THE CONVERSE OF THE THEOREM CONCERNING THE DIVISION
OF A PLANE BY AN OPEN CURVE*

BY

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In his paper, "On the foundations of plane analysis situs,"† R. L. Moore proved that if \( l \) is an open curve‡ and \( S \) is the set of all points, then \( S - l = S_1 + S_2 \), where \( S_1 \) and \( S_2 \) are connected point sets such that every arc from a point of \( S_1 \) to a point of \( S_2 \) contains at least one point of \( l \).§ Clearly the sets \( S_1 \) and \( S_2 \) are non-compact.|| Professor Moore's theorem is proved on the basis of his set of axioms \( \Sigma_3 \). Thus the theorem is true in certain spaces which are neither metrical, descriptive, nor separable.

This theorem for open curves is analogous to the theorem of Jordan,¶ that a simple closed curve lying wholly within a plane decomposes the plane into an inside and an outside region. The converse of this theorem for simple closed curves was first formulated by Schoenflies,** who makes use of metrical properties in his proof. A different proof has been given by Lennes,†† who uses straight lines. R. L. Moore has pointed out that, on the basis of \( \Sigma_3 \), an argument similar in large part to that of Lennes can be carried through with the use of arcs and closed curves.

The object of the present paper is to show that the converse of the open curve theorem holds in spaces satisfying \( \Sigma_3 \). The statement of the converse theorem is as follows:

† These Transactions, vol. 17 (1916), pp. 132-64.
‡ An open curve is defined by Moore as a closed connected set of points \( M \) such that if \( P \) is a point of \( M \), then \( M - P \) is the sum of two mutually exclusive connected sets of points, neither of which contains a limit point of the other. See R. L. Moore, loc. cit., p. 159.
|| Fréchet calls a set of points \( M \) compact if every infinite subset of \( M \) has at least one limit point. Cf. M. Fréchet, Sur quelques points du calcul fonctionnel, Rendiconti del Circolo Matematico di Palermo, vol. 22 (1906), § 9. A set of points which does not possess this property is said to be non-compact.
Suppose $K$ is a closed set of points and that $S - K = S_1 + S_2$, where $S_1$ and $S_2$ are non-compact point sets such that (1) every two points of $S_i$ ($i = 1, 2$) can be joined by an arc lying entirely in $S_i$; (2) every arc joining a point of $S_1$ to a point of $S_2$ contains a point of $K$; (3) if $O$ is a point of $K$ and $P$ is a point not belonging to $K$, then $P$ can be joined to $O$ by an arc having no point except $O$ in common with $K$. Every point set $K$ that satisfies these conditions is an open curve.

In order to prove our theorem, we shall first establish the truth of several lemmas.

**Lemma A.** Suppose $J$ is a simple closed curve* such that (1) $A$ and $B$, two distinct points of $K$, both lie on $J$ and (2) $J - A - B = AF_1 B + AF_2 B$ where $AF_1 B$ is a subset of $S_1$ while $AF_2 B$ is a subset of $S_2$. The interior $I$ of $J$ must contain at least one point of $K$.

**Proof.** Suppose Lemma A is false. Then $I$ contains only points of $S_1 + S_2$. Let $P$ be any point of $I$. Join $P$ to $F_1$ by an arc lying except for $F_1$ entirely in $I$. Join $P$ to $F_2$ in the same manner. The point set, $PF_1 + PF_2$, contains as a subset an arc $F_1 X F_2$, which contains no points of $K$. But this is contrary to our assumption concerning arcs from a point in $S_1$ to a point in $S_2$.

**Lemma B.** Under the same hypothesis as in Lemma A, if $[X]$ denotes the set of all points of $K$, which are in $I$, then $[X] + A + B$ is a simple continuous arc from $A$ to $B$.

