SOME GEOMETRIC CHARACTERIZATIONS OF ISOTHERMAL NETS ON A CURVED SURFACE*

BY

GABRIEL M. GREEN

INTRODUCTION

If a surface S, whose equations are

$$x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v),$$

be referred to an orthogonal net of parameter curves, its first fundamental form is

$$ds^2 = Edu^2 + Gdv^2,$$

where E and G are functions of u and v. This orthogonal net is called *isothermal*, if by a proper transformation $\bar{u} = \phi(u)$, $\bar{v} = \psi(v)$ of the parameters the first fundamental form may be reduced to

$$ds^2 = \bar{\lambda}(\bar{u}, \bar{v})(d\bar{u}^2 + d\bar{v}^2).$$

A necessary and sufficient condition that such a reduction may be effected upon the form as first written is, that the equation

$$\frac{\partial^2}{\partial u \partial v} \log\left(\frac{E}{G}\right) = 0 \tag{1}$$

be satisfied identically.

So far as the writer is aware, a purely geometric interpretation of this analytic condition does not appear in the literature. Of course the characterization of an isothermal net as a net which divides the surface into infinitesimal squares is not a geometric one.† The fact that these nets have long played a very important part in geometric investigations may lend interest to a purely geometric characterization thereof. In what follows, we present two such characterizations, each of which has its own peculiar

* Presented to the Society, December 27, 1916.

† Nor can Kasner's elegant characterization of an isothermal net, as one for which the two-parameter family of isogonal trajectories form a linear system, be regarded as geometric. Cf. his note, *A characteristic property of isothermal systems of curves*, Mathematische Annalen, vol. 59 (1904), pp. 352–4.

advantages: one is very easily derived, while the other, although it requires more complicated analytic machinery, leads to a more elegant result.

It should be noted, moreover, that the second characterization mentioned applies to all interesting cases, whereas the first breaks down when the isothermal net consists of the lines of curvature, that is, when the surface is isothermic. Darboux, however, has given a very elegant, though more complicated, geometric characterization of isothermic surfaces,* in connection with his solution of the problem of Ribaucour, in terms of the envelope of a certain two-parameter family of spheres. The related transformations D_m of isothermic surfaces, due to Darboux, permit the characterization of such surfaces in terms of the theory of rectilinear congruences.

1. The relation R. Conjugate congruences

Let X, Y, Z be the direction cosines of a line l in space. If X, Y, Z be given as functions of u, v, then through each point $P(x, y, z)$ of the surface S passes a line l, and the totality of these lines form a congruence which we shall denote by Γ. We shall assume that the line l which corresponds to a point P of S does not lie in the plane tangent to S at P, in other words, that *the determinant*

$$\Delta \equiv \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & X \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & Y \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & Z \end{vmatrix}$$

does not vanish for any point of the surface S.

Let us write the equations

$$\begin{aligned} \frac{\partial^2 x}{\partial u^2} &= a\frac{\partial x}{\partial u} + b\frac{\partial x}{\partial v} + cX, \\ \frac{\partial^2 x}{\partial u \partial v} &= a'\frac{\partial x}{\partial u} + b'\frac{\partial x}{\partial v} + c'X \\ \frac{\partial^2 x}{\partial v^2} &= a''\frac{\partial x}{\partial u} + b''\frac{\partial x}{\partial v} + c''X, \end{aligned} \tag{2}$$

the coefficients of which are functions of u, v which shall be determined as follows. In the first of equations (2), for instance, replace x, X successively by y, Y and by z, Z, thus obtaining three equations in all, which may be

* *Sur les surfaces isothermiques*, Annales scientifiques de l'école normale supérieure, 3e série, vol. 16 (1899), pp. 491–508, in particular p. 504.

solved for a, b, c since the determinant Δ is nowhere zero. The coefficients of the second and third of equations (2) may be calculated in the same way. These equations are obviously fundamental in the study of the congruence Γ in its relation to the surface S. They are, in fact, very similar to the Gauss equations for the surface S, to which they reduce if X, Y, Z are the direction cosines of the normal to S.

