ON BOUNDARY VALUE PROBLEMS IN LINEAR DIFFERENTIAL EQUATIONS IN GENERAL ANALYSIS*

BY

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In a paper On a theory of linear differential equations in general analysis,† we considered the solution of the general linear differential equations

\begin{align*}
(A) & \quad M_1(\eta) = D\eta - \alpha - J\alpha\eta = 0, \\
(B) & \quad M_2(\eta) = D\eta - J\alpha\eta = 0, \\
(C) & \quad M_2(\eta) - \alpha_0 = D\eta - J\alpha\eta - \alpha_0 = 0,
\end{align*}

and their adjoints

\begin{align*}
(A') & \quad N_1(\eta) = D\eta + \alpha + J\eta\alpha = 0, \\
(B') & \quad N_2(\eta) = D\eta + J\eta\alpha = 0, \\
(C') & \quad N_2(\eta) - \alpha_0 = D\eta + J\eta\alpha - \alpha_0 = 0.
\end{align*}

In the case of equations (A), (B), and (C) we found‡ that the general solution of class $\mathcal{G}'$ is expressible in the form

$$\eta = \kappa + J\eta_0\kappa + \eta_1,$$

in which $\eta_1$ is a particular solution of the equation in question, $\eta_0$ is a solution of equation (A) whose Fredholm determinant is not zero, and $\kappa$ is any function of the class $\mathcal{G}$. Similarly, the general solution of equations (A'), (B'), and (C') has the form

$$\eta = \kappa + J\eta_0\kappa + \eta_1.$$

We found further that there is a unique solution of each of these equations which satisfies an initial condition of the form

$$\eta(x_1) = \kappa_0,$$

where $x_1$ is any element of $\mathcal{X}$, and $\kappa_0$ any function of $\mathcal{K}$.

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* Presented to the Society, April 6, 1917.
† These Transactions, vol. 18 (1917), pp. 73–96. This paper will be referred to as I in the sequel. With the exception of an additional condition on $J$ in §§ 2 and 3, the postulates and properties of the classes $\mathcal{M}'$, $\mathcal{M}'$, $\mathcal{G}$, $\mathcal{G}'$, and $\mathcal{R}$, and the functions $\alpha$, $\eta$, and $\kappa$, and the operators $J$ and $D$ in the present paper are the same as in I.
‡ Cf. I, loc. cit., pp. 84, 86, 87.

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In Section 1 of this paper we consider the solutions of the equations (A), (B), and (C), and (A'), (B'), and (C'), whose values at two or more elements of \( \mathfrak{X} \) satisfy a linear relation, i.e., a linear boundary condition. Section 2 is devoted to the definition of adjoint systems of boundary conditions. Section 3 derives the usual theorems concerning the interrelations of solutions of adjoint systems. They are in the main similar to those derived in the first three sections of the paper by Bôcher on *Applications and generalizations of the concept of adjoint systems* and include them as special cases. We have not given particular instances of the theory developed in the following pages. It is an easy matter to construct some of the more important ones along the lines outlined in §§12–14 of I.† We might remark, however, that if we let \( \mathfrak{P}' = \mathfrak{P}'' = [0 \leq p \leq 1] \) and \( \mathfrak{M}' = \mathfrak{M}'' = \) the class of all continuous functions on the interval \((0, 1)\) and the operator \( J \) be the definite integral \( \int_0^1 dp \), we have below a theory of linear boundary value problems for linear integro-differential equations.

1. **Boundary conditions.** The boundary conditions which we consider are linear in the values of the function \( \eta \) at two points \( x_1 \) and \( x_2 \) of the class \( \mathfrak{X} \). The relations are further chosen so that the substitution of the general solution of the differential equations (A), (B), or (C) reduces the determination of the function \( \kappa \) to the solution of a linear general integral equation of the second or Fredholm type, i.e., one to which the Fredholm-Moore‡ theory is applicable. Such boundary conditions have the form§

\[
S(\eta) = c\eta(x_1) + J_1 \eta(x_1) + d\eta(x_2) + J_2 \eta(x_2) = 0,
\]

\[
S(\eta) = \sigma_0,
\]

in which \( \sigma_1, \sigma_2, \) and \( \sigma_0 \) are functions of the class \( \mathfrak{S} \), and \( c \) and \( d \) are constants for which \( c + d \neq 0 \). By dividing by \( c + d \), we get equivalent conditions with \( c + d = 1 \). We assume in the sequel that \( c \) and \( d \) satisfy this relation. In so far as the expression \( S(\eta) + \sigma_1 + \sigma_2 \) is of frequent occurrence, we denote it by \( S_0(\eta) \).

