EXISTENCE THEOREMS FOR THE GENERAL REAL SELF-ADJOINT LINEAR SYSTEM OF THE SECOND ORDER*

BY

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INTRODUCTION

Sturm† in 1836 established many fundamental theorems concerning the properties of solutions of the linear differential equation (1) below, and the system formed by (1) and the boundary conditions (3), of which the oscillation theorems of § 1 of the present paper are immediate consequences.

In the special case of periodic conditions Mason‡ and Bôcher§ with more or less generality established an oscillation theorem.

Birkhoff|| extended their work to the general self-adjoint linear boundary conditions (see (5) and (1)), where, however, he assumed $K \equiv 1$, and that λ does not enter into the boundary conditions, and established an oscillation theorem for $u_p(x)$, the solution corresponding to the pth characteristic number.

It will be the object of this paper to generalize these results to the most general real, self-adjoint linear system of the second order, where K and the coefficients of the boundary conditions are functions of λ, by extending Bôcher's and Birkhoff's methods,¶ which are based on the application of Sturm's theorems to this system.**

* Presented to the Society, under a different title, April 26, 1913.

† Journal de mathématiques, vol. 1 (1836), pp. 106–186.

‡ Mathematische Annalen, vol. 58 (1904), these Transactions, vol. 7 (1906), pp. 337–360.

§ Comptes Rendus, vol. 140 (1905), p. 928.

|| These Transactions, vol. 10 (1909), pp. 259–270.

¶ Some of the theorems obtained by Birkhoff have been worked out independently and by other methods by Haupt. See Dissertation, "*Ueber Oszillationstheoreme*," Teubner, Leipzig, 1911. See also Mathematische Annalen, vol. 76 (1914), pp. 67–104.

After a report of my results had been published (see Bulletin of the American Mathematical Society, vol. 19 (1913), p. 502), the case where K is a function of λ and the coefficients of the boundary conditions are constants, together with the corresponding system in difference equations was partially treated by Fort, "*Linear difference and differential equations*," American Journal of Mathematics, vol. 39 (1917), pp. 1–26.

** The writing of this paper has been furthered by stimulus and suggestions from Professors Birkhoff and Bôcher, to whom I desire to express my cordial appreciation.

1. On a Sturmian boundary problem

Given the second order linear differential equation,

$$\frac{d}{dx}\left[K(x,\lambda)\frac{du}{dx}\right] - G(x,\lambda)u = 0, \tag{1}$$

let us consider the following linear combinations of a function $u(x,\lambda)$ and its first derivative with regard to x, $u'(x,\lambda)$,

$$\begin{aligned} L_i[u(x,\lambda)] &= \alpha_i(\lambda)u(x,\lambda) - \beta_i(\lambda)K(x,\lambda)u'(x,\lambda) \\ M_i[u(x,\lambda)] &= \gamma_i(\lambda)u(x,\lambda) + \delta_i(\lambda)K(x,\lambda)u'(x,\lambda). \end{aligned} \quad (i = 0, 1), \tag{2}$$

By the sets of conditions (A), (B), for the equation (1) and the linear combinations (2) shall be meant the following.

Conditions (A).

I. $K(x,\lambda)$ and $G(x,\lambda)$ are continuous* real functions of x and λ, for all real values of x in the interval

$$(X) \qquad (a \leqq x \leqq b)$$

and for all real values of λ in the interval

$$(\Lambda) \qquad (\mathcal{L}_1 < \lambda < \mathcal{L}_2).\dagger$$

II. $K(x,\lambda)$ is always positive in (X, Λ).

III. Throughout (Λ), the eight coefficients $\alpha_i, \cdots, \delta_i$ of (2) are continuous real functions of λ, and

$$|\alpha_i| + |\beta_i| > 0, \qquad |\gamma_i| + |\delta_i| > 0.$$

IV. (I) K and G always decrease (or at least do not increase) as λ increases.

(II) Either $\beta_i \equiv 0$, or $\beta_i \not\equiv 0$, in which case the quotient α_i/β_i decreases (or at least does not increase) as λ increases. Also either $\delta_i \equiv 0$, or $\delta_i \not\equiv 0$, in which case the quotient γ_i/δ_i decreases (or at least does not increase) as λ increases.

V. For any arbitrary λ, there shall exist an x for which K or G actually decreases as λ increases, unless for this value $\beta_i \not\equiv 0$ (or $\delta_i \not\equiv 0$) and α_i/β_i (or γ_i/δ_i) actually decreases for this value of λ as λ increases.

Conditions (B).

Such further conditions on the coefficients of the system as will ensure the correctness of Sturm's Theorem of Oscillation for the system (1), (3). Various sets of conditions have been worked out by Professor Bôcher in Chapter III,

* The existence and continuity of $\partial K / \partial x$ is commonly required, but this is not necessary. See Weyl, Mathematische Annalen, vol. 68 (1910), p. 221 and Bôcher, these Transactions, vol. 14 (1913), pp. 412–418.

† In particular, $\mathcal{L}_1$ and $\mathcal{L}_2$ may be $-\infty$ or $+\infty$ respectively.

paragraphs 13–15 of his recent book, *Leçons sur les Méthodes de Sturm* (Paris, 1917).

Consider now the sturmian system consisting of the equation (1) and conditions

$$(3)\qquad \begin{aligned} L_0[u(a)] &= 0, \\ M_0[u(a)] &= 0. \end{aligned}$$

If $u(x, \lambda)$ is a solution of (1), (different from zero), satisfying the first boundary condition of (3) and if we have either

$$\frac{\gamma_0}{\delta_0} < \frac{\gamma_1}{\delta_1} \qquad \text{or} \qquad \frac{\gamma_0}{\delta_0} > \frac{\gamma_1}{\delta_1},$$

if $\gamma_0 \delta_1 - \delta_0 \gamma_1$ is not zero in (Λ), and if the expressions involved satisfy (A), then it follows from Sturm's Oscillation Theorem that

$$M_0[u(b)] \qquad \text{and} \qquad M_1[u(b)]$$

change sign when they vanish and their roots separate one another, and also that the roots of $u(x, \lambda)$ decrease as λ increases.*

Concerning the sturmian system (1), (3) we have the following results,† provided (A) and (B) are satisfied:

I. There exist an infinite number of characteristic values‡ for the system in (Λ), having a single cluster point at $\mathcal{L}_2$.

II. If we arrange these values $\lambda_0, \lambda_1, \cdots$ in order of increasing magnitude and denote the corresponding characteristic functions by $U_0(x), U_1(x), \cdots$, then $U_n(x)$ will vanish precisely n times on $a < x < b$.

III. If $u(x, \lambda)$ be the particular solution satisfying

$$L_0[u(a)] = 0$$

for which

$$u(a, \lambda) = \beta_0(\lambda), \qquad K(a, \lambda) u'(a, \lambda) = \alpha_0(\lambda),$$

then for $x > a$

$$\lim_{\lambda = \mathfrak{L}_1} K(x, \lambda) u'(x, \lambda) = \infty, \qquad \lim_{\lambda = \mathfrak{L}_1} u(x, \lambda) = \infty,$$

$$\lim_{\lambda = \mathfrak{L}_1} \frac{K(x, \lambda) u'(x, \lambda)}{u(x, \lambda)} = \infty.$$

* Loc. cit., pp. 139, 143.

† See Bôcher, *Encyklopädie der mathematischen Wissenschaften*, II *A*, 7*a*.

‡ A value λ_i of λ is said to be a characteristic value of a system such as (1), (3), if when $\lambda = \lambda_i$, this system has a solution not identically zero. Any such solution is termed a characteristic function.