The equations

$$\xi = x + \mu \frac{\partial x}{\partial u}, \qquad \eta = y + \mu \frac{\partial y}{\partial u}, \qquad \zeta = z + \mu \frac{\partial z}{\partial u} \tag{3}$$

define the coördinates of a point $M(\xi, \eta, \zeta)$ on the tangent to the curve $v =$ const. at P. If P be allowed to move along a curve $u =$ const. on S, the corresponding lines PM generate a ruled surface $R^{(u)}$, and the point M traces a curve on this ruled surface. The tangent at any point of this curve traced by M has its direction cosines proportional to $\partial\xi/\partial v$, $\partial\eta/\partial v$, $\partial\zeta/\partial v$, where for instance

$$\frac{\partial \xi}{\partial v} = \frac{\partial x}{\partial v} + \frac{\partial \mu}{\partial v}\frac{\partial x}{\partial u} + \mu \frac{\partial^2 x}{\partial u \partial v}.$$

Using the second of equations (2), we obtain

$$\frac{\partial \xi}{\partial v} = \left(\frac{\partial \mu}{\partial v} + \mu a'\right)\frac{\partial x}{\partial u} + (1 + \mu b')\frac{\partial x}{\partial v} + \mu c' X. \tag{4}$$

The tangent to the curve traced by the point M as v alone varies lies in the plane determined by the lines l and PM if and only if

$$\mu = -\frac{1}{b'}. \tag{5}$$

We have therefore determined geometrically a unique point M on each line PM, which we shall call the *minus first transform* of the point P, and which is defined by the equations

$$\xi = x - \frac{1}{b'}\frac{\partial x}{\partial u}, \qquad \eta = y - \frac{1}{b'}\frac{\partial y}{\partial u}, \qquad \zeta = z - \frac{1}{b'}\frac{\partial z}{\partial u}. \tag{6}$$

In the same way, we may determine the *first transform* of the point P as a point M' lying on the tangent to the curve $u =$ const. at P, and defined by the equations

$$\xi' = x - \frac{1}{a'}\frac{\partial x}{\partial v}, \qquad \eta' = y - \frac{1}{a'}\frac{\partial y}{\partial v}, \qquad \zeta' = z - \frac{1}{a'}\frac{\partial z}{\partial v}. \tag{7}$$

Our designation of the points M and M' as the minus first and first transforms of the point P is intended to indicate the analogy to the Laplace trans-

forms of a conjugate net. In fact, if the parametric net on S is conjugate, $c' = 0$, the second of equations (2) becomes one of the Laplace type, and the points M and M' defined by equations (6) and (7) are actually the minus first and first Laplace transforms of the point P.

If the parametric net is conjugate, the points M and M' just determined are the same no matter what the congruence Γ may be. If, however, the parametric net is not conjugate, the points M and M' change with the congruence Γ. The following considerations will show why this is so. For a non-conjugate parametric net, the ruled surface $R^{(u)}$ formed by the tangents to the curves $v =$ const. along a fixed curve $u =$ const. is a skew ruled surface, and therefore a plane through a generator thereof is tangent to the ruled surface at a definite point of that generator. From the way in which the point M was obtained, it is obvious that *the point M is the point at which the plane determined by the lines l and PM is tangent to the ruled surface $R^{(u)}$*. A similar characterization obtains for the point M', and so different points M and M' would be determined by different congruences Γ.

The line l' which joins the points M and M' we shall call the *line conjugate to the line l*, and the congruence Γ' which is formed by the lines l' the *congruence conjugate to the congruence* Γ. If the parametric net is non-conjugate, it is easy to see that a given congruence Γ' can have but one congruence Γ associated with it, so that the congruences Γ and Γ' may in this case be described as conjugate to each other. For, the line l' intersects the parametric tangents in points M and M', and the line l is then determined as the line of intersection of the two planes tangent at M and M' to the two parametric ruled surfaces $R^{(u)}$ and $R^{(v)}$ respectively. If the parametric net is conjugate, however, a congruence Γ will have a unique conjugate congruence Γ', but a congruence Γ'—of which there exists only one for a given conjugate net—will be conjugate to any congruence Γ. In the case of a conjugate net, Wilczynski* has called the line joining the minus first and first Laplace transforms of a point P of the surface the *ray* of the point P, and the congruence formed of these lines the *ray congruence*.