In a similar way, we consider relative to the adjoint differential equations \( (A'), (B'), \) and \( (C') \) boundary conditions of the form:

\[
T(\tilde{\eta}) = c\tilde{\eta}(x_1) + J_1 \tilde{\eta}(x_1) \tau_1 + d\tilde{\eta}(x_2) + J_2 \tilde{\eta}(x_2) \tau_2 = 0,
\]

\[
T(\tilde{\eta}) = \tau_0,
\]

† Loc. cit., pp. 87–96.
§ If the classes \( \mathfrak{P}' = \mathfrak{P}'' = \{1, 2, \ldots, n\} \) and \( J = \Sigma \), these conditions reduce to the boundary conditions considered by Bounitzky: Journal de Mathématiques, ser. 6, vol. 5 (1909), p. 68.
where $\tau_1, \tau_2,$ and $\tau_0$ are functions of the class $\mathcal{R},$ and $c + d = 1.$ We set

$$T_0(\tilde{\eta}) = T(\tilde{\eta}) + \tau_1 + \tau_2.$$ 

The following propositions are easily obtained by making the required substitutions and rearranging the terms properly.

1. $S(\eta_1 + \eta_2) = S(\eta_1) + S(\eta_2),$ 
2. $S(J\eta \kappa) = JS(\eta) \kappa,$ 
3. $S(\kappa + J\eta \kappa) = \kappa + JS_0(\eta) \kappa,$ 
4. $S_0(\eta_1 + \eta_2) = S_0(\eta_1) + S(\eta_2),$ 
5. $S_0(\kappa + \eta + J\eta \kappa) = \kappa + S_0(\eta) + JS_0(\eta) \kappa,$ 

$(1')$ $T(\tilde{\eta}_1 + \tilde{\eta}_2) = T(\tilde{\eta}_1) + T(\tilde{\eta}_2),$

$(2')$ $T(J\kappa \tilde{\eta}) = J\kappa T(\tilde{\eta}),$

$(3')$ $T(\kappa + J\kappa \tilde{\eta}) = \kappa + J\kappa T_0(\tilde{\eta}),$

$(4')$ $T_0(\tilde{\eta}_1 + \tilde{\eta}_2) = T_0(\tilde{\eta}_1) + T(\tilde{\eta}_2),$

$(5')$ $T_0(\kappa + \tilde{\eta} + J\kappa \tilde{\eta}) = \kappa + T_0(\tilde{\eta}) + J\kappa T_0(\tilde{\eta}).$

These propositions immediately give rise to the following theorems relating to the systems

1. $M_1(\eta) = D\eta - \alpha - J\alpha \eta,$ 
2. $M_2(\eta) = D\eta - J\alpha \eta = 0,$ 
3. $M_2(\eta) = \alpha_0,$

1'. $N_1(\tilde{\eta}) = D\tilde{\eta} + \alpha + J\tilde{\eta} \alpha,$

2'. $N_2(\tilde{\eta}) = D\tilde{\eta} + J\tilde{\eta} \alpha = 0,$

3'. $N_2(\eta) = \alpha_0,$

Theorem I. A necessary and sufficient condition that the system (1) [(1')] has a solution is that the Fredholm determinant of $S_0(\eta_0) [T_0(\eta_0)]$ be different from zero, $\eta_0 [\tilde{\eta}_0]$ being a solution of equation (A) [(A')] whose Fredholm determinant is not zero.

For if $\eta_0$ is a particular solution of $M_1(\eta) = 0,$ whose Fredholm determinant is not zero, the general solution can be written

$$\eta = \kappa + \eta_0 + J\eta_0 \kappa,$$

and so $\kappa$ must satisfy the equation

$$S_0(\eta) = \kappa + S_0(\eta_0) + JS_0(\eta_0) \kappa = 0.$$
This has the form of a reciprocal relation. It has a solution and one only if the Fredholm determinant of \( S_0(\eta_0) \) is not zero.\(^*\)

If \( S^{-1}(\eta_0) \) is the reciprocal of \( S_0(\eta_0) \), then we have the

**Corollary.** If a solution \( \eta \) of class \( S' \) of the system (1) exists, it has the form

\[
\eta = S^{-1}(\eta_0) + \eta_0 + J\eta_0 S^{-1}(\eta_0).
\]