**Lemma B** can be proved by methods similar to those of Lennes.

**Lemma C.** No subset of $K$ is a simple closed curve.

**Proof.** Suppose some subset $J$ of $K$ is a simple closed curve. The point set $S_1$ cannot lie entirely in $I$, the interior of $J$. For if every point of $S_1$ is in $I$, then every infinite subset of $S_1$ must have at least one limit point. This is contrary to the supposition that $S_1$ is non-compact. In like manner, $S_2$ is not entirely in $I$.

No point of $K$ belongs to $I$. For suppose a point $F$ of $K$ belongs to $I$. Let $G$ be a point of $S_1$ not in $I$. Every arc from $F$ to $G$ must contain a

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* If $A$ and $B$ are distinct points, a simple continuous arc from $A$ to $B$ is defined by Lennes as a bounded, closed, connected set of points containing $A$ and $B$ but containing no connected proper subset that contains both $A$ and $B$. See N. J. Lennes, loc. cit., p. 308. In the present paper “arc” and “simple continuous arc” will be used synonymously. A simple closed curve is a set of points composed of two arcs $AXB$ and $AYB$ that have no common point except $A$ and $B$.

† If $AF_1 B$ is a simple continuous arc, $AF_1 B$ denotes the point set $AF_1 B - A - B$.

point of $K$, different from $F$, namely a point of $J$. But this is contrary to hypothesis. Hence $I$ is a subset of $S_1 + S_2$.

Suppose $I$ contains a point $P$ of $S_1$ and a point $Q$ of $S_2$. Then $P$ and $Q$ can be joined by an arc lying entirely in $I$ and therefore containing no point of $K$. But this is contrary to hypothesis.

Suppose $I$ contains a point $L$ of $S_1$. Let $G$ denote any point of $S_1$ not in $I$. Since $J$ consists entirely of points of $K$, $G$ is not on $J$. Every arc from $L$ to $G$ must contain a point of $K$, namely, a point of $J$. But this is contrary to hypothesis. Hence $I$ contains no points of $S_1$.

In like manner $I$ contains no point of $S_2$.

Thus the supposition that $J$, a subset of $K$, is a simple closed curve leads to a contradiction.

**Lemma D.** Suppose $J_1$ and $J_2$ are simple closed curves such that (1) $A$ and $B$, two distinct points of $K$, lie on both $J_1$ and $J_2$ and (2) $J_i - A - B = \overline{AF_iB}$ + $\overline{AF_iB}$ ($i = 1, 2$) where $\overline{AF_iB}$ is a subset of $S_i$ while $\overline{AF_iB}$ is a subset of $S_2$. Under these conditions the subset of $K$ within $J_1$ is the same as the subset of $K$ within $J_2$.

**Proof.** Let $K_1$ denote the set of all points of $K$ within $J_1$, while $K_2$ denotes the set of all points of $K$ within $J_2$. By Lemma B, $K_1 + A + B$ is a simple continuous arc from $A$ to $B$, as is also the point set, $K_2 + A + B$.

Suppose Lemma D is false. Four cases may arise:

Case I. $K_1$ is a proper subset of $K_2$. Then $K_2 + A + B$ contains a proper connected subset $K_1 + A + B$, which contains both $A$ and $B$. But this is contrary to the definition of a simple continuous arc from $A$ to $B$.

Case II. $K_2$ is a proper subset of $K_1$. Impossible as in Case I.

Case III. $K_2$ consists of two non-vacuous point sets, $G_1$ and $G_2$, where $G_1$ denotes those points of $K_2$ which are points of $K_1$, while $G_2$ denotes those points of $K_2$ which are not points of $K_1$.

By hypothesis, $K_2$ contains no point of $J_1$. Hence all points of $K_2$ must lie within $J_1$. For suppose a point $P$ of $K_2$ is without $J_1$. Then $K_2$ can be divided into two mutually exclusive sets, $M_1$ and $M_2$, where $M_1$ is the set of all points of $K_2$ within $J_1$ while $M_2$ is the set of all points of $K_2$ without $J_1$. Manifestly neither of these sets can contain a limit point of the other. Hence the supposition that $K_2$ contains points without $J_1$ leads to a contradiction.