We may notice that a very important special case of a system satisfying (A) and (B) is formed by the equation

$$\frac{d}{dx}\left[k(x)\frac{du}{dx}\right] - [l(x) - \lambda g(x)]u = 0$$

together with boundary conditions in which the coefficients are constants, provided $k(x) > 0$, $g(x) > 0$. The interval (Λ) is $(-\infty, +\infty)$.

If however g changes sign, and $l \geqq 0$, $\alpha_0 \beta_0 \geqq 0$, $\gamma_0 \delta_0 \geqq 0$, the equation may be put in the form

$$\frac{d}{dx}\left[\frac{k(x)}{|\lambda|}\frac{du}{dx}\right] - \left[\frac{l(x)}{|\lambda|} - (\operatorname{sgn}\lambda)g(x)\right]u = 0.$$

If the parameter λ be replaced by the new parameter $\bar{\lambda} = |\lambda|$ the conditions (A) and (B) are satisfied for the interval $(0, +\infty)$ for (Λ). By this device, due to Bôcher,* the case in which $g(x)$ changes sign can be dealt with.

2. The generalized self-adjoint boundary problem. Existence of an infinite set of characteristic values

Consider the system consisting of the differential equation (1) together with the boundary conditions

$$\begin{aligned} L_0[u(a)] &= M_0[u(b)], \\ L_1[u(a)] &= M_1[u(b)], \end{aligned} \tag{4}$$

where L_i and M_i are defined by (2). Throughout this paper we shall suppose that the system (1), (4) satisfies the set of conditions (A), § 1, and that the conditions (4) do not reduce to the sturmian type for any λ, and that these conditions are *self-adjoint*

$$\alpha_0 \beta_1 - \beta_0 \alpha_1 \equiv -(\gamma_0 \delta_1 - \delta_0 \gamma_1).$$

Now the quantity $\alpha_0 \beta_1 - \beta_0 \alpha_1$ is never zero in this case, since the conditions (4) are then reducible by linear combination to the sturmian type. By division of the first equation (4) by this quantity, we may, without loss of generality or effect upon (A), (B), § 1, take

$$\alpha_0 \beta_1 - \beta_0 \alpha_1 \equiv -(\gamma_0 \delta_1 - \delta_0 \gamma_1) \equiv -1. \tag{5}$$

We shall consider only real solutions of (1), (4).

Let $u_0(x, \lambda)$ and $u_1(x, \lambda)$ denote the two linearly independent solutions of (2) satisfying the conditions

* Cf. Bôcher, *Proceedings of The Fifth International Congress of Mathematicians*, Cambridge, England, vol. 1 (1912), p. 173.

$$
(6) \qquad \begin{aligned} L_0[u_0(a)] &= 0, \qquad L_1[u_0(a)] = 1, \\ L_0[u_1(a)] &= 1, \qquad L_1[u_1(a)] = 0. \end{aligned}
$$

Solving the first two equations, we have

$$
(7) \qquad K(a,\lambda)u_0'(a,\lambda) = \alpha_0(\lambda), \qquad u_0(a,\lambda) = \beta_0(\lambda);
$$

and solving the second pair,

$$
(8) \qquad K(a,\lambda)u_1'(a,\lambda) = -\alpha_1(\lambda), \qquad u_1(a,\lambda) = -\beta_1(\lambda).
$$

By (A), α_0 and β_0 may not vanish together, so that (7) and (8) also determine u_0 and u_1 as functions of x, linearly independent for every value of λ. Abel's formula for (1) gives

$$
(9) \qquad u_0 u_1' - u_1 u_0' = -\frac{1}{K}.
$$

Also direct computation and simplification by (6) and (9) show that

$$
(10) \qquad \begin{aligned} L_0[u_0]L_1[u_1] - L_0[u_1]L_1[u_0] &= -1, \\ M_0[u_0]M_1[u_1] - M_0[u_1]M_1[u_0] &= -1. \end{aligned}
$$

THEOREM. *A necessary and sufficient condition that there exist a solution u not identically zero satisfying* (4) *when $\lambda = l$ is that $\phi(l) = 0$, where*

$$
(11) \qquad \phi(\lambda) = -2 + M_1[u_0(b,\lambda)] + M_0[u_1(b,\lambda)].
$$

Proof. The general solution of (1) is

$$
u(x,\lambda) = c_0(\lambda)u_0(x,\lambda) + c_1(\lambda)u_1(x,\lambda).
$$

A simple reduction by means of (6) shows that a necessary and sufficient condition that $u(x,\lambda)$ not identically zero satisfy (4) is that c_0 and c_1 satisfy

$$
\begin{aligned} c_0\{-M_0[u_0(b)]\} + c_1\{1 - M_0[u_1(b)]\} &= 0, \\ c_0\{1 - M_1[u_0(b)]\} + c_1\{-M_1[u_1(b)]\} &= 0, \end{aligned}
$$

and are not both zero; this is possible if and only if

$$
(12) \qquad \phi(\lambda) = \begin{vmatrix} -M_0[u_0(b)] & 1 - M_0[u_1(b)] \\ 1 - M_1[u_0(b)] & -M_1[u_1(b)] \end{vmatrix} = 0.
$$

By (10) this reduces to (11).

The equation $\phi(\lambda) = 0$ is the *characteristic equation,* that is, an equation which has for its roots all the characteristic values and no other roots. We may note that the quantities c_0 and c_1 are determined uniquely, except for a constant multiplier, provided not all the elements of the determinant (12) vanish. A value, l, for which $\phi(l) = 0$ but not all the elements of (12) vanish, is said to be a *simple characteristic value.*

If all the elements of (12) are zero, then c_0 and c_1 are both arbitrary and there will exist two linearly independent solutions of the system (1), (4). A value $\lambda = l$, such that

$$(13)\qquad \begin{aligned} M_0[u_0(b)] &= 0, & M_0[u_1(b)] &= 1, \\ M_1[u_0(b)] &= 1, & M_1[u_1(b)] &= 0, \end{aligned}$$

is called a *double characteristic value.**

Consider the auxiliary sturmian system

$$(14)\qquad \begin{aligned} L_0[u(a)] &= 0, \\ M_0[u(b)] &= 0. \end{aligned}$$

We shall assume that this system satisfies conditions (B) of § 1 in addition to (A). Let the characteristic values of this system be $\lambda_0, \lambda_1, \cdots$, ordered so that

$$\lambda_0 < \lambda_1 < \lambda_2 \cdots.$$

EXISTENCE THEOREM. *If the system* (1), (4) *satisfies conditions* (A), (B), *there exists at least one characteristic value of the system* (1), (4) *between any two characteristic values of the auxiliary sturmian system* (14).

To prove this we first note that we have already determined u_0 so as to satisfy the first of conditions (14), and the only solution which can be so determined differs from u_0 merely by a constant factor. Consequently, the characteristic values of the system (14) are the roots of the equation

$$(15)\qquad M_0[u_0(b, \lambda)] = 0.$$

Hence we have from (10) for $\lambda = \lambda_i$,

$$M_0[u_1(b, \lambda_i)]\, M_1[u_0(b, \lambda_i)] = 1 \qquad (i = 0, 1, 2, \cdots).$$

Reducing (11) by means of this last relation, we have

$$(16)\qquad \phi(\lambda_i) = \frac{1}{M_1[u_0(b, \lambda_i)]}\{1 - M_1[u_0(b, \lambda_i)]\}^2$$

so that at $\lambda = \lambda_i$, $\phi(\lambda_i)$ has the same sign as $M_1[u_0(b, \lambda_i)]$ except when $M_1[u_0(b, \lambda_i)] = 1$, that is when $\phi(\lambda_i) = 0$.