Denoting, for convenience, by \( \eta_0 [\hat{\eta}_0] \) a solution of equation (A) [(A')] whose Fredholm determinant is not zero, we get in a similar way by using Propositions (3) [(3')]:

**Theorem II.** A necessary and sufficient condition that there exist a solution of the system (2) [(2')] which is not identically zero, is that the Fredholm determinant of \( S_0(\eta_0) [T_0(\hat{\eta}_0)] \) be zero. If \( \mu_1, \ldots, \mu_n [\mu''_1, \ldots, \mu''_n] \) are a complete set of linearly independent solutions of

\[
\mu' + JS_0(\eta_0)\mu' = 0, \quad [\mu'' + JT_0(\eta_0) = 0],
\]

then the general solution of the system (2) [(2')] can be written

\[
\eta = \sum_{m=1}^{n} (\mu'_m + J\eta_0 \mu'_m) \mu''_m, \quad \hat{\eta} = \sum_{m=1}^{n} \mu''_m (\mu''_m + J\mu''_m \hat{\eta}_0),
\]

where \( \mu''_m [\mu'_m] \) are any \( n \) functions of the class \( M'' [M'] \).

In such a case, the system (2) [(2')] is said to have \( n \)-fold compatibility.

Using propositions (1) and (3) [(1') and (3')], and denoting by \( \eta_1 [\hat{\eta}_1] \) a particular solution of equation (C) [(C')], we have

**Theorem III.** The system (3) [(3')] has a unique solution if the Fredholm determinant of \( S_0(\eta_0) [T_0(\hat{\eta}_0)] \) is not zero. If this determinant is zero, then a necessary and sufficient condition for the existence of a solution of this system is that

\[
J_\kappa (S(\eta_1) - \sigma_0) = 0, \quad [J_\kappa (T(\hat{\eta}_1) - \tau_0) = 0],
\]

for every solution \( \kappa \) of the homogeneous equation

\[
\kappa + J_\kappa S_0(\eta_0) = 0, \quad [\kappa + J_\kappa T_0(\hat{\eta}_0) = 0].
\]

A similar set of propositions and theorems can be derived if the boundary conditions are linear relations in the values of the solution of the differential equations at \( n \) points of \( \mathcal{X} : x_1, \ldots, x_n \). For the equation (B) the conditions take the form

\[
S(\eta) = \sum_{m=1}^{n} (c_m \eta(x_m) + J\sigma_m \eta(x_m)) = 0, \quad S(\eta) = \sigma_0,
\]

where \( \sigma_1, \ldots, \sigma_n \), and \( \sigma_0 \) are any functions of \( \mathcal{X} \) and \( c_1, \ldots, c_n \) a set of constants for which

\[
c_1 + c_2 + \cdots + c_n = 1.\]

2. On adjoint systems. Before taking up the definition of adjoint systems, it will be necessary to add a further postulate on the operator $J$. In this and the succeeding paragraphs, we shall assume that $J$ has the following property:

If $J\mu'' \mu' = 0$ for every $\mu'$ of the class $\mathcal{M}'$, then $\mu'' = 0$,

and

if $J\mu'' \mu' = 0$ for every $\mu''$ of the class $\mathcal{M}''$, then $\mu' = 0$.

This property corresponds to the definite property $P_0$ of Moore* when $\mathcal{B}' = \mathcal{B}'' = \mathcal{B}$, and $\mathcal{M}' = \mathcal{M}'' = \mathcal{M}$. We shall therefore call it the definite property $P_0$. It is equivalent to the following property, which is the form in which we shall have occasion to apply it.

If $Jk\kappa_0 = 0$ or $Jk_\kappa \kappa = 0$ for every $\kappa$ of the class $\mathcal{B}$, then $\kappa_0 = 0$.

We observe that $J$ has this definite property in the finite case, and also the instances $(a)$, $(b)$, $(c)$ but not $(d)$ of §.†

Consider now the system

$$M_2(\eta) = D\eta - J\alpha \eta,$$

$$S_1(\eta) = c_1 \eta(x_1) + J\sigma_{11} \eta(x_1) + d_1 \eta(x_2) + J\sigma_{12} \eta(x_2).$$

