 = 0,$$

(93)

$$\sum \xi\, \frac{\partial X'_1}{\partial \nu} + \omega_1 \frac{\partial W'_1}{\partial \nu} = 0.$$

Moreover, the condition

$$\sum \xi\, \frac{\partial X'_1}{\partial u} + \omega_1 \frac{\partial W'_1}{\partial u} = 0$$

follows from

$$\sum \frac{\partial x}{\partial \nu} \frac{\partial X}{\partial u} + \frac{\partial w}{\partial \nu} \frac{\partial W}{\partial u} = 0,$$

which is a consequence of (85) and their derivatives.

In like manner it can be shown that the tangential coordinates $X''_1, Y''_1, Z''_1, W''_1$ of the locus of $T_2$ are of the form

$$X''_1 = \int X \left( \frac{\partial \mu_1}{\partial u} - \beta \mu_1 \right) du + \mu_1 \left( \frac{\partial X}{\partial \nu} + \alpha X \right) dv.$$

(94)

From (5), (86), (89) and (90) we have

$$\theta_1 = \sum X_1 x + W_1 w,$$

(95)

$$\lambda_1 = \sum x_1 X + w_1 W.$$

When the values of $\lambda_1$ and $\theta_1$ from (89) and (86) are substituted in (4) and (83), the resulting equations are reducible to

$$\sigma_1 = \int \left( \sum X'_1 \frac{\partial x'_1}{\partial u} + W'_1 \frac{\partial w'_1}{\partial u} \right) du + \left( \sum X'_1 \frac{\partial x'_1}{\partial \nu} + W'_1 \frac{\partial w'_1}{\partial \nu} \right) dv,$$

$$\overline{\sigma}_1 = \int \left( \sum x'_1 \frac{\partial X'_1}{\partial u} + w'_1 \frac{\partial W'_1}{\partial u} \right) du + \left( \sum x'_1 \frac{\partial X'_1}{\partial \nu} + w'_1 \frac{\partial W'_1}{\partial \nu} \right) dv.$$

Hence by a suitable choice of the additive constants of integration we have

$$\sigma_1 + \overline{\sigma}_1 = \sum X'_1 x'_1 + W'_1 w'.$$

(96)

In consequence of these results we have from (5) and (90)

$$\sum X_1 x_1 + W_1 w_1 = 0.$$

Since also

$$\sum \xi_2 X'_1 + \omega_2 W'_1 = 0,$$
it follows from (9), (90), (91) and (93) that
\[ \sum X_1 \frac{\partial x_1}{\partial u} + W_1 \frac{\partial w_1}{\partial u} = 0, \quad \sum X_1 \frac{\partial x_1}{\partial v} + W_1 \frac{\partial w_1}{\partial v} = 0. \]
Hence equations (9) and (84) define the same transformation $T$ of $S$.

By making use of the results of § 5, we can obtain the equations of the theorem of permutability of transformations $T$ from the standpoint of tangential coordinates. The functions $\lambda_{12}$ and $\lambda_{21}$ must satisfy
\[ \frac{\partial \lambda_{ij}}{\partial u} = -\frac{\partial}{\partial u} \left( \frac{\lambda_j}{\lambda_i} \right), \quad \frac{\partial \lambda_{ij}}{\partial v} = -\frac{\partial}{\partial v} \left( \frac{\lambda_j}{\lambda_i} \right) \quad (i = 1, 2, \quad i \neq j). \]
The functions $\tilde{\sigma}_{12}, \tilde{\sigma}_{21}, \tilde{\tau}_{12}, \tilde{\tau}_{21}$ are given by
\[ \tilde{\tau}_{12} = \tilde{\tau}_{21}, \quad \tilde{\tau}_{12} = \frac{\lambda_2 \lambda_{21} \tau_1}{\lambda_1} + \frac{\lambda_1 \lambda_{12} \tau_2}{\lambda_2} - \lambda_{12} \lambda_{21}, \]
\[ \tilde{\sigma}_{12} = \tilde{\sigma}_{21} = -\left( \frac{\lambda_2 \lambda_{21} \sigma_1}{\lambda_1} + \frac{\lambda_1 \lambda_{12} \sigma_2}{\lambda_2} + \lambda_{12} \lambda_{21} \right), \]
and the tangential coordinates of $S_{12}$, namely $X_{12}, Y_{12}, Z_{12}, W_{12}$, are of the form
\[ \lambda_1 \lambda_{12} X_{12} = \lambda_2 \lambda_{21} X_1 + \lambda_1 \lambda_{12} X_2 - \lambda_{12} \lambda_{21} X. \]