We may apply the results of Sturm to the functions $M_0[u_0(b, \lambda)]$ and $M_1[u_1(b, \lambda)]$, so that these functions must change sign when they vanish and their roots separate one another.† Equation (15) taken in conjunction

* Hence at a double characteristic value we have by solving (13),

$$\begin{aligned} K(b, l)\, u_0'(b, l) &= \gamma_0, & u_0(b, l) &= -\delta_0, \\ K(b, l)\, u_1'(b, l) &= -\gamma_1, & u_1(b, l) &= \delta_1. \end{aligned}$$

† See § 1.

with this fact establishes that if $M_1[u_0(b,\lambda)]$ is positive at $\lambda_0, \lambda_2, \cdots$, it is negative at $\lambda_1, \lambda_3, \cdots$ or *vice versa*. Hence $\phi(\lambda)$ is positive or zero at $\lambda_0, \lambda_2, \cdots$, and negative at $\lambda_1, \lambda_3, \cdots$; or else $\phi(\lambda)$ is positive or zero at $\lambda_1, \lambda_3, \cdots$, and negative at $\lambda_0, \lambda_2, \cdots$. In either case $\phi(\lambda)$ has at least one root between λ_p and λ_{p+1}. This proves the theorem and carries with it the existence of an infinite set of characteristic values for the system (1), (4) under the conditions stated.

3. Some properties of the function $\phi(\lambda)$

In considering the uniqueness of the characteristic values, we shall need information concerning the behavior of $\phi(\lambda)$ near simple and double characteristic values.

We now introduce the further condition

$$(C) \qquad \begin{vmatrix} \alpha_0 & \beta_0 & \gamma_0 & \delta_0 \\ \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_0' & \beta_0' & \gamma_0' & \delta_0' \\ \alpha_1' & \beta_1' & \gamma_1' & \delta_1' \end{vmatrix} \leqq 0.$$

This condition is unaltered by linear combination of the two boundary conditions.

THEOREM I′. *If (A), (B), (C) are satisfied in the vicinity of $\lambda = l$, a simple characteristic value, $\phi(\lambda)$ changes sign in such wise that $d\phi/d\lambda$ has the same sign as $M_0[u_0(b,\lambda)]$ or $-M_1[u_1(b,\lambda)]$, provided that $K, G, \alpha_0, \cdots \delta_1$ have continuous derivatives with respect to λ of the first order, and for $\lambda = l$ either K_λ or G_λ is negative for at least one value of x in $a < x \leqq b$.*

THEOREM II′. *If (A), (B), (C) are satisfied in the vicinity of $\lambda = l$, a double characteristic value, $\phi(\lambda)$ preserves a negative sign, while $M_0[u_0(b,\lambda)]$ and $-M_1[u_1(b,\lambda)]$ change from positive to negative as λ increases through l, provided that $K, G, \alpha_0, \cdots \delta_1$ have continuous derivatives with regard to λ of the first two orders, and for $\lambda = l$ either K_λ or G_λ is negative for at least one value of x in $a < x \leqq b$.*

Theorem I′ will be proved if it is shown that $d\phi/d\lambda$ has the same sign as $M_0[u_0(b,\lambda)]$ or $-M_1[u_1(b,\lambda)]$ at $\lambda = l$. We proceed to compute $d\phi/d\lambda$. For this purpose, it will be convenient to introduce the following system of equations:

$$(17) \qquad \begin{aligned} z' &= G(x,\lambda)u, \\ u' &= H(x,\mu)z, \\ \alpha_i(\lambda)u(a) - \beta_i(\mu)z(a) &= \gamma_i(\lambda)u(b) + \delta_i(\mu)z(b) \quad (i = 0, 1), \end{aligned}$$

where

$$H(x,\mu) = 1/K(x,\mu).$$

If we eliminate z from the equations (17) we obtain

$$(17') \qquad \frac{d}{dx}\left[K(x,\mu)\frac{du}{dx}\right] - G(x,\lambda)u = 0,$$

$$L_i[u(a)] = M_i[u(b)] \qquad (i = 0, 1),$$

where by L_0, L_1, M_0, M_1 are denoted the same expressions as in (2) save that $\beta_0(\lambda)$, $\beta_1(\lambda)$, $\delta_0(\lambda)$, $\delta_1(\lambda)$ are replaced by $\beta_0(\mu)$, $\beta_1(\mu)$, $\delta_0(\mu)$, $\delta_1(\mu)$ respectively. Evidently if we write $\lambda = \mu$, the equations (17′) become precisely (1), (4).

If now we define $u_0(x;\lambda,\mu)$, $u_1(x;\lambda,\mu)$ as the solutions of the first equation (17′) satisfying modified conditions like (7), (8),

$$K(a,\lambda)u_0'(a;\lambda,\mu) = \alpha_0(\lambda), \qquad u_0(a;\lambda,\mu) = \beta_0(\mu),$$

$$K(a,\lambda)u_1'(a;\lambda,\mu) = -\alpha_1(\lambda), \qquad u_1(a;\lambda,\mu) = -\beta_1(\mu),$$

and if, as in (11), we define

$$\phi(\lambda,\mu) = M_1[u_0(b;\lambda,\mu)] + M_0[u_1(b;\lambda,\mu)] - 2,$$

then we have clearly

$$\frac{d\phi(\lambda)}{d\lambda} = \left[\frac{\partial\phi(\lambda,\mu)}{\partial\lambda} + \frac{\partial\phi(\lambda,\mu)}{\partial\mu}\right]_{\mu=\lambda},$$

in consequence of the obvious identity $\phi(\lambda) = \phi(\lambda,\lambda)$.

If, moreover, we write

$$z_0(x;\lambda,\mu) = K(x,\mu)u_0'(x;\lambda,\mu), \qquad z_1(x;\lambda,\mu) = K(x,\mu)u_1'(x;\lambda,\mu),$$

it is seen that (u_0, z_0) and (u_1, z_1) may be defined as the solutions of the first two equations (17) which fulfil the further conditions

$$z_0(a;\lambda,\mu) = \alpha_0(\lambda), \qquad u_0(a;\lambda,\mu) = \beta_0(\mu),$$

$$z_1(a;\lambda,\mu) = -\alpha_1(\lambda), \qquad u_1(a;\lambda,\mu) = -\beta_1(\mu).$$

We may now give $\phi(\lambda,\mu)$ the form:

$$\phi(\lambda,\mu) = \gamma_1(\lambda)u_0(b;\lambda,\mu) + \delta_1(\mu)z_0(b;\lambda,\mu)$$
$$+ \gamma_0(\lambda)u_1(b;\lambda,\mu) + \delta_0(\mu)z_1(b;\lambda,\mu) - 2.$$

By virtue of the symmetry in u and z on the one hand, and in λ and μ on the other, in these equations and in (17), it will be necessary to compute merely $\partial\phi/\partial\lambda$ to obtain $\partial\phi/\partial\mu$. For then replacing λ by μ, G by H, α_i by $-\beta_j$, γ_i by δ_j, u_i by z_j $(i \neq j)$, we shall have $\partial\phi/\partial\mu$.

It is worth noting that the equations (5) may be used in simplifying $d\phi/d\lambda$

for $\lambda = \mu$, since these remain unaltered if α_i be replaced by $-\beta_j$, γ_i by δ_j, β_i by $-\alpha_j$, and δ_i by γ_j.