We assume that the expression $S_1(\eta)$ is what we shall call linearly independent, i.e., there exists another linear combination of $\eta(x_1)$ and $\eta(x_2)$

$$S_2(\eta) = c_2 \eta(x_1) + J\sigma_{21} \eta(x_1) + d_2 \eta(x_2) + J\sigma_{22} \eta(x_2),$$

with $c_1 d_2 - c_2 d_1 \neq 0$, such that the equations $S_1(\eta) = 0$ and $S_2(\eta) = 0$ have the unique solution

$$\eta(x_1) = 0 \quad \text{and} \quad \eta(x_2) = 0.‡$$

We can always determine the constants $a$ and $b$ with $b \neq 0$ so that in $S'_2 = aS_1 + bS_2$,

$$c'_2 = c_1 - 1, \quad d'_2 = 2 - c_1,$$

i.e.,

$$c'_2 + d'_2 = 1, \quad c_1 d'_2 - c'_2 d_1 = 1.$$

Evidently $S_1$ and $S'_2$ will be completely equivalent to $S_1$ and $S_2$, and we shall assume in the sequel that in the $S_2$ chosen, $c_2$ and $d_2$ have the character of $c'_2$ and $d'_2$.

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‡ If $\mathcal{B}' = \mathcal{B}'' = (1, \cdots, n)$, and $J = \sum_{\kappa=1}^{n}$ then this condition actually reduces to the linear independence of the boundary conditions $S_1$. A discussion of the above definition for the case of linear integral expressions by L. J. Rouse will appear shortly.
On account of the assumption relative to $S_1$ and $S_2$, it is possible to solve the equations

\begin{align*}
  c_1 \eta(x_1) + J\sigma_{11} \eta(x_1) + d_1 \eta(x_2) + J\sigma_{12} \eta(x_2) &= S_1, \\
  c_2 \eta(x_1) + J\sigma_{21} \eta(x_1) + d_2 \eta(x_2) + J\sigma_{22} \eta(x_2) &= S_2,
\end{align*}

for $\eta(x_1)$ and $\eta(x_2)$ in terms of $S_1$ and $S_2$. If we substitute in the expression

\[
  J\tilde{\eta}(x_2) \eta(x_2) - J\tilde{\eta}(x_1) \eta(x_1)
\]

and collect the coefficients of $S_1$ and $S_2$, these coefficients will be linear expressions in $\eta(x_1)$ and $\eta(x_2)$ of the form

\begin{align*}
  c_1 \eta(x_1) + J\sigma_{11} \eta(x_1) + d_1 \eta(x_2) + J\sigma_{12} \eta(x_2) &= T_1(\eta), \\
  c_2 \eta(x_1) + J\sigma_{21} \eta(x_1) + d_2 \eta(x_2) + J\sigma_{22} \eta(x_2) &= T_2(\eta).
\end{align*}

We therefore have the identity

\[
  J\tilde{\eta}(x_2) \eta(x_2) - J\tilde{\eta}(x_1) \eta(x_1) = JT_1(\eta) S_2(\eta) - JT_2(\eta) S_1(\eta),
\]

and we note that on account of the condition $J^{\rho}$, the $T_1(\eta)$ and $T_2(\eta)$ will be uniquely defined, if $S_1$ and $S_2$ are given as functions of $\eta(x_1)$ and $\eta(x_2)$.

We call the expression $N_2(\eta) = D\tilde{\eta} + J\tilde{\eta} \alpha$ together with the $T_1(\eta)$ so defined an adjoint system of system (1).*

In order to obtain the relations between the constants and functions in $S_1$, $S_2$ and $T_1$, $T_2$, we can proceed as outlined in the definition. The same relations result however if we equate the coefficients in the defining identity. This yields

\begin{align*}
  c_1' c_2 - c_2' c_1 &= -1, \\
  d_1' c_2 - d_2' c_1 &= 0, \\
  c_1' d_2 - c_2' d_1 &= 0, \\
  d_1' d_2 - d_2' d_1 &= 1;
\end{align*}

from which we conclude that

\begin{align*}
  c_1' &= d_1, \\
  c_2' &= c_1, \\
  c_2' &= d_2, \\
  d_1' &= c_2.
\end{align*}

We find further

\begin{align*}
  d_1 \sigma_{21} - d_2 \sigma_{11} + c_2 \tau_{11} - c_1 \tau_{21} + J(\tau_{11} \sigma_{21} - \tau_{21} \sigma_{11}) &= 0, \\
  d_1 \sigma_{22} - d_2 \sigma_{12} + d_2 \tau_{11} - d_1 \tau_{21} + J(\tau_{11} \sigma_{22} - \tau_{21} \sigma_{12}) &= 0,
\end{align*}