As all points of $K_2$ lie within $J_1$ and are points of $K$, then, by definition of $K_1$, either (1) $K_2$ is the same as $K_1$, which is contrary to supposition or (2) $K_2$ is a proper subset of $K_1$, which is contrary to Case II.

**Case IV.** $K_1$ and $K_2$ have no common point. The point set, $K_1 + K_2 + A + B$, is a simple closed curve composed entirely of points of $K$. But this is contrary to Lemma C.

**Definition 1.** The points $A$, $B$ and $X$ [$A \neq B$] of $K$ are said to be in
the order $AXB$, if, and only if, $X$ is within some simple closed curve $J$, such that (1) $A$ and $B$ are on $J$ and (2) $J - A - B = AF_1B + AF_2B$, where $AF_1B$ is a subset of $S_1$, while $AF_2B$ is a subset of $S_2$.

That the set of all points $[X]$ such that $AXB$ is the same for every such closed curve $J$ follows at once from Lemma D.

Lemma E. If $A$ and $B$ $[A \neq B]$ are points of $K$, then there exists a point $X$ of $K$ such that $AXB$.

Lemma F. If $ABC$, then $CBA$.

Lemma G. If $ABC$, then $A \neq B$ and $B \neq C$.

Lemma H. If $ABC$, then not $BAC$.

Proof. Construct a simple closed curve $J$ such that (1) $A$ and $C$ are on $J$ and (2) $J - A - C = AF_1C + AF_2C$, where $AF_1C$ is a subset of $S_1$ while $AF_2C$ is a subset of $S_2$. By Definition 1 and Lemma D, $B$ is within $J$. Join $B$ to $F_1$ by an arc lying except for $B$ entirely in $S_1$. The arc $BF_1$ has no point in common with the arc $AF_2C$, which, except for $A$ and $C$, contains only points of $S_2$. Let $G_1$ denote the first point that the arc $BF_1$ has in common with the arc $AF_1C$. Join $B$ to $F_2$ by an arc lying except for $B$ entirely in $S_2$ and let $G_2$ denote the first point of the arc $BF_2$ which is on the arc $AF_2C$. The point set, $G_1B + G_2B$, is a simple continuous arc, lying except for its end points, $G_1$ and $G_2$, entirely within $J$ and containing only the point $B$ of $K$. The point set, $G_1AG_2$, is without the closed curve $BG_1CG_2B$.

Hence, by Lemma D and Definition 1, not $BAC$.

Lemma I. There is but one arc of $K$ from $A$ to $C$.

Proof. If there were two such arcs, their sum would contain as a subset at least one simple closed curve. But this is contrary to Lemma C.

Lemma J. A necessary and sufficient condition that three distinct points, $A, B,$ and $C$ of $K$ should be in the order $ABC$, is that $B$ should be on the $K$-arc from $A$ to $C$.

Lemma K. If $A, B,$ and $C$ $[A \neq B, B \neq C, C \neq A]$ are points of $K$, then either $ACB$, $CBA$, or $BAC$.

Proof. Suppose $ACB$ is false. Let $J$ be a simple closed curve such that (1) $A$ and $B$ are on $J$, and (2) $J - A - B = AF_1B + AF_2B$, where $AF_1B$ is a subset of $S_1$ while $AF_2B$ is a subset of $S_2$. By Definition 1 and Lemma D, the point $C$ is without $J$. Join $C$ to $F_1$ by an arc lying except for $C$ entirely in $S_1$. Join $C$ to $F_2$ by an arc lying except for $C$ entirely in $S_2$. Let $G_1$ denote the first point which the arc $CF_1$ has in common with the arc $AF_1B$. Let $G_2$ denote the first point which the arc $CF_2$ has in common with the arc $AF_2B$. The point set, $CG_1 + CG_2$, is a simple continuous arc from $G_1$ to $G_2$, lying except for its end points entirely without $J$ and contain-

*See R. L. Moore, loc. cit., Theorem 24, p. 141.
ing only the point $C$ of $K$. Hence either (1) $B$ is within $CG_1 AG_2 C$ or (2) $A$ is within $CG_1 BG_2 C$.* In Case I, $CBA$ while in Case II, $CAB$.