Now we have

$$\frac{\partial \phi}{\partial \lambda} = \gamma_0 \frac{\partial u_1(b)}{\partial \lambda} + \delta_0 \frac{\partial z_1(b)}{\partial \lambda} + \gamma_1 \frac{\partial u_0(b)}{\partial \lambda} + \delta_1 \frac{\partial z_0(b)}{\partial \lambda} + \gamma_0' u_1(b) + \gamma_1' u_0(b), \tag{18}$$

where the accent indicates differentiation with respect to λ, and the arguments λ, μ are omitted in $u_0, z_0, u_1, z_1, \gamma_0, \gamma_1, \delta_0, \delta_1$. If we differentiate the first two equations (17) for $u = u_0$ with regard to λ, we obtain

$$\frac{d}{dx}\left[K(x,\mu)\frac{\partial u_0'}{\partial \lambda}\right] - G(x,\lambda)\frac{\partial u_0}{\partial \lambda} = G_\lambda(x,\lambda)u_0.^* \tag{19}$$

We may also notice that $\partial u_0/\partial\lambda$ satisfies the initial conditions

$$\frac{\partial u_0(a)}{\partial \lambda} = 0, \qquad \frac{\partial z_0(a)}{\partial \lambda} = K(a,\mu)\frac{\partial}{\partial \lambda}u_0'(a) = \alpha_0'(\lambda).$$

The equation (19) is a linear, non-homogeneous equation of the second order. If the right hand side is replaced by zero, u_0 and u_1 are two linearly independent solutions of the resulting homogeneous equation. Lagrange's Method of Variation of Parameters yields as the general solution

$$\frac{\partial u_0(x)}{\partial \lambda} = \int_a^x \frac{u_0(x)u_1(\xi) - u_1(x)u_0(\xi)}{u_0(\xi)u_1'(\xi) - u_1(\xi)u_0'(\xi)} \frac{G_\lambda(\xi)u_0(\xi)}{K(\xi,\mu)} d\xi + C_0(\lambda)u_0(x) + C_1(\lambda)u_1(x),$$

which, after simplifying by (9) and solving for C_i to satisfy the initial conditions, becomes

$$\left.\frac{\partial u_0(x)}{\partial \lambda}\right]_{\mu=\lambda} = -\int_a^x [u_0(x)u_1(\xi) - u_1(x)u_0(\xi)]G_\lambda(\xi)u_0(\xi)d\xi - \beta_1\alpha_0' u_0(x) - \beta_0\alpha_0' u_1(x). \tag{20}$$

Also by differentiating as to x and multiplying by $K(x,\mu)$, we find

$$\left.\frac{\partial z_0(x)}{\partial \lambda}\right]_{\mu=\lambda} = -\int_a^x [z_0(x)u_1(\xi) - z_1(x)u_0(\xi)]G_\lambda(\xi)u_0(\xi)d\xi - \beta_1\alpha_0' z_0(x) - \beta_0\alpha_0' z_1(x). \tag{21}$$

In exactly similar fashion, we determine

$$\left.\frac{\partial u_1(x)}{\partial \lambda}\right]_{\mu=\lambda} = -\int_a^x [u_0(x)u_1(\xi) - u_1(x)u_0(\xi)]G_\lambda(\xi)u_1(\xi)d\xi + \beta_1\alpha_1' u_0(x) + \beta_0\alpha_1' u_1(x), \tag{22}$$

* $G_\lambda = \dfrac{\partial G(x,\lambda)}{\partial \lambda}$; $\quad K_\mu = \dfrac{\partial K(x,\mu)}{\partial \mu}$.

$$\left.\frac{\partial z_1(x)}{\partial \lambda}\right]_{\mu=\lambda} = -\int_a^x [z_0(x)u_1(\xi) - z_1(x)u_0(\xi)]G_\lambda(\xi)u_1(\xi)d\xi + \beta_1\alpha_1' z_0(x) + \beta_0\alpha_1' z_1(x). \tag{23}$$

By means of (18), (20), (21), and (22) we may write

$$\begin{aligned}\left.\frac{\partial\phi}{\partial\lambda}\right]_{\mu=\lambda} = &-\int_a^b \{M_0[u_0(b)]u_1(\xi) - M_0[u_1(b)]u_0(\xi)\}G_\lambda(\xi)u_1(\xi)d\xi \\ &+ \gamma_0' u_1(b) + \gamma_1' u_0(b) \\ &-\int_a^b \{M_1[u_0(b)]u_1(\xi) - M_1[u_1(b)]u_0(\xi)\}G_\lambda(\xi)u_0(\xi)d\xi \\ &+ \{M_0[u_0(b)]\beta_1\alpha_1' + M_0[u_1(b)]\beta_0\alpha_1' \\ &- M_1[u_0(b)]\beta_1\alpha_0' - M_1[u_1(b)]\beta_0\alpha_0'\}.\end{aligned}$$

We may write out $\partial\phi/\partial\mu$ from symmetry, and combining for $\mu = \lambda$ with $\partial\phi/\partial\lambda$, we obtain

$$\begin{aligned}\frac{d\phi}{d\lambda} = &-\int_a^b F[u_1(\xi), u_0(\xi)]G_\lambda(\xi)d\xi \\ &-\int_a^b F[u_1'(\xi), u_0'(\xi)]K_\lambda(\xi)d\xi + R,\end{aligned} \tag{24}$$

where

$$F[s, t] = M_0[u_0(b)]s^2 + \{M_1[u_0(b)] - M_0[u_1(b)]\}st - M_1[u_1(b)]t^2,$$

and

$$\begin{aligned}R = &M_0[u_0(b)](\alpha_1'\beta_1 - \alpha_1\beta_1') + M_1[u_0(b)](\alpha_1\beta_0' - \beta_1\alpha_0') \\ &+ M_0[u_1(b)](\alpha_1'\beta_0 - \alpha_0\beta_1') - M_1[u_1(b)](\alpha_0'\beta_0 - \alpha_0\beta_0') \\ &+ \gamma_1' u_0(b) + \delta_1' z_0(b) + \gamma_0' u_1(b) + \delta_0' z_1(b).\end{aligned}$$

If $\alpha_0 \cdots \delta_1$ are all constants, then $R \equiv 0$, and $d\phi/d\lambda$ reduces to the sum of two integrals, each of which has an integrand whose first factor is a quadratic form. The common discriminant of these forms is

$$D = \{M_1[u_0(b)] - M_0[u_1(b)]\}^2 + 4M_0[u_0(b)]M_1[u_1(b)].$$

From (10) we have

$$M_0[u_0(b)]M_1[u_1(b)] = -1 + M_0[u_1(b)]M_1[u_0(b)]$$

or

$$D = \{M_0[u_1(b)] + M_1[u_0(b)]\}^2 - 4.$$

The discriminant breaks up into two factors, one of which is precisely $\phi(\lambda)$ by (11). Hence at $\lambda = l$ the discriminant vanishes.

Now at a simple value, not all the coefficients of the quadratic forms can

vanish (see (10) and (13)); furthermore, u_0 and u_1 are linearly independent. Hence the first factors of the integrands will vanish only at isolated points of the interval $a < x \leqq b$ for $\lambda = l$. But either G_λ or K_λ is negative for at least one value of x in $a < x \leqq b$, and zero when not negative. Hence the sign of $d\phi/d\lambda$ is that of $M_1[u_1(b)]$ or $-M_0[u_0(b)]$ when $\alpha_0 \cdots \delta_1$ are constants.