If equations similar to (95) are to be satisfied, we must have
\[ \lambda_{12} = \sum X_1 x_{12} + W_1 w_{12} = \sum X_1 x_2 + W_1 w_2 - \theta_{21}, \]
\[ \lambda_{21} = \sum X_2 x_{12} + W_2 w_{12} = \sum X_2 x_1 + W_2 w_1 - \theta_{12}. \]

When these equations are differentiated, we find that the resulting equations are satisfied in virtue of the preceding formulas. Hence we may take $\lambda_{12}$ and $\lambda_{21}$ as given by (100).

Equations similar to (5) and (90) are
\[ x_1' = x_{12} + \frac{\sigma_{12}}{\lambda_{12}} x_1 = x_2 - \frac{\theta_{21}}{\theta_1} x - \frac{x_1}{\sigma_1} \left( \frac{\theta_1}{\theta_2} + \theta_{21} \right), \]
\[ X_1' = X_{12} + \frac{\sigma_{12}}{\lambda_{12}} X_1 = X_2 - \frac{\lambda_{21}}{\lambda_1} X - \frac{X_1}{\sigma_1} \left( \frac{\lambda_1}{\lambda_2} \frac{\sigma_2}{\lambda_{21}} + \lambda_{21} \lambda_1 \right). \]
From these equations, (98) and (100) we obtain
\[ \sum x_1' \lambda_{12} + W_1' \lambda_{12} = \sigma_{12} + \tilde{\sigma}_{12}. \]
Consequently when $\lambda_{12}$ and $\lambda_{21}$ have the values (100), the expressions (99) are the tangential coordinates of $S_{12}$ whose point coordinates are given by (45).

From the form of (99) we are led at once to

**Theorem 8.** When $S, S_1, S_2, S_{12}$ form a quatern for transformations $T$, four corresponding tangent planes meet in a point.
During the remainder of this section we assume that the point coördinates are cartesian and that \( X', Y', Z' \) are the direction-cosines of the normal to \( S \). Consequently \( -W \) is the distance from the origin to the tangent plane.

From (58) it follows that \( x'_1 \) as it appears in (89) is equal to \( x'o' + x'^{(0)} \) and \( w'_1 = \sigma'_1 \). Hence (89) may be replaced by

\[
\lambda_1 = Xx'^{(0)} + Yy'^{(0)} + Zz'^{(0)}.
\]

Consequently \( \lambda_1 \) is the distance from the origin to the tangent plane to \( S'^{(0)} \).

We note that \( \lambda_1 \) and \( \mu_1 \) determine a transformation of \( S'^{(1)} \). If \( X_1, Y_1, Z_1 \) are the direction-cosines of the normal to the transform \( S_{1}' \), and \( -W_{1}' \) the distance from the origin to the tangent plane to \( S_{1}' \), equations (84) are replaced by

\[
\frac{\partial}{\partial u} \left( \frac{X_1}{W_1'} \right) = -\frac{\tau_1}{\lambda_1} \frac{\partial}{\partial u} \left( \frac{X}{\lambda_1} \right), \quad \frac{\partial}{\partial v} \left( \frac{X_1}{W_1'} \right) = \frac{\sigma_1}{\lambda_1} \frac{\partial}{\partial v} \left( \frac{X}{\lambda_1} \right).
\]