We have elsewhere† spoken of the relation between the congruences Γ and Γ' as the relation R. It is obviously a purely projective relation, and we have in fact found the notion of the greatest importance in the general theory of surfaces and rectilinear congruences.

* E. J. Wilczynski, *The general theory of congruences*, these Transactions, vol. 16 (1915), pp. 311–327.

† In three communications to the Society, abstracts of which have appeared in the Bulletin as follows: vol. 21 (1914–15), pp. 484–5, vol. 22 (1915–16), p. 274, and vol. 23 (1916–17), pp. 73–4.

2. The developables of the congruence Γ'

The congruence Γ' formed of the lines MM' contains two one-parameter families of developables, which we now proceed to determine. Any curve on the surface S may be defined by equations of the form $u = u(t)$, $v = v(t)$. We seek those functions $u(t)$ and $v(t)$ which define the curves on S which correspond to the developables of Γ'. We note first of all that if the line MM' generates a developable, then the tangent to the curve traced by M, the tangent to the curve traced by M', and the generator MM' lie in a plane. The tangent to the curve traced by M has its direction cosines proportional to $d\xi/dt$, $d\eta/dt$, $d\zeta/dt$, where ξ, η, ζ have the values given by equations (6) when u and v are replaced by the functions $u(t)$ and $v(t)$ respectively. We have of course

$$\frac{d\xi}{dt} = \frac{\partial \xi}{\partial u}\frac{du}{dt} + \frac{\partial \xi}{\partial v}\frac{dv}{dt}, \qquad \text{etc.,}$$

where

$$\begin{aligned} \frac{\partial \xi}{\partial u} &= \left[1 - \frac{a}{b'} - \frac{\partial}{\partial u}\left(\frac{1}{b'}\right)\right]\frac{\partial x}{\partial u} - \frac{b}{b'}\frac{\partial x}{\partial v} - \frac{c}{b'}X, \\ \frac{\partial \xi}{\partial v} &= -\frac{1}{b'}\left(\frac{\partial}{\partial v}\log b' + a'\right)\frac{\partial x}{\partial u} - \frac{c'}{b'}X. \end{aligned} \tag{8}$$

Equations (8) are obtained directly from equations (6), use being made of the first two of equations (2). We have therefore

$$\begin{aligned} \frac{d\xi}{dt} = \left\{\left(1 - \frac{a}{b'} + \frac{1}{b'^2}\frac{\partial b'}{\partial u}\right)\frac{du}{dt} - \frac{1}{b'^2}\left(\frac{\partial b'}{\partial v} + a'b'\right)\frac{dv}{dt}\right\}\frac{\partial x}{\partial u} \\ - \frac{b}{b'}\frac{du}{dt}\frac{\partial x}{\partial v} - \frac{1}{b'}\left(c\frac{du}{dt} + c'\frac{dv}{dt}\right)X, \end{aligned}$$

the same equation subsisting, of course, if ξ be replaced successively by η and ζ, and x at the same time by y and z, X by Y and Z. Similarly,

$$\begin{aligned} \frac{d\xi'}{dt} = -\frac{a''}{a'}\frac{dv}{dt}\frac{\partial x}{\partial u} + \left\{\left(1 - \frac{b''}{a'} + \frac{1}{a'^2}\frac{\partial a'}{\partial v}\right)\frac{dv}{dt}\right. \\ \left. - \frac{1}{a'^2}\left(\frac{\partial a'}{\partial u} + a'b'\right)\frac{du}{dt}\right\}\frac{\partial x}{\partial v} - \frac{1}{a'}\left(c''\frac{dv}{dt} + c'\frac{du}{dt}\right)X. \end{aligned}$$

We have also

$$\xi - \xi' = -\frac{1}{b'}\frac{\partial x}{\partial u} + \frac{1}{a'}\frac{\partial x}{\partial v}, \qquad \text{etc.}$$