Now suppose that $\alpha_0 \cdots \delta_1$ are functions of λ. Let us determine under what conditions R will have the same sign as that of the two integrals of (24). A necessary condition that the last line of R be expressible in the form

$$C_0 M_0[u_0] + D_0 M_1[u_0] + D_1 M_0[u_1] + C_1 M_1[u_1]$$

is that

$$\begin{aligned} \gamma_1' &= C_0 \gamma_0 + D_0 \gamma_1, \\ \delta_1' &= C_0 \delta_0 + D_0 \delta_1, \\ \gamma_0' &= D_1 \gamma_0 + C_1 \gamma_1, \\ \delta_0' &= D_1 \delta_0 + C_1 \delta_1, \end{aligned}$$

whence solving we have

$$\begin{aligned} C_0 &= \gamma_1' \delta_1 - \gamma_1 \delta_1', \qquad & C_1 &= \gamma_0 \delta_0' - \gamma_0' \delta_0, \\ D_0 &= \gamma_0 \delta_1' - \gamma_1' \delta_0, \qquad & D_1 &= \gamma_0' \delta_1 - \gamma_1 \delta_0'. \end{aligned}$$

Since $D_0 = -D_1$ by (5), and since the coefficients of $M_0[u_1(b)]$ and $M_1[u_0(b)]$ in the first two lines of R are equal for the same reason, we have

$$R = \mathcal{A} M_1[u_1] + \mathcal{B}\{M_0[u_1] - M_1[u_0]\} - \mathcal{C} M_0[u_0],$$

where

$$\begin{aligned} \mathcal{A} &= -\left\{ \beta_0^2 \frac{d}{d\lambda}\left(\frac{\alpha_0}{\beta_0}\right) + \delta_0^2 \frac{d}{d\lambda}\left(\frac{\gamma_0}{\delta_0}\right) \right\}, \\ \mathcal{B} &= (\alpha_0' \beta_1 - \alpha_1 \beta_0') + (\gamma_0' \delta_1 - \gamma_1 \delta_0'), \\ \mathcal{C} &= -\left\{ \beta_1^2 \frac{d}{d\lambda}\left(\frac{\alpha_1}{\beta_1}\right) + \delta_1^2 \frac{d}{d\lambda}\left(\frac{\gamma_1}{\delta_1}\right) \right\}. \end{aligned}$$

The coefficients $\mathcal{A}$ and $\mathcal{C}$ are positive or zero in consequence of (A), § 1. But we have

$$M_0[u_1] + M_1[u_0] = 2$$

at $\lambda = l$. Combining with (10) we find

$$M_0[u_1] - M_1[u_0] = \pm 2\sqrt{-M_0[u_0] M_1[u_1]}.$$

Let $u^2 = \pm M_1[u_1]$ and $v^2 = \mp M_0[u_0]$; then it is clear that

$$R = \pm[\mathcal{A}u^2 + 2\mathcal{B}uv + \mathcal{C}v^2].$$

The necessary and sufficient condition that this quadratic form be semi-definite is that

$$\mathcal{A}\mathcal{C} - \mathcal{B}^2 \geqq 0,$$

which may be written

$$(D') \quad \begin{vmatrix} \alpha_0\beta_0' - \beta_0\alpha_0' + \gamma_0\delta_0' - \delta_0\gamma_0' & \alpha_0\beta_1' - \beta_0\alpha_1' + \gamma_0\delta_1' - \delta_0\gamma_1' \\ \alpha_1\beta_0' - \beta_1\alpha_0' + \gamma_1\delta_0' - \delta_1\gamma_0' & \alpha_1\beta_1' - \beta_1\alpha_1' + \gamma_1\delta_1' - \delta_1\gamma_1' \end{vmatrix} \geqq 0,$$

since differentiation of (5) shows that the second element of the first row of this determinant is equal to $\mathcal{B}$.

The determinant in (D') may be written in the form

$$\begin{Vmatrix} \begin{vmatrix} \alpha_0 & \beta_0 \\ \alpha_0' & \beta_0' \end{vmatrix} + \begin{vmatrix} \gamma_0 & \delta_0 \\ \gamma_0' & \delta_0' \end{vmatrix} & \begin{vmatrix} \alpha_0 & \beta_0 \\ \alpha_1' & \beta_1' \end{vmatrix} + \begin{vmatrix} \gamma_0 & \delta_0 \\ \gamma_1' & \delta_1' \end{vmatrix} \\ \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_0' & \beta_0' \end{vmatrix} + \begin{vmatrix} \gamma_1 & \delta_1 \\ \gamma_0' & \delta_0' \end{vmatrix} & \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_1' & \beta_1' \end{vmatrix} + \begin{vmatrix} \gamma_1 & \delta_1 \\ \gamma_1' & \delta_1' \end{vmatrix} \end{Vmatrix}$$

$$= \begin{vmatrix} \alpha_0 & \beta_0 \\ \alpha_0' & \beta_0' \end{vmatrix}\begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_1' & \beta_1' \end{vmatrix} - \begin{vmatrix} \alpha_0 & \beta_0 \\ \alpha_1' & \beta_1' \end{vmatrix}\begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_1' & \beta_1' \end{vmatrix} + \begin{vmatrix} \gamma_0 & \delta_0 \\ \gamma_0' & \delta_0' \end{vmatrix}\begin{vmatrix} \gamma_1 & \delta_1 \\ \gamma_1' & \delta_1' \end{vmatrix}$$

$$- \begin{vmatrix} \gamma_0 & \delta_0 \\ \gamma_1' & \delta_1' \end{vmatrix}\begin{vmatrix} \gamma_1 & \delta_1 \\ \gamma_0' & \delta_0' \end{vmatrix} + \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_1' & \beta_1' \end{vmatrix}\begin{vmatrix} \gamma_0 & \delta_0 \\ \gamma_0' & \delta_0' \end{vmatrix} + \begin{vmatrix} \alpha_0 & \beta_0 \\ \alpha_0' & \beta_0' \end{vmatrix}\begin{vmatrix} \gamma_1 & \delta_1 \\ \gamma_1' & \delta_1' \end{vmatrix}$$

$$- \begin{vmatrix} \alpha_0 & \beta_0 \\ \alpha_1' & \beta_1' \end{vmatrix}\begin{vmatrix} \gamma_1 & \delta_1 \\ \gamma_0' & \delta_0' \end{vmatrix} - \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_0' & \beta_0' \end{vmatrix}\begin{vmatrix} \gamma_0 & \delta_0 \\ \gamma_1' & \delta_1' \end{vmatrix}.$$

But by expanding by Laplace's development we have

$$\begin{vmatrix} \alpha_0 & \beta_0 & \alpha_0 & \beta_0 \\ \alpha_1 & \beta_1 & \alpha_1 & \beta_1 \\ \alpha_0' & \beta_0' & \alpha_0' & \beta_0' \\ \alpha_1' & \beta_1' & \alpha_1' & \beta_1' \end{vmatrix} = 2\left\{ \begin{vmatrix} \alpha_0 & \beta_0 \\ \alpha_1 & \beta_1 \end{vmatrix}\begin{vmatrix} \alpha_0' & \beta_0' \\ \alpha_1' & \beta_1' \end{vmatrix} - \begin{vmatrix} \alpha_0 & \beta_0 \\ \alpha_0' & \beta_0' \end{vmatrix}\begin{vmatrix} \alpha_1' & \beta_1' \\ \alpha_1 & \beta_1 \end{vmatrix} \right.$$

$$\left. + \begin{vmatrix} \alpha_0 & \beta_0 \\ \alpha_1' & \beta_1' \end{vmatrix}\begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_0' & \beta_0' \end{vmatrix} \right\} = 0.$$

But since the system is self-adjoint,

$$\begin{vmatrix} \gamma_0 & \delta_0 \\ \gamma_1 & \delta_1 \end{vmatrix}\begin{vmatrix} \alpha_0' & \beta_0' \\ \alpha_1' & \beta_1' \end{vmatrix} = \begin{vmatrix} \alpha_0 & \beta_0 \\ \alpha_1' & \beta_1' \end{vmatrix}\begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_0' & \beta_0' \end{vmatrix} - \begin{vmatrix} \alpha_0 & \beta_0 \\ \alpha_0' & \beta_0' \end{vmatrix}\begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_1' & \beta_1' \end{vmatrix}.$$