*This definition is a generalization of the definition due to Birkhoff, these Transactions, vol. 9 (1908), p. 173. Bounitzky (loc. cit., p. 73) gives a definition of adjoint which may be generalized as follows: $S_1(\eta)$ and $T_1(\tilde{\eta})$ are adjoint if for every $\eta(x_1)$, $\eta(x_2)$, $\eta(x_1)$, $\eta(x_2)$, for which $S_1(\eta) = 0$ and $T_1(\tilde{\eta}) = 0$, we have

\[
  J\tilde{\eta}(x_2) \eta(x_2) - J\tilde{\eta}(x_1) \eta(x_1) = 0.
\]

If $S_1$ and $T_1$ are adjoint according to our definition they will also be according to the Bounitzky definition, and conversely on account of $J^{\rho}$. The Bounitzky definition does not however seem to lend itself so readily to the derivation of the results of this section and the next.
These equalities express the fact that the system of kernels
\[ c_1 \tau_{21} - c_2 \tau_{11}, \quad d_1 \tau_{21} - d_2 \tau_{11}, \]
\[ c_2 \tau_{12} - c_1 \tau_{22}, \quad d_2 \tau_{12} - d_1 \tau_{22}, \]
are the reciprocals of the system of kernels
\[ d_2 \sigma_{11} - d_1 \sigma_{21}, \quad d_2 \sigma_{12} - d_1 \sigma_{22}, \]
\[ c_1 \sigma_{21} - c_2 \sigma_{11}, \quad c_1 \sigma_{22} - c_2 \sigma_{12}. \]
If we add the last four equations, we get the following symmetrical relation
\[ \sigma_{22} + \sigma_{21} + \tau_{11} + \tau_{12} + J (\tau_{11} + \tau_{12}) (\sigma_{22} + \sigma_{21}) \]
\[ = \sigma_{11} + \sigma_{12} + \tau_{21} + \tau_{22} + J (\tau_{21} + \tau_{22}) (\sigma_{11} + \sigma_{12}). \]
With the aid of these relations we verify without much difficulty that
\[ T_{01}(\hat{\eta}) + S_{02}(\eta) + JT_{01}(\hat{\eta})S_{02}(\eta) - T_{02}(\hat{\eta}) - S_{01}(\eta) \]
\[ - JT_{02}(\hat{\eta})S_{01}(\eta) = \hat{\eta}(x_2) + \eta(x_2) + J\hat{\eta}(x_2)\eta(x_2) \]
\[ - \hat{\eta}(x_1) - \eta(x_1) - J\hat{\eta}(x_1)\eta(x_1), \]
where
\[ T_{01}(\hat{\eta}) = T_1(\hat{\eta}) + \tau_{11} + \tau_{12}, \quad T_{02}(\hat{\eta}) = T_2(\hat{\eta}) + \tau_{21} + \tau_{22}, \]
\[ S_{01}(\eta) = S_1(\eta) + \sigma_{11} + \sigma_{12}, \quad S_{02}(\eta) = S_2(\eta) + \sigma_{21} + \sigma_{22}. \]
We therefore have
\[ \text{Theorem I. If } S_1(\eta) \text{ and } T_1(\hat{\eta}) \text{ are adjoint, i.e., if we have identically} \]
\[ J\hat{\eta}(x_2)\eta(x_2) - J\hat{\eta}(x_1)\eta(x_1) = JT_1S_2 - JT_2S_1, \]
then we also have identically
\[ \hat{\eta}(x_2) + \eta(x_2) + J\hat{\eta}(x_2)\eta(x_2) - \hat{\eta}(x_1) - \eta(x_1) - J\hat{\eta}(x_1)\eta(x_1) \]
\[ = T_{01} + S_{02} + JT_{01}S_{02} - T_{02} - S_{01} - JT_{02}S_{01}. \]
The same relations between the \( \sigma \) and \( \tau \) in the light of their reciprocal character give
\[ \text{Theorem II. If } T_1 \text{ and } T_2 \text{ are adjoint to } S_1 \text{ and } S_2 \text{ then from} \]
\[ T_1(\hat{\eta}) = 0 \quad \text{and} \quad T_2(\hat{\eta}) = 0 \]
it follows uniquely that
\[ \hat{\eta}(x_1) = 0 \quad \text{and} \quad \hat{\eta}(x_2) = 0, \]
i.e., \( T_1(\hat{\eta}) \) is also linearly independent.
The element of arbitrariness which enters into the definition of the adjoint
$T_1$ of $S_1$ is taken care of in the following

**Theorem III.** If $S_2$ be replaced by any other $S'_2$ which has the same character
as $S_2$, and if we denote by $T_1$ and $T'_1$ the corresponding adjoint expressions, then
there exist functions $\kappa_1$ and $\kappa'_1$ of the class $\mathcal{R}$ such that

$$T_1 = T'_1 + JT'_1 \kappa_1$$
and

$$T'_1 = T_1 + JT_1 \kappa'_1,$$

i. e., $T_1$ and $T'_1$ are essentially equivalent.