**Lemma L.** If $A$, $B$, and $C$ ($A \neq B$, $B \neq C$, $C \neq A$) are points of $K$, in the order $ABC$, then the $K$-arc $AC$ $=$ the $K$-arc $AB$ $+$ the $K$-arc $BC$.

**Proof.** Consider the figure described in the proof of Lemma $H$. The interior of $J = G_1 BG_2 \cap$ the interior of $AG_1 BG_2 A \cap$ the interior of $BG_1 CG_2 B$.† The point set $G_1 BG_2$ contains only the point $B$ of $K$. The points $A$ and $B$, together with those points of $K$ that lie within $AG_1 BG_2 A$, form the $K$-arc $AB$, while the points $B$ and $C$, together with those points of $K$ that lie within $BG_1 CG_2 B$, form the $K$-arc $BC$. But the points $A$ and $C$ together with the points of $K$ within $J$ form the $K$-arc $AC$.

Hence the $K$-arc $AC$ $=$ the $K$-arc $AB$ $+$ the $K$-arc $BC$.

**Lemma M.** If $ABC$ and $BDC$, then $ADC$.

**Proof.** By Lemma $L$, the $K$-arc $AC$ $=$ $K$-arc $AB$ $+$ $K$-arc $BC$. By Lemma $J$, $D$ is on the $K$-arc $BC$. Hence $D$ is on the $K$-arc $AC$. Thus, by Lemma $J$, $ADC$.

**Lemma N.** If $ABC$ and $ADC$ and $B \neq D$, then $ABD$ or $DBC$.

**Proof.** By Lemma $L$, the $K$-arc $AC$ $=$ $K$-arc $AD$ $+$ $K$-arc $DC$. By Lemma $J$, $B$ is on the $K$-arc $AC$. Hence, as $B \neq D$, then either $B$ is on the $K$-arc $AD$ or $B$ is on the $K$-arc $DC$. Hence, by Lemma $J$, either $ABD$ or $DBC$.

It can easily be shown that the following Lemmas $O$–$S$ are logical consequences of Lemmas $F$, $H$, $G$, $K$, $M$, and $N$.

**Lemma O.** If $ABC$ and $ADC$ and $B \neq D$, then $ABD$ or $ADB$.

**Lemma P.** If $ABC$ and $BCD$, then $ABD$.

**Lemma Q.** If $ABC$ and $BDC$, then $ABD$.

**Lemma R.** If $ABC$ and $DBC$ and $A \neq D$, then $ABD$ or $DAB$.

**Lemma S.** If $ABC$ and $DBC$, and $A \neq D$, then $ADC$ or $DAC$.

**Lemma T.** If $P$ is a point of $K$, which is contained in some segment of $K$ and $M$ is a subset of $K$, then $P$ is a limit point of $M$, if, and only if, every segment of $K$, containing $P$ contains at least one point of $M$ distinct from $P$.

**Proof.** Suppose $P$ is a limit point of $M$. Let $APB$ denote any segment of $K$ that contains $P$. Construct a simple closed curve $J$ such that (1) $A$ and $B$ are on $J$ and (2) $J - A - B = AF_1 B + AF_2 B$, where $AF_1 B$ is a subset of $S_1$ while $AF_2 B$ is a subset of $S_2$. All points of the segment $APB$ lie within $J$. As $P$ is a limit point of $M$, the interior of $J$ must contain at least one point $\overline{P}$ of $M$, different from $P$. All points of $M$ are points of $K$. Hence, as $APB$ contains all points of $K$ in the interior of $J$, $\overline{APB}$ contains a point of $M$ distinct from $P$.