But \( X_1, Y_1, Z_1 \) are the direction-cosines of the normal to \( S_{1}' \) also. Moreover, the function \( W_1 \) is given by

\[
\frac{\partial}{\partial u} \left( \frac{W_1}{W_1'} \right) = -\frac{\tau_1}{\lambda_1} \frac{\partial}{\partial u} \left( \frac{W}{\lambda_1} \right), \quad \frac{\partial}{\partial v} \left( \frac{W_1}{W_1'} \right) = \frac{\sigma_1}{\lambda_1} \frac{\partial}{\partial v} \left( \frac{W}{\lambda_1} \right).
\]

11. Transformations \( S \) of conjugate systems with equal tangential invariants

When equation (81) has equal invariants, it can be written

\[
\frac{\partial^2 \lambda_1}{\partial u \partial v} + \frac{\partial \log \sqrt{\rho}}{\partial u} \frac{\partial \lambda_1}{\partial u} + \frac{\partial \log \sqrt{\rho}}{\partial v} \frac{\partial \lambda_1}{\partial v} + \gamma \lambda_1 = 0.
\]

Analogously to the results of § 9 we have that \( \mu_1 = 2\rho \lambda_1 \) is a solution of the adjoint of this equation. For this value we have

\[
\sigma_1 = \tau_1 = \lambda_1^2 \rho,
\]

so that the tangential equation of the transform is

\[
\frac{\partial^2 \lambda_1}{\partial u \partial v} - \frac{\partial}{\partial u} \log \sqrt{\rho} \lambda_1 \frac{\partial \lambda_1}{\partial u} - \frac{\partial}{\partial v} \log \sqrt{\rho} \lambda_1 \frac{\partial \lambda_1}{\partial v} + \gamma_1 \lambda_1 = 0,
\]

which also has equal invariants.

If we put

\[
\varphi_1 = \lambda_1 \sqrt{\rho}, \quad \varphi'_1 = \lambda_1' / \sqrt{\rho} \lambda_1,
\]

these equations are equivalent to

\[
\frac{\partial^2 \varphi_1}{\partial u \partial v} = \left( \frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial u \partial v} - \gamma \right) \varphi_1, \quad \frac{\partial^2 \varphi'_1}{\partial u \partial v} = \left( \sqrt{\rho} \lambda_1 \frac{\partial^2}{\partial u \partial v} \frac{1}{\sqrt{\rho} \lambda_1} - \gamma_1 \right) \varphi'_1.
\]
These equations are satisfied respectively by the functions
\[ \nu_1 = \sqrt{\rho} X, \quad \nu_2 = \sqrt{\rho} Y, \quad \nu_3 = \sqrt{\rho} Z; \]
\[ \tilde{\nu}_1 = X'/\sqrt{\rho}\lambda_1, \quad \tilde{\nu}_2 = Y'/\sqrt{\rho}\lambda_1, \quad \tilde{\nu}_3 = Z'/\sqrt{\rho}\lambda_1. \]
In terms of these functions equations (84) are reducible to
\[ \frac{\partial}{\partial u} (\tilde{\nu}_i \partial_1) = \left( \frac{\partial \nu_i}{\partial u} - \nu_i \frac{\partial \partial_1}{\partial u} \right), \]
\[ \frac{\partial}{\partial v} (\tilde{\nu}_i \partial_1) = - \left( \frac{\partial \nu_i}{\partial v} - \nu_i \frac{\partial \partial_1}{\partial v} \right) \quad (i = 1, 2, 3). \]

Since the tangential equation of $S$ has equal invariants, there exists a surface $\Sigma$ with this spherical representation of its asymptotic lines. Its point coordinates $\xi, \eta, \zeta$ are given by the Lelieuvre formulas
\[ \frac{\partial \xi}{\partial u} = \nu_2 \frac{\partial \nu_3}{\partial u} - \nu_3 \frac{\partial \nu_2}{\partial u}, \quad \frac{\partial \eta}{\partial v} = - \nu_2 \frac{\partial \nu_3}{\partial v} + \nu_3 \frac{\partial \nu_2}{\partial v}. \]