Also

$$\begin{vmatrix} \alpha_0 & \beta_0 \\ \alpha_1 & \beta_1 \end{vmatrix}\begin{vmatrix} \gamma_0' & \delta_0' \\ \gamma_1' & \delta_1' \end{vmatrix} = \begin{vmatrix} \gamma_0 & \delta_0 \\ \gamma_1' & \delta_1' \end{vmatrix}\begin{vmatrix} \gamma_1 & \delta_1 \\ \gamma_0' & \delta_0' \end{vmatrix} - \begin{vmatrix} \gamma_0 & \delta_0 \\ \gamma_0' & \delta_0' \end{vmatrix}\begin{vmatrix} \gamma_1 & \delta_1 \\ \gamma_1' & \delta_1' \end{vmatrix}.$$

Hence

$$(D') \quad - \begin{vmatrix} \gamma_0 & \delta_0 \\ \gamma_1 & \delta_1 \end{vmatrix}\begin{vmatrix} \alpha_0' & \beta_0' \\ \alpha_1' & \beta_1' \end{vmatrix} - \begin{vmatrix} \alpha_0 & \beta_0 \\ \alpha_1 & \beta_1 \end{vmatrix}\begin{vmatrix} \gamma_0' & \delta_0' \\ \gamma_1' & \delta_1' \end{vmatrix} + \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_1' & \beta_1' \end{vmatrix}\begin{vmatrix} \gamma_0 & \delta_0 \\ \gamma_1' & \delta_1' \end{vmatrix}$$

$$+ \begin{vmatrix} \alpha_0 & \beta_0 \\ \alpha_0' & \beta_0' \end{vmatrix}\begin{vmatrix} \gamma_1 & \delta_1 \\ \gamma_1' & \delta_1' \end{vmatrix} - \begin{vmatrix} \alpha_0 & \beta_0 \\ \alpha_1' & \beta_1' \end{vmatrix}\begin{vmatrix} \gamma_1 & \delta_1 \\ \gamma_0' & \delta_0' \end{vmatrix} - \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_0' & \beta_0' \end{vmatrix}\begin{vmatrix} \gamma_0 & \delta_0 \\ \gamma_1' & \delta_1' \end{vmatrix} \geqq 0.$$

But this is the Laplace development by the first two columns of the determinant

$$-\begin{vmatrix} \alpha_0 & \beta_0 & \gamma_0 & \delta_0 \\ \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_0' & \beta_0' & \gamma_0' & \delta_0' \\ \alpha_1' & \beta_1' & \gamma_1' & \delta_1' \end{vmatrix},$$

which reduces (D') to (C).

Hence if (C) is satisfied, R, and also $d\phi/d\lambda$, has the same sign as $M_1[u_1(b)]$ or $-M_0[u_0(b)]$ at $\lambda = l$. If $\lambda = l$ is a simple characteristic value, it follows that $\phi(\lambda)$ changes sign at $\lambda = l$, having for $\lambda > l$ the same sign as $M_1[u_1(b)]$ or $-M_0[u_0(b)]$, and for $\lambda < l$ the opposite sign, at least if the functions $K, G, \alpha_0, \cdots, \delta_1$ have continuous derivatives with regard to λ. Thus Theorem I′ is proved.

The first part of Theorem II′ will be proved if we show that $d^2\phi/d\lambda^2$ is negative at a double value. The formula obtained above for $d\phi/d\lambda$ shows that $d\phi/d\lambda$ vanishes at a double value $\lambda = l$ in consequence of equations (13). Also, if we differentiate the second identity (10) twice as to λ, we obtain at such a value

$$\frac{\partial^2 M_0[u_1]}{\partial\lambda^2} + \frac{\partial^2 M_1[u_0]}{\partial\lambda^2} = \frac{\partial M_0[u_0]}{\partial\lambda}\frac{\partial M_1[u_1]}{\partial\lambda} - \frac{\partial M_0[u_1]}{\partial\lambda}\frac{\partial M_1[u_0]}{\partial\lambda}.$$

But the left-hand member of this equation is precisely $d^2\phi/d\lambda^2$ while the right-hand member can be evaluated explicitly by use of (20), (21), (22), (23). Hence at such a double value we have

$$\begin{aligned}\frac{d^2\phi}{d\lambda^2} = -\int_a^b\int_a^b [&f^2(\xi)g^2(\eta) + f_1^2(\xi)g_1^2(\eta) + f^2(\xi)g_1^2(\eta) + f_1^2(\eta)g^2(\xi) \\ &- f(\xi)g(\xi)f(\eta)g(\eta) - f_1(\xi)g_1(\xi)f_1(\eta)g_1(\eta) \\ &- 2f(\xi)g(\xi)f_1(\eta)g_1(\eta)]\,d\xi d\eta + \bar{R},\end{aligned}$$

where

$$f = u_0\sqrt{-G_\lambda}, \qquad f_1 = u_0'\sqrt{-K_\lambda},$$

$$g = u_1\sqrt{-G_\lambda}, \qquad g_1 = u_1'\sqrt{-K_\lambda},$$

$$\begin{aligned}\bar{R} = -(\mathcal{A}\mathcal{C} - \mathcal{B}^2) - (E_0 + F_0)&\left(-\int_a^b g^2(\xi)\,d\xi - \int_a^b g_1^2(\xi)\,d\xi\right) \\ &- (E_1 + F_1)\left(-\int_a^b f^2(\xi)\,d\xi - \int_a^b f_1^2(\xi)\,d\xi\right),\end{aligned}$$

$$E_i = \alpha_i'\beta_i - \alpha_i\beta_i', \qquad F_i = \gamma_i'\delta_i - \gamma_i\delta_i'.$$

Now by (A), § 1, the quantities G_λ, K_λ, E_i, F_i are negative or zero, so that $\bar{R}$ is negative or zero.

It remains to determine the sign of the integral terms in $d^2\phi/d\lambda^2$. By interchanging the variables of integration in the proper manner, these terms take the form

$$-\frac{1}{2}\int_a^b\int_a^b \{[f(\xi)g(\eta)-f(\eta)g(\xi)]^2+[f_1(\xi)g_1(\eta)-f_1(\eta)g_1(\xi)]^2 \\ +2[f(\xi)g_1(\eta)-f_1(\eta)g(\xi)]^2\}d\xi d\eta,$$

which is negative, since either f and g or f_1 and g_1 are linearly independent.

It is obvious, therefore, that $d^2\phi/d\lambda^2$ is negative at a double characteristic value. Hence $\phi(\lambda)$ preserves a negative sign at a double value.

Finally, we observe that a direct computation of $dM_0[u_0(b,\lambda)]/d\lambda$ and $dM_1[u_1(b,\lambda)]/d\lambda$ at such a double value, l, using (13), (20), (21), (22), (23) yields for $\lambda = l$,

$$\frac{dM_0[u_0(b,\lambda)]}{d\lambda}=\int_a^b f^2(\xi)d\xi+\int_a^b g^2(\xi)d\xi-(E_0+F_0),$$

$$\frac{dM_1[u_1(b,\lambda)]}{d\lambda}=-\int_a^b f_1^2(\xi)d\xi-\int_a^b g_1^2(\xi)d\xi+E_1+F_1.$$

These expressions are positive and negative, respectively, which proves the second part of Theorem II′.