* For a proof of this statement see R. L. Moore, loc. cit., Theorem 27, pp. 144–5.
Suppose every segment containing $P$ contains at least one point of $M$ distinct from $P$. Let $APB$ be any segment of $K$ containing $P$. Let us assume that $P$ is not a limit point of $M$. Then there exists a region $R$ containing $P$ and containing neither $A$, $B$, nor any point of $M$ other than $P$. Let $X$ denote the first point of the $K$-arc $PA$ on $R'$, the boundary of $R$. Let $Y$ denote the first point of the $K$-arc $PB$ on $R'$. The $K$-segment $XPY$ is within $R$. Hence the supposition that $R$ contains no point of $M$ different from $P$ leads to a contradiction.

**Lemma U.** If $A$ and $B$ ($A \neq B$) are points of $K$, then $B$ is not a limit point of the set of all points $\{C\}$ such that $BAC$.

**Proof.** Suppose $B$ is a limit point of the set of all points $\{C\}$ such that $BAC$. Put about $B$ a region $R$. Infinitely many points of $\{C\}$ must lie within $R$. Of these points we can select a sequence $\bar{C}_1, \bar{C}_2, \cdots$, having $B$ as their only limit point and lying in $R$.* Call $\bar{C}_1, \bar{C}_1$. The $K$-arc $AC_1$ cannot contain all the points $\bar{C}_2, \bar{C}_3, \bar{C}_4, \cdots$. For suppose it did. Then $A$ as $B$ is a limit point of $\bar{C}_2, \bar{C}_3, \bar{C}_4, \cdots$, $B$ is a limit point of the point set $AC_1$. But the set $AC_1$ is closed. Hence $B$ must belong to the point set $AC_1$. By Lemma G and Definition 1, $A \neq B \neq C_1$. Hence $ABC_1$. But by Lemma H, if $BAC_1$, then not $ABC_1$.

For no values of $i$ and $j$ ($i \neq j$) can $B$ lie on the $K$-arc $\bar{C}_i, \bar{C}_j$. For suppose for $i = k$ and $j = n$, $B$ is on the $K$-arc $\bar{C}_k, \bar{C}_n$. As $\bar{C}_k \neq B \neq \bar{C}_n$, then by Lemma J, $\bar{C}_k B\bar{C}_n$. By Lemma Q, $\bar{C}_k B\bar{C}_n$ and $BAC_1$ imply that $\bar{C}_k BA$ and hence, by Lemma F, $AB\bar{C}_k$. But, by Lemma H, $BAC_1$ implies not $AB\bar{C}_k$.

Let $C_2$ denote that point of the set $\bar{C}_2, \bar{C}_3, \bar{C}_4, \cdots$ of least subscript which does not lie on the $K$-arc $AC_1$. Let $C_3$ denote that point of least subscript of the set $\bar{C}_3, \bar{C}_4, \bar{C}_5, \cdots$, which does not lie on the $K$-arc $AC_2$. Continue this process and obtain an infinite sequence of points $C_1, C_2, C_3, \cdots$, and an infinite sequence of $K$-arcs $AC_1, C_1C_2, \cdots$. Let $M$ denote the set of all points of the arcs of this sequence. The point $B$ is the only limit point of the set $C_1, C_2, C_3, \cdots$. In view of the above lemmas concerning order on $K$, it is clear that the points $B, A, C_1, C_2, \cdots$, are in the order $BAC_1C_2 \cdots C_n C_{n+1} \cdots$.

Either the point $B$ is the only limit point of $M$, not contained in $M$, or there is at least one other limit point of $M$, not in $M$.