Similar equations in the functions $\nu_i$ give the point coördinates of a surface $\Sigma_1$ with the same spherical representation of its asymptotic lines as the parametric system on $S_1$. Moreover, equations (84') are the condition that $\Sigma$ and $\Sigma_1$ be the focal surfaces of a $W$-congruence.†

From the theory of $W$-congruences it follows that, if $X, Y, Z$ and $X_1, Y_1, Z_1$ are the direction-cosines (not merely direction-parameters), of the normals to $\Sigma$ and $\Sigma_1$ respectively, then
\[ \nu_1 = \sqrt{\rho} X, \quad \tilde{\nu}_1 = \sqrt{\rho_1} X_1, \]
where $-1/\rho^2$ and $-1/\rho_1^2$ are the gaussian curvatures of $\Sigma$ and $\Sigma_1$ respectively. Hence (84') may be written
\[ \frac{\partial}{\partial u} (\sqrt{\rho_1} \lambda_1 X_1) = \rho \lambda_1^2 \frac{\partial}{\partial u} \left( \frac{X}{\lambda_1} \right), \quad \frac{\partial}{\partial v} (\sqrt{\rho_1} \lambda_1 X_1) = - \rho \lambda_1^2 \frac{\partial}{\partial v} \left( \frac{X}{\lambda_1} \right). \]

Comparing these equations with (102), we have
\[ W_1^{(1)} = \frac{1}{\sqrt{\rho_1} \lambda_1}, \]
and equations (103) become
\[ \frac{\partial}{\partial u} (\sqrt{\rho_1} \lambda_1 W_1) = - \rho \lambda_1^2 \frac{\partial}{\partial u} \left( \frac{W}{\lambda_1} \right), \quad \frac{\partial}{\partial v} (\sqrt{\rho_1} \lambda_1 W_1) = \rho \lambda_1^2 \frac{\partial}{\partial v} \left( \frac{W}{\lambda_1} \right). \]

But these are the equations of transformations $\Omega$ of conjugate systems with equal tangential invariants previously found by the author.‡

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* E., p. 193. A reference of this sort is to the author's *Differential Geometry*.
† E., p. 419.
‡ M', §§ 1, 3.
special surfaces $S^{(i)}$ and $S_1^{(i)}$, treated in § 7, were used in the discussion of the transformations $\Omega$.

12. THE EXTENDED THEOREM OF PERMUTABILITY

In this section we extend the theorem of permutability so as to involve a group of eight surfaces. Let $S_1$, $S_2$, $S_3$ be three transforms of $S$ by means of functions $\theta_i$, $\phi_i$ for $i = 1$, $2$, $3$ respectively. Applying the theorem of permutability to the three pairs of these surfaces, we get three new surfaces $S_{12}$, $S_{23}$, $S_{31}$. We recall that $S_{ij} = S_{ji}$. Since $S_{12}$ and $S_{13}$ are transforms of $S_1$, there exists a family of surfaces $S'$, for each of which $S_1$, $S_{12}$, $S_{13}$, $S'$ form a quatern. It is our purpose to show that one of these surfaces $S'$ is such that $S_2$, $S_{12}$, $S_{23}$, $S'$ form a quatern; and likewise $S_3$, $S_{13}$, $S_{23}$, $S'$.

We denote by $\theta'_{i2}$, $\phi'_{i2}$ the functions transforming $S_{12}$ into $S'$. The equations analogous to the first of (35) and (36) are

$$
(104) \quad \theta'_{12} \theta'_{12} w'_{12} = w_{13} (\theta_{13} \theta'_{13} + \theta_{12} \theta'_{12} - \theta'_{12} \theta'_{13}),
$$

$$
\theta_{12} \theta'_{12} w'_{12} x' = w_{13} (\theta_{13} \theta'_{13} x_{12} + \theta_{12} \theta'_{12} x_{13} - \theta'_{13} \theta'_{12} x_{1}).
$$