We wish now to write a necessary and sufficient condition that the line whose direction cosines are proportional to $d\xi/dt$, $d\eta/dt$, $d\zeta/dt$, the line whose direction cosines are proportional to $d\xi'/dt$, $d\eta'/dt$, $d\zeta'/dt$, and the

line whose direction cosines are proportional to $\xi - \xi'$, $\eta - \eta'$, $\zeta - \zeta'$, be parallel to the same plane. The condition sought is easily seen to be the vanishing of the determinant formed from the nine coefficients of $\partial x/\partial u$, $\partial x/\partial v$, X in the above expressions for $d\xi/dt$, $d\xi'/dt$, $\xi - \xi'$. When this determinant is expanded, the condition is found to be

$$A\left(\frac{du}{dt}\right)^2 + 2B\frac{du}{dt}\frac{dv}{dt} + C\left(\frac{dv}{dt}\right)^2 = 0, \tag{9}$$

where

$$A = c'\left[b'(b' - a) + \frac{\partial b'}{\partial u} - a'b\right] + c\left(\frac{\partial a'}{\partial u} + a'b'\right),$$

$$C = -c'\left[a'(a' - b'') + \frac{\partial a'}{\partial v} - a''b'\right] - c''\left(\frac{\partial b'}{\partial v} + a'b'\right),$$

$$B = \frac{1}{2}\left\{c''\left[b'(b' - a) + \frac{\partial b'}{\partial u} - a'b\right] + c'\left(\frac{\partial a'}{\partial u} - \frac{\partial b'}{\partial v}\right)\right.$$
$$\left. - c\left[a'(a' - b'') + \frac{\partial a'}{\partial v} - a''b'\right]\right\}.$$

Equation (9) is a quadratic differential equation of the first order, the solutions of which determine the curves on the surface S which correspond to the developables of the congruence Γ' of lines MM'. The asymptotic net of the surface S is moreover defined by the quadratic differential equation

$$D\left(\frac{du}{dt}\right)^2 + 2D'\frac{du}{dt}\frac{dv}{dt} + D''\left(\frac{dv}{dt}\right)^2 = 0, \tag{10}$$

where D, D', D'' have the usual significance as the coefficients of the second fundamental form of S. Each of the quadratics (9) and (10) determines a pair of directions at any point of the surface S, and *these pairs separate each other harmonically if and only if*

$$AD'' - 2BD' + CD = 0. \tag{11}$$

The left-hand member of this equation is the simultaneous invariant of the two binary forms (9), (10). But two tangents are conjugate if they separate harmonically the asymptotic tangents. Therefore the net of curves defined by equation (9) is a conjugate net if and only if equation (11) holds.

Now, the coefficients c, c', c'' of equations (2) are easily seen to be proportional to D, D', D''; in fact,

$$c : D = c' : D' = c'' : D'' = \sqrt{EG - F^2}/\Delta,$$

where Δ is the determinant introduced at the beginning of § 1. If the values

of c, c', c'' be substituted in the expressions for A, B, C, and the resulting quantities then substituted in equation (11), this equation reduces essentially to

$$(DD'' - D'^2)\left(\frac{\partial a'}{\partial u} - \frac{\partial b'}{\partial v}\right) = 0. \tag{12}$$

The first factor on the left vanishes identically only for a developable surface. But for such a surface the only conjugate nets are nets one component family of which consists of the straight line generators. We may therefore state the following theorem:

THEOREM. *The developables of the congruence* Γ' *of lines* MM' *correspond to a conjugate net on the surface* S *if and only if*

$$\frac{\partial a'}{\partial u} - \frac{\partial b'}{\partial v} = 0. \tag{13}$$

If the parametric net is conjugate, equation (13) expresses the fact that the net has equal Laplace-Darboux invariants. The characterization of a conjugate net with equal invariants as a net for which the developables of the ray congruence correspond to a conjugate net on the surface, is due to Wilczynski.*

We have found a geometric property which is expressed by equation (13) when the surface S is not developable. We proceed now to obtain another geometric property which covers the case of a developable surface, but which on the other hand becomes trivial if the parametric net on the surface is conjugate.