**Case I.** Suppose some point $E$, not in $M$, and different from $B$, is a limit point of $M$.

As $M$ is a subset of $K$ and as $K$ is closed, $E$ is a point of $K$. Hence, as $A \neq B, B \neq E, E \neq A$, it follows, by Lemma $K$, that either $EAB, ABE$, or $BEA$.

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*See R. L. Moore, loc. cit., Theorems 8 and 9, p. 134.*
Since the $K$-segment $AB$ contains no points of $M$, it follows, by Lemma $T$, that the order $BEA$ does not hold.

Suppose $EAB$. Then since $A$ is a limit point of $M$ and $M$ is connected, it easily follows, with the help of Lemma $T$, that $M$ is a subset of the $K$-arc $EB$. No point $P$ of $M$ is in the order $APB$. Hence $M$ is a subset of the $K$-arc $EA$. The $K$-arc $EA$ is closed. Hence $B$, which is not on this arc, cannot be a limit point of $M$. Thus the supposition that $E, A, B$ are in the order $EAB$ leads to a contradiction.

Suppose $ABE$. Then, since every point $P$ of $M$ is in the order $BAP$, it follows that no point of $M$ is on the $K$-segment $EA$. Hence, by Lemma $T$, $B$ is not a limit point of $M$. Hence the supposition that $ABE$ leads to a contradiction.

**Case II.** Suppose $B$ is the only limit point of $M$, not in $M$. In Case II it can easily be shown that the point set $M + B$ is a simple continuous arc from $A$ to $B$ having no points except $A$ and $B$ in common with the $K$-arc $AXB$. But this is contrary to Lemma $C$, as $M + AXB$ is a simple closed curve which is a subset of $K$.

Hence the supposition that $B$ is a limit point of $[C]$ leads to a contradiction.

**Lemmas $V$.** If $A$ and $B$ $[A \neq B]$ are points of $K$, then there exists a point $E$ of $K$ such that $ABE$.

**Proof.** Construct a simple closed curve $J$ such that (1) $A$ and $B$ are on $J$ and (2) $J - A - B = AF_1B + AF_2B$, where $AF_1B$ is a subset of $S_1$ while $AF_2B$ is a subset of $S_2$. By Lemma $U$, $B$ is not a limit point of the set of all points $[C]$ such that $BAC$. Hence we can put about $B$ a region $R$, containing no point $C$ such that $BAC$ and containing no point of the arc $F_1AF_2$. There exists a segment $MXN$ such that (1) $M$ and $N$ are on $J$, (2) the segment $MBN$ of the curve $J$ is within $R$ and (3) $MXN$ and the interior of $MXNBM$ are in $R$ and without $J$.* The points $M$ and $N$ cannot both be points of $S_1$. For if $M$ and $N$ are both points of $S_1$, then that arc of $J$ from $M$ to $N$ which contains $B$, will also contain $A$. But this is contrary to our choice of $R$. In like manner $M$ and $N$ are not both points of $S_2$.

As the arc $MXN$ joins a point of $S_1$ to a point of $S_2$ it must contain at least one point $E$ of $K$. As $E$ is without $J$, the order $AEB$ does not hold. As $E$ is within $R$, the order $EAB$ does not hold.

Hence, by Lemma $K$, $ABE$.

We are now in a position to prove our theorem. That $K$ is connected follows at once from Lemmas $A$ and $B$.

Let $P$ be any point of $K$. That $K - P = K_1 + K_2$, where $K_1$ and $K_2$ are two mutually exclusive connected point sets neither of which contains a limit point of the other may be proved as follows. Take $A$ any point of $K$,

different from $P$. Let $K_1$ denote the set of all points $[X]$ of $K$ such that $X = A$, $PXA$ or $PAX$, while $K_2$ denotes the set of all points $[Y]$ such that $APY$. That $K_2$ is not vacuous is a consequence of Lemma $V$. That $K_1$ and $K_2$ are connected follows with the aid of Lemma $B$, while it may be proved, with the aid of Lemma $U$, that neither of these sets contains a limit point of the other.

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