In consequence of (36) and an analogous expression for $x_{13}$ the second of (104) is reducible to

$$
\theta_{12} \theta'_{12} w'_{12} x' = x_1 \left( w_{12} \theta'_{12} w_{13} \theta'_{13} - \frac{\theta_2 \theta_{21}}{\theta_1 \theta_{12}} w_{13} \theta'_{13} w_2 \theta_{13} - \frac{\theta_3 \theta_{21}}{\theta_1 \theta_{13}} w_{12} \theta'_{12} w_3 \theta_{12} \right)
$$

$$
- x_2 w_{13} \theta'_{13} w_2 \theta_{13} - x_3 w_{12} \theta'_{12} w_3 \theta_{12} + \frac{x}{\theta_1} (w_2 \theta_{21} w_{13} \theta'_{13} \theta_{13} + w_3 \theta_{31} w_{12} \theta'_{12} \theta_{12}).
$$

In deriving these equations, we looked upon $S'$ as a transform of $S_{12}$ which in turn is a transform of $S_1$. Looking upon $S_{12}$ as a transform of $S_2$ we get the analogous equations

$$
\theta_{21} \theta'_{21} w'_{21} = w_{23} (\theta_{21} \theta'_{21} + \theta_{23} \theta'_{23} - \theta'_{21} \theta'_{23}),
$$

$$
\theta_{21} \theta'_{21} w'_{21} x' = w_{23} (\theta_{21} \theta'_{21} x_{12} + \theta_{23} \theta'_{23} x_{23} - \theta'_{21} \theta'_{23} x_2).
$$

The latter equation is reducible to

$$
\theta_{21} \theta'_{21} w'_{21} x' = - x_1 w_{23} \theta'_{23} w_1 \theta_{23} - x_3 w_{21} \theta_{21} w_3 \theta_{21}
$$

$$
+ x_2 \left( w_{21} \theta'_{21} w_{23} \theta'_{23} - w_{21} \theta'_{21} w_3 \theta_{21} \frac{\theta_3 \theta_{32}}{\theta_2 \theta_{23}} \theta_3 \theta_{22} \theta_{21} \theta_{12}
$$

$$
- w_{23} \theta'_{23} w_1 \theta_{22} \frac{\theta_1 \theta_{12}}{\theta_2 \theta_{21}} \right)
$$

$$
+ \frac{x}{\theta_2} (w_3 \theta_{32} w_{21} \theta'_{21} \theta_{21} + w_1 \theta_{12} w_{23} \theta'_{23} \theta_{23}).
$$
From their definition it follows that $\theta_{21}$ and $w_{21}$ are the same functions as $\theta_{12}$ and $w_{12}$ respectively. Making use of this fact, we eliminate $x'$ from (105) and (106). In the reduction we note that from (35) we have

$$w_1 w_{12} \theta_{12} = w_2 w_{21} \theta_2 \theta_{21}.$$  

The resulting equation is of the form

$$Ax_1 + Bx_2 + Cx = 0,$$

where $A$, $B$, and $C$ are determinate functions. These functions must equal zero, since equations similar to the above hold also in the $y$’s and $z$’s. This gives the three equations

$$w_1 \theta_{12} w_{13} \theta_{13} - \theta_2 w_{12} \theta_{13} w_{2} \theta_{21} \theta_{12} - \theta_3 w_{12} \theta_{13} w_{3} \theta_{31} \theta_{13} + \theta_2 w_{22} \theta_{23} \theta_{23} = 0,$$

$$w_2 \theta_{21} w_{23} \theta_{23} - \theta_3 w_{21} \theta_{23} w_{3, \theta_{32}} \theta_{21} - \theta_1 w_{23} \theta_{23} w_{1} \theta_{12} \theta_{21} + \theta_1 w_{13} \theta_{13} \theta_{13} = 0,$$

$$w_{13} \theta_{13} w_{1} \theta_{13} w_{2} \theta_{21} + w_{12} \theta_{12} w_{1} \theta_{12} w_{3} \theta_{31} - w_{21} \theta_{21} w_{2} \theta_{21} w_{3} \theta_{32} - w_{23} \theta_{23} w_{1} \theta_{12} w_{2} \theta_{23} = 0.$$