The totality of points $M(\xi, \eta, \zeta)$ form a surface S_{-1}, which we may call the *minus first transform* of S, and the points $M'(\xi', \eta', \zeta')$ likewise form a surface S_1, the *first transform* of S. From the way in which the point M was obtained, it is clear that the plane of the lines l and PM is tangent to the surface S_{-1} at M. Consequently the tangent at M to the curve $u =$ const. on S_{-1} intersects the line l in a point whose coördinates we shall now find. The said tangent has direction cosines proportional to $\partial\xi/\partial v$, $\partial\eta/\partial v$, $\partial\zeta/\partial v$, these derivatives being given by the second of equations (8):

$$\frac{\partial \xi}{\partial v} = -\frac{1}{b'}\left(\frac{\partial}{\partial v}\log b' + a'\right)\frac{\partial x}{\partial u} - \frac{c'}{b'}X. \tag{14}$$

The required point of intersection must have for its x-coördinate an expression of the form $\xi + \nu\partial\xi/\partial v$, and also of the form $x + \omega X$. By combining equation (14) with the first of equations (6), it is readily seen that the point of

* Loc. cit., p. 319.

intersection has for its x-coördinate

$$\xi - \frac{1}{\dfrac{\partial}{\partial v}\log b' + a'}\frac{\partial \xi}{\partial v} = x + \frac{c'}{b'\left(\dfrac{\partial}{\partial v}\log b' + a'\right)}X. \tag{15}$$

Similarly, the tangent at M' to a curve $v = \text{const.}$ on S_1 meets the corresponding line l in the point whose x-coördinate is

$$\xi' - \frac{1}{\dfrac{\partial}{\partial u}\log a' + b'}\frac{\partial \xi'}{\partial u} = x + \frac{c'}{a'\left(\dfrac{\partial}{\partial u}\log a' + b'\right)}X. \tag{16}$$

Examination of the right-hand members of equations (15) and (16) shows that *the points in which the line l is cut by the corresponding tangent to a curve $u = \text{const.}$ on S_{-1} and the corresponding tangent to a curve $v = \text{const.}$ on S_1 coincide if and only if*

$$\frac{\partial a'}{\partial u} = \frac{\partial b'}{\partial v}. \tag{13}$$

If the parametric net on S is conjugate, $c' = 0$, and condition (13) is not at all necessary. In fact, in this case the two points of intersection on l coincide with the point P, always.

3. Geometric characterization of isothermal nets

We shall now suppose that the parametric net on the surface S is an orthogonal net, and that the line l is normal to S at P. Then equations (2) become the Gauss equations of the surface, and in addition $F = 0$. Some of the coefficients in the Gauss equations are

$$c = D, \qquad c' = D', \qquad c'' = D'',$$

$$a' = \frac{1}{2}\frac{\partial}{\partial v}\log E, \qquad b' = \frac{1}{2}\frac{\partial}{\partial u}\log G.$$

The condition that the parametric net be isothermal, i. e., equation (1), may therefore be written

$$\frac{\partial a'}{\partial u} = \frac{\partial b'}{\partial v},$$

which is precisely the equation for which we have already found geometric interpretations. Using the results of the preceding paragraphs, we may therefore state the following geometric characterizations of isothermal nets:

An orthogonal net on a non-developable surface is an isothermal net, if and only if the developables of the congruence which is conjugate to the congruence of normals of the surface correspond to a conjugate net on the surface.

If an orthogonal net on any surface S is not conjugate—i. e., does not consist of the lines of curvature of S—then the net is isothermal if and only if each normal to the surface is cut in the same point by the corresponding tangent to the curve u = const. on the minus first transform of S, and by the corresponding tangent to the curve v = const. on the first transform of S.

If the lines of curvature form an isothermal net, the surface is isothermic; therefore, in virtue of the theorem of Wilczynski, *a surface is isothermic if and only if the developables of the ray congruence of its lines of curvature correspond to a conjugate net on the surface.*

HARVARD UNIVERSITY
March, 1917