We proceed to state and prove the above theorems in a more general form, after introducing the condition

$$(C)\qquad \begin{vmatrix} \alpha_0 & \beta_0 & \gamma_0 & \delta_0 \\ \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \Delta\alpha_0 & \Delta\beta_0 & \Delta\gamma_0 & \Delta\delta_0 \\ \Delta\alpha_1 & \Delta\beta_1 & \Delta\gamma_1 & \Delta\delta_1 \end{vmatrix} \leqq 0.$$

THEOREM I. *If* (A), (B), (C) *are satisfied in the vicinity of* $\lambda = l$, *a simple characteristic value,* $\phi(\lambda)$ *changes sign in such wise that*

$$\frac{\phi(l+\Delta\lambda)}{\Delta\lambda}=\frac{\Delta\phi}{\Delta\lambda}$$

has the same sign as $M_0[u_0(b,\lambda)]$ *or* $-M_1[u_1(b,\lambda)]$.

THEOREM II. *If* (A), (B), (C) *are satisfied in the vicinity of* $\lambda = l$, *a double characteristic value,* $\phi(\lambda)$ *preserves a negative sign, while* $M_0[u_0(b,\lambda)]$ *and* $-M_1]u_1(b,\lambda)]$ *change from positive to negative as* λ *increases through* l.

Suppose that Theorem I did not hold for the case where the derivatives fail to exist. For instance, suppose that $\phi(\lambda)$ did not have the specified sign at $\lambda = l+\Delta\lambda > l$.

Consider modified values of $K, G, \alpha_0 \cdots \delta_1$ as follows:

$$
(25)\qquad
\begin{aligned}
\bar{K}(x,\lambda) &= K(x,l) + \frac{(\lambda - l)}{\Delta\lambda}[K(x,l+\Delta\lambda) - K(x,l)],\\
\bar{G}(x,\lambda) &= G(x,l) + \frac{(\lambda - l)}{\Delta\lambda}[G(x,l+\Delta\lambda) - G(x,l)],\\
\bar{\alpha}_0(\lambda) &= \alpha_0(l) + \frac{(\lambda - l)}{\Delta\lambda}[\alpha_0(l+\Delta\lambda) - \alpha_0(l)],\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
\bar{\delta}_1(\lambda) &= \delta_1(l) + \frac{\lambda - l}{\Delta\lambda}[\delta_1(l+\Delta\lambda) - \delta_1(l)].
\end{aligned}
$$

The system with the modified coefficients will satisfy all the conditions as to derivatives and will coincide with the given system for $\lambda = l$ and for $\lambda = l + \Delta\lambda$. Let $\bar{u}_0$, $\bar{u}_1$ denote the two linearly independent solutions of the system (2), (6), whose coefficients are defined by (25). Let $\bar{\phi}(\lambda) = 0$ be the characteristic equation for this system. Now it is clear that $\phi(l) = \bar{\phi}(l) = 0$ and $\bar{\phi}(l + \Delta\lambda) = \phi(l + \Delta\lambda)$. The above conclusion holds for $\bar{\phi}(\lambda)$, so that at $\lambda = l$, $\bar{\phi}$ changes sign and for $\lambda > l$, $\bar{\phi}(\lambda)$ has the same sign as $\bar{M}_1[\bar{u}_1(b)]$ or $-\bar{M}_0[\bar{u}_0(b)]$. By hypothesis, however, $\phi(\lambda)$ does not have this sign at $\lambda = l + \Delta\lambda$. Hence $\bar{\phi}(\lambda)$ must vanish in the interval $(l, l + \Delta\lambda)$ for $\Delta\lambda$ arbitrarily small, i. e., $\bar{\phi}(l') = 0$, where $l < l' < l + \Delta\lambda$. But by Theorem I, for two successive values for which $\bar{\phi}(\lambda) = 0$, $\bar{M}_0[\bar{u}_0(b)]$ and $\bar{M}_1[\bar{u}_1(b)]$ must vanish in (l, l'):

$$\bar{M}_0[\bar{u}_0(b,\lambda_1)] = 0, \qquad \bar{M}_1[\bar{u}_1(b,\lambda_2)] = 0 \quad (l < \lambda_1, \lambda_2 \leqq l').$$

We infer that

$$M_0[u_0(b,l)] = 0, \qquad M_1[u_1(b,l)] = 0,$$

since $\bar{M}_0[\bar{u}_0(b,\lambda)]$ and $\bar{M}_1[\bar{u}_1(b,\lambda)]$ are continuous functionals of $\bar{K}$, $\bar{G}$, $\bar{\alpha}_0, \cdots, \bar{\delta}_1$ and since for $\Delta\lambda = 0$, these reduce to $K, G, \alpha_0, \cdots, \delta_1$ respectively.*

Combining these equations with the equation $\phi(l) = 0$ and (10), we find further that

$$M_1[u_0(b,l)] = 1, \qquad M_0[u_1(b,l)] = 1,$$

i. e., $\lambda = l$ is a double characteristic value contrary to hypothesis.

For $\lambda < l$, a similar discussion may be made.

In order to prove Theorem II, consider the system with the modified coefficients (25), which satisfies the condition as to derivatives required by Theorem II′. The values of $\bar{K}$, $\bar{G}$, $\bar{\alpha}_0, \cdots, \bar{\delta}_1$, coincide with those of K, G, $\alpha_0, \cdots, \delta_1$ respectively at the double value $\lambda = l$ and at $\lambda = l + \Delta\lambda$.

* Cf. Bôcher, these Transactions, vol. 3 (1902), p. 208.

Let $\bar{\phi}(\lambda) = 0$ denote the corresponding characteristic equation so that

$$\phi(l) = \bar{\phi}(l), \qquad \phi(l + \Delta\lambda) = \bar{\phi}(l + \Delta\lambda).$$

If $\phi(\lambda)$ does not preserve a negative sign in the vicinity of l, we may assume that $\phi(l + \Delta\lambda)$ is positive or zero. Hence $\phi(\lambda)$ must vanish for l' inasmuch as $\lambda = l$ is a double value for (25) and $d^2\bar{\phi}/d\lambda^2$ is negative at $\lambda = l$. At the first value l' of λ ($l' > l$) for which $\bar{\phi}(\lambda)$ vanishes, it is clear that $d\bar{\phi}/d\lambda \geqq 0$ since $\bar{\phi}(\lambda)$ is negative in the vicinity of l. Therefore, as we have seen, $\bar{M}_1[\bar{u}_1(b, \lambda)]$ is negative for λ greater than but nearly equal to l. Hence this function vanishes between l and l'. This leads to a contradiction for $\Delta\lambda$ sufficiently small just as in the proof of Theorem I.

Likewise if $M_0[u_0(b, \lambda)]$ or $M_1[u_1(b, \lambda)]$ have not the stated sign near a double value of λ, we consider $\bar{M}_0[\bar{u}_0(b, \lambda)]$ or $\bar{M}_1[\bar{u}_1(b, \lambda)]$ and prove that these functions vanish between l and $l + \Delta\lambda$ where $\Delta\lambda$ is arbitrarily small. Then, by allowing $\Delta\lambda$ to approach zero, we are led to a contradiction.

Thus Theorems I and II are established.

4. On conditions sufficient for uniqueness

We are now ready to state a theorem concerning the uniqueness of the characteristic values.