It is readily found that these equations are equivalent to the three

$$\theta_i w_{ij} \theta_{ij} = \theta_j w_j \theta_{ji} + \theta_i w_j \theta_{jk} - \theta_k w_j \theta_{ji} \quad \left( i + j + k = 3 \right).$$

When these values are substituted in (105), the result is reducible to

$$\Phi x' = x_1 (\theta_2 \theta_{31} + \theta_3 \theta_{21} - \theta_1 \theta_{23} \theta_{32}) + x_2 (\theta_1 \theta_{13} + \theta_3 \theta_{12} \theta_{31} - \theta_2 \theta_{12} \theta_{31}) + x_3 (\theta_1 \theta_{23} \theta_{21} + \theta_2 \theta_{12} \theta_{13} - \theta_3 \theta_{21} \theta_{12}) + x (\theta_1 \theta_{23} \theta_{31} + \theta_{23} \theta_{21} \theta_{32}),$$

where

$$\Phi = \frac{\theta_1 \theta_{12} w_{12} \theta_{12} w_{12}}{w_2 w_3}$$

$$= \theta_1 (\theta_{13} \theta_{32} + \theta_{12} \theta_{23} - \theta_{32} \theta_{23}) + \theta_2 (\theta_{31} \theta_{23} + \theta_{21} \theta_{13} - \theta_{13} \theta_{31}) + \theta_3 (\theta_{12} \theta_{31} + \theta_{23} \theta_{21} - \theta_{12} \theta_{21}) + \theta_{13} \theta_{32} \theta_{21} + \theta_{12} \theta_{23} \theta_{31}.$$
It remains for us to show that the functions $\theta'_{ij}$ as given by (108) satisfy equations analogous to (26), namely

$$
(111) \quad \frac{\partial}{\partial u} (w_{ij} \theta'_{ij}) = \tau_{ij} \frac{\partial}{\partial u} \left( \frac{\theta_{ik}}{\theta_{ij}} \right), \quad \frac{\partial}{\partial v} (w_{ij} \theta'_{ij}) = -\sigma_{ij} \frac{\partial}{\partial v} \left( \frac{\theta_{ik}}{\theta_{ij}} \right).
$$

We know that this is true, since (108) for $i = 1$, $j = 2$, $k = 3$ follows from (36) when $x_{12}, x_{1}, x_{2}, x$ are replaced by $\theta'_{12}, \theta'_{13}, \theta'_{23}, \theta'_{3}$ respectively; and these results are general. Hence we have the extended theorem of permutability:

**Theorem 9.** If $S$, $S_1$, $S_2$, $S_{12}$; $S$, $S_2$, $S_3$, $S_{23}$; $S$, $S_3$, $S_1$, $S_{13}$ are three quaterns of surfaces, a surface $S'$ can be found, without quadrature, such that $S_1$, $S_{12}$, $S_{13}$, $S'$; $S_2$, $S_{12}$, $S_{23}$, $S'$; $S_3$, $S_{13}$, $S_{23}$, $S'$ are quaterns.

13. **Relations between transformations $T$ and radial transformations**

If $\omega$ is a solution of equation (18), the surface $\bar{S}$ whose coordinates $\bar{x}$, $\bar{y}$, $\bar{z}$, are given by

$$
(112) \quad \bar{x} = \frac{x}{\omega}, \quad \bar{y} = \frac{y}{\omega}, \quad \bar{z} = \frac{z}{\omega},
$$

is referred to a conjugate system. In fact, the point equation of $\bar{S}$ is

$$
(113) \quad \frac{\partial^2 \bar{\theta}}{\partial u \partial v} + \left( a + \frac{\partial \log \omega}{\partial v} \right) \frac{\partial \bar{\theta}}{\partial u} + \left( b + \frac{\partial \log \omega}{\partial u} \right) \frac{\partial \bar{\theta}}{\partial v} = 0.
$$

We say that $\bar{S}$ is obtained from $S$ by a radial transformation, since the line joining any pair of corresponding points on $S$ and $\bar{S}$ passes through a point—origin.

If $\theta_1$ is a solution of (18), then $\bar{\theta}_1 = \theta_1/\omega$ is a solution of (113). Also it can be shown that if $\phi_1$ is a solution of the adjoint equation of (18), then $\bar{\phi}_1 = \phi_1 \omega$ is a solution of the adjoint of (113).

We consider the transformation $T$ of $S$ by means of these functions $\bar{\theta}_1$ and $\bar{\phi}_1$. If $\tau_1$ and $\sigma_1$ denote functions analogous to $\tau_1$ and $\sigma_1$, it is readily found that to within additive constants we have

$$
\tau_{1} = \tau_{1}, \quad \bar{\sigma}_{1} = \sigma_{1}.
$$

Assuming these values, we have from equations (20) and the analogous ones,

$$
\frac{\partial}{\partial u} (\bar{x}_1 \bar{w}_1) = \tau_{1} \frac{\partial}{\partial u} \left( \frac{\bar{x}}{\theta_1} \right), \quad \frac{\partial}{\partial v} (\bar{x}_1 \bar{w}_1) = -\bar{\sigma}_1 \frac{\partial}{\partial v} \left( \frac{\bar{x}}{\theta_1} \right),
$$

by integration

$$
(114) \quad \bar{x}_1 \bar{w}_1 = x_1 w_1
$$

to within an additive constant. Also $\bar{w}_1$ is given by

$$
(115) \quad \frac{\partial \bar{w}_1}{\partial u} = \tau_{1} \frac{\partial}{\partial u} \left( \frac{1}{\theta_1} \right) = \tau_{1} \frac{\partial}{\partial u} \left( \frac{\omega}{\theta_1} \right), \quad \frac{\partial \bar{w}_1}{\partial v} = -\bar{\sigma}_1 \frac{\partial}{\partial v} \left( \frac{\omega}{\theta_1} \right).
$$
Evidently there exists a function $\omega_1$ defined by

$$
\frac{\partial}{\partial u} (\omega_1 w_1) = \tau_1 \frac{\partial}{\partial u} \left( \frac{\omega_1}{\theta_1} \right), \quad \frac{\partial}{\partial v} (\omega_1 w_1) = -\sigma_1 \frac{\partial}{\partial v} \left( \frac{\omega_1}{\theta_1} \right),
$$

which equations are similar to (20). Comparing (115) and (116), we note that $\omega_1$ can be chosen so that

$$
\omega_1 w_1 = \tilde{w}_1,
$$

and consequently we have from (114)

$$
\tilde{x}_1 = \frac{x_1}{\omega_1}, \quad \tilde{y}_1 = \frac{y_1}{\omega_1}, \quad \tilde{z}_1 = \frac{z_1}{\omega_1}.
$$

Hence we have

**Theorem 10.** If two surfaces $S$ and $\bar{S}$ are in the relation of a radial transformation and $S_1$ is a $T$ transform of $S$, a surface $\bar{S}_1$ can be found by a quadrature which is a radial transform of $S_1$ and a $T$ transform of $\bar{S}$.

When in particular $\omega = \theta_1$, then $\theta_1 = 1$, and consequently $S$ and $\bar{S}$ are parallel. Now in all generality we take

$$
\omega_1 = -1, \quad \omega_1 = -1/\omega_1.
$$

Therefore, we have

**Theorem 11.** A transformation $T$ is equivalent to the combination of an axial, a parallel and an axial transformations.

Consider now a general quatern of surfaces $S, S_1, S_2, S_{12}$. From (116), (26) and analogous equations it follows that the functions

$$
\omega = \theta_2, \quad \omega_1 = \theta_{12}, \quad \omega_2 = \frac{1}{w_2},
$$

determine axial transformations of $S, S_1$ and $S_2$ respectively, into $\bar{S}, \bar{S}_1, \bar{S}_2$.

The equations determining the axial transform of $S_{12}$ as of the pair $S_1$ and $S_{12}$, are

$$
\frac{\partial}{\partial u} (\omega_{12} w_{12}) = \tau_{12} \frac{\partial}{\partial u} \left( \frac{\omega_1}{\theta_{12}} \right), \quad \frac{\partial}{\partial v} (\omega_{12} w_{12}) = -\sigma_{12} \frac{\partial}{\partial v} \left( \frac{\omega_1}{\theta_{12}} \right).
$$

In consequence of the preceding equations we may take $\omega_{12} = 1/w_{12}$.