Theorem Concerning Uniqueness. *For a system* (1), (4), *satisfying* (A), (B), (C), *there exists one and only one characteristic value between every pair of characteristic numbers of the auxiliary sturmian system* (1), (3). *If* λ_0, λ_1, λ_2, $\cdots$ *are the ordered characteristic numbers of this system* (1), (3), *and* l_0, l_1, l_2, $\cdots$ *are the ordered characteristic numbers of the system* (1), (4) (*account being taken of their multiplicity*), *the following cases are possible*:

I_a: $\quad \mathcal{L}_1 < \lambda_0 \leqq l_0 < \lambda_1 < l_1 \leqq \lambda_2 \leqq l_2 < \lambda_3 < l_3 < \cdots \mathcal{L}_2,$

I_b: $\quad \mathcal{L}_1 < l_0 \leqq \lambda_0 \leqq l_1 < \lambda_1 < l_2 \leqq \lambda_2 \leqq l_3 < \cdots \mathcal{L}_2,$

II_a: $\quad \mathcal{L}_1 < l_0 < \lambda_0 < l_1 \leqq \lambda_1 \leqq l_2 < \lambda_2 < l_3 \leqq \lambda_3 \leqq \cdots \mathcal{L}_2,$

II_b: $\quad \mathcal{L}_1 < \lambda_0 < l_0 \leqq \lambda_1 \leqq l_1 < \lambda_2 < l_2 \leqq \lambda_3 \leqq l_3 < \cdots \mathcal{L}_2.$

The conditions for these cases are respectively:

I_a: $\quad M_1[u_0(b, \lambda_0)] > 0, \qquad \phi(\mathcal{L}_1 + \epsilon) > 0,$

I_b: $\quad M_1[u_0(b, \lambda_0)] > 0, \qquad \phi(\mathcal{L}_1 + \epsilon) < 0,$

II_a: $\quad M_1[u_0(b, \lambda_0)] < 0, \qquad \phi(\mathcal{L}_1 + \epsilon) > 0,$

II_b: $\quad M_1[u_0(b, \lambda_0)] < 0, \qquad \phi(\mathcal{L}_1 + \epsilon) < 0.$

Proof. Case I. If $M_1[u_0(b, \lambda_0)] > 0$, then by (16), ϕ is positive or zero

at $\lambda_0, \lambda_2, \lambda_4, \cdots$ and ϕ is negative at $\lambda_1, \lambda_3, \cdots$. Hence there exist at least two values (l_1, l_2) in each double interval $(\lambda_{2p-1}, \lambda_{2p+1})$ such that

$$\lambda_{2p-1} < l_1 \leqq \lambda_{2p} \leqq l_2 < \lambda_{2p+2}.$$

If there were additional values l_i in a double interval, there would be at least two more, or four in all, since $\phi(\lambda_{2p-1})$ and $\phi(\lambda_{2p+1})$ have the same sign. If we suppose that there is no double value, then at least two simple values must fall within $(\lambda_{2p-1}, \lambda_{2p})$ or $(\lambda_{2p}, \lambda_{2p+1})$ and at two successive such values $\Delta\phi/\Delta\lambda$ has opposite signs. However, by Theorem I of § 3, $\Delta\phi/\Delta\lambda$ would have the same sign as $-M_0[u_0(b, \lambda)]$ at both of these values. This is impossible, since the sign of $M_0[u_0(b, \lambda)]$ does not change in this interval. Suppose now that a double value exists, necessarily for $\lambda = \lambda_{2p}$ by (13). Since $\phi(\lambda_{2p-1})$ and $\phi(\lambda_{2p+1})$ are both negative and $\phi(\lambda)$ is negative near λ_{2p} by Theorem II, § 3, there will be, if there are other roots in $(\lambda_{2p-1}, \lambda_{2p+1})$, two simple roots lying in one of the intervals $(\lambda_{2p-1}, \lambda_{2p})$ or $(\lambda_{2p}, \lambda_{2p+1})$. But this has already been proved impossible.

To complete the discussion of Case I, it needs only to be shown that if $\phi(\mathcal{L}_1 + \epsilon) > 0$, we are led to a single value l_0 of l in the interval $(\mathcal{L}_1, \lambda_1)$ such that $\lambda_0 \leqq l_0 \leqq \lambda_1$, while if $\phi(\mathcal{L}_1 + \epsilon) < 0$, we are led to two roots l_0, l_1, such that $\mathcal{L}_1 < l_0 \leqq \lambda_0 \leqq l_1 < \lambda_1$. For then we have Case I_a and Case I_b respectively.

When $\phi(\mathcal{L}_1 + \epsilon) > 0$, suppose first that $\phi(\lambda_0) > 0$. There is one and only one root in (λ_0, λ_1) since $\Delta\phi/\Delta\lambda$ has one and the same sign at all such roots which are simple. And there is no root in $(\mathcal{L}_1, \lambda_0)$, for if there were *one* root, there would necessarily be *two* which is clearly impossible. Thus we have Case I_a.

The possibility $\phi(\lambda_0) = 0$ may be excluded. For in this case λ_0 would be a double value by (10), (11), and (16). Also $\phi(\lambda)$ is negative near λ_0 by Theorem II, § 3. Hence there must be a simple characteristic value for $\lambda < \lambda_0$ at which $\Delta\phi/\Delta\lambda$ is negative. Consequently by Theorem I, § 3, $M_0[u_0(b, \lambda)]$ is negative for this value. However, by Theorem II, § 3, $M_0[u_0(b, \lambda)]$ is positive near λ_0 for $\lambda < \lambda_0$, so that $M_0[u_0(b, \lambda)]$ changes sign for $\lambda < \lambda_0$, which is impossible.

When $\phi(\mathcal{L}_1 + \epsilon) < 0$, we may apply precisely the same reasoning to the interval $(\mathcal{L}_1, \lambda_1)$, as we have done to $(\lambda_{2p-1}, \lambda_{2p+1})$ to prove that there are two roots l_0, l_1 restricted as in Case I_b.

Case II. If $M_1[u_0(b, \lambda)] < 0$, then by (16), ϕ is negative at $\lambda_0, \lambda_2, \cdots$ and ϕ is positive or zero at $\lambda_1, \lambda_3, \cdots$. It follows exactly as in Case I, that there exist two values l', l'', in each double interval $(\lambda_{2p}, \lambda_{2p+2})$ such that

$$\lambda_{2p} < l' \leqq \lambda_{2p+1} \leqq l'' < \lambda_{2p+2}.$$

If $\phi(\mathcal{L}_1 + \epsilon) < 0$, we have Case II_b, since there are no characteristic values in $(\mathcal{L}_1, \lambda_0)$. If $\phi(\mathcal{L}_1 + \epsilon) > 0$, we have only the possibility of a single characteristic value $l_0 < \lambda_0$, and Case II_a arises.

On the basis of the foregoing Uniqueness Theorem, we can at once state the following *Oscillation Theorem* in the form of a corollary.

COROLLARY. *If the system* (1), (4) *satisfies* (A), (B), (C), *the characteristic function* $u_p(x)$ *belonging to the pth characteristic value will vanish* $p-2$, $p-1$, p, $p+1$, *or* $p+2$ *times on* $a < x < b$.

Proof. For the system (1), (14), the auxiliary sturmian characteristic values, λ_p, are given by

$$M_0[u_0(b, \lambda)] = 0.$$

The Uniqueness Theorem states that l_p lies on the interval $(\lambda_{p-1}, \lambda_{p+1})$. Now by Sturm's Oscillation Theorem for the system (1), (14), $u_0(x, \lambda_p)$ vanishes exactly p times on $a < x < b$ and so $u_0(x, \lambda)$ will vanish $p-1$, p, or $p+1$ times on $a < x < b$ for $\lambda_{p-1} \leqq \lambda \leqq \lambda_{p+1}$, or $u_0(x, l_p)$ will have $p-1$, p, or $p+1$ zeros on this same interval of the x-axis. But the zeros of $u_p(x)$ and $u_0(x, l_p)$ separate one another or else coincide, so that $u_p(x)$ will have on $a < x < b$ either $p-2$, $p-1$, p, $p+1$, or $p+2$ zeros.

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