In order to show that the same function $\omega_{12}$ determines the axial transform of $S_{12}$, as of the pair $S_2$ and $S_{12}$, the following equations must be satisfied:

$$
\frac{\partial}{\partial u} \left( \frac{w_{21}}{w_{12}} \right) = \tau_{21} \frac{\partial}{\partial u} \left( \frac{1}{w_2 \theta_{21}} \right), \quad \frac{\partial}{\partial v} \left( \frac{w_{21}}{w_{12}} \right) = -\sigma_{21} \frac{\partial}{\partial v} \left( \frac{1}{w_2 \theta_{21}} \right).
$$

When the values of $w_{12}, w_{21}, \sigma_{21}$ and $\tau_{21}$, as given by (35) and analogous equations are substituted, it is found that (119) are satisfied. Hence the surfaces
$S$, $\bar{S}_1$, $\bar{S}_2$, $\bar{S}_{12}$ form a quaternion. Their coördinates are of the form

$$
\bar{x} = \frac{x}{\theta_2}, \quad \bar{x}_1 = \frac{x_1}{\theta_{12}}, \quad \bar{x}_2 = x_2 w_2, \quad \bar{x}_{12} = x_{12} w_{12},
$$
the transformation functions being

$$
\bar{\theta}_1 = \frac{\theta_1}{\theta_2}, \quad \bar{\phi}_1 = \theta_2 \phi_1, \quad \bar{\omega}_1 = w_1 \theta_{12};
$$
$$
\bar{\theta}_2 = 1, \quad \bar{\phi}_2 = \theta_2 \phi_2, \quad \bar{\omega}_2 = 1;
$$
$$
\bar{\theta}_{12} = 1, \quad \bar{\phi}_{12} = \theta_{12} \phi_{12}, \quad \bar{\omega}_{12} = 1;
$$
$$
\bar{\theta}_{21} = w_2 \theta_{21}, \quad \bar{\phi}_{21} = \frac{\phi_{21}}{w_2}, \quad \bar{\omega}_{21} = \frac{w_{21}}{w_{12}}.
$$

These functions satisfy equations analogous to (35).

In a similar manner we get a second quaternion by using $\theta_1$ for the axial transformation of $S$. Hence we have

**Theorem 12.** When a quaternion of surfaces is known, two other quaterns each containing two pairs of parallel surfaces can be found without quadrature, and these surfaces are axial transforms of the surfaces of the given quaternion.

As a matter of fact axial transformations can be looked upon as special types of transformations $T$. For if we take

$$
\tau_2 = -\sigma_2 = 1, \quad \theta_2 = \omega - 1, \quad w_2 = \frac{1}{\theta_2} + 1 = \frac{\omega}{\omega - 1},
$$
equations of the form (19) and (20) with subscripts 2 instead of 1 are integrable in the form

$$
x_2 = \frac{x}{\omega}, \quad y_2 = \frac{y}{\omega}, \quad z_2 = \frac{z}{\omega}.
$$

Let us apply the theorem of permutability to the case in which $S_2$ is given as above. One solution of (37) is

$$
\theta_{21} = \frac{\theta_1}{\omega_2 \theta_2} = \frac{\theta_1}{\omega}.
$$

We find also

$$
\tau_{12} = -\sigma_{12} = 1, \quad \sigma_{21} = \sigma_1, \quad \tau_{21} = \tau_1,
$$
$$
w_{12} \theta_{12} = 1 + \theta_{12}, \quad w_{21} = w_1 (1 + \theta_{12}).
$$

Hence if we put $\omega_1 = \theta_{12} + 1$, equation (36) in this case reduces to (118). Consequently the theorem of permutability is equally true when an axial transformation is used. It is readily shown also that theorem 12 can be established by means of the generalized results of § 11.