For if we solve the relations

$$S_1 = c_1 \eta(x_1) + J\sigma_{11} \eta(x_1) + d_1 \eta(x_2) + J\sigma_{12} \eta(x_2),$$

$$S_2 = c_2 \eta(x_1) + J\sigma_{21} \eta(x_1) + d_2 \eta(x_2) + J\sigma_{22} \eta(x_2)$$

for $\eta(x_1)$ and $\eta(x_2)$, we find

$$\eta(x_1) = d_2 S_1 - d_1 S_2 + J\tau_{21} S_1 - J\tau_{11} S_2,$$

$$\eta(x_2) = c_1 S_2 - c_2 S_1 - J\tau_{22} S_1 + J\tau_{12} S_2.$$

Substituting these values in

$$S_2' = c_2 \eta(x_1) + J\sigma'_{21} \eta(x_1) + d_2 \eta(x_2) + J\sigma'_{22} \eta(x_2),$$

we find

$$S'_2 = S_2 + J (k_2 S_1 + \kappa_1 S_2),$$

where $\kappa_1$ and $\kappa_2$ are functions of the class $\mathcal{R}$, depending on $\sigma'_{21}, \sigma'_{22}, \tau_{11}, \tau_{12},$
$\tau_{21}$, and $\tau_{22}$. We therefore have

$$J\tilde{\eta}(x_2) \eta(x_2) - J\tilde{\eta}(x_1) \eta(x_1) = JT'_1 S'_2 - JT'_1 S_1$$

$$= JT'_1 (S_2 + Jk_2 S_1 + J\kappa_1 S_2) - JT'_1 S_1$$
$$= J (T'_1 + JT'_1 \kappa_1) S_2 - J (T'_2 - JT'_1 \kappa_2) S_1;$$

from which on account of the uniqueness of $T_1$ it follows that

$$T_1 = T'_1 + JT'_1 \kappa_1.$$

In a similar way we obtain the other relation of the theorem.

3. On the solutions of adjoint systems. Before taking up the relations
which exist between solutions of adjoint systems we note the following lemmas.

**Lemma I.** If $\eta$ is a solution of equation $(A)$ and $\tilde{\eta}$ a solution of the adjoint
equation $(A')$, then we have for every set of adjoint expressions $S_1, S_2, T_1,$ and $T_2$

$$T_{01}(\tilde{\eta}) + S_{02}(\eta) + JT_{01}(\tilde{\eta}) S_{02}(\eta) = T_{02}(\tilde{\eta}) + S_{01}(\eta) + JT_{02}(\tilde{\eta}) S_{01}(\eta).$$
For by Theorem IV of § 11 of I,* we have for every pair of solutions \( \eta \) and \( \hat{\eta} \) of the adjoint equations \((A)\) and \((A')\)

\[
\hat{\eta}(x_2) + \eta(x_2) + J\hat{\eta}(x_2)\eta(x_2) = \hat{\eta}(x_1) + \eta(x_1) + J\hat{\eta}(x_1)\eta(x_1).
\]

Our lemma is then an immediate consequence of Theorem I of the preceding section.

**Lemma II.** If \( \eta \) is any solution of the system \((2)\) \[ (2') \]

\[
M_2(\eta) = 0, \quad S_1(\eta) = 0, \quad [N_2(\hat{\eta}) = 0, \quad T_1(\hat{\eta}) = 0],
\]

then \( S_2(\eta) \ [T_2(\hat{\eta})] \) is a solution of the equation

\[
S_2(\eta) + JT_{01}(\hat{\eta})S_2(\eta) = 0, \quad [T_2(\hat{\eta}) + JT_2(\hat{\eta})S_01(\eta) = 0],
\]

where \( \hat{\eta} \ [\eta] \) is any solution of equation \((A')\) \[ (A) \].

Let \( \eta \) be a solution of system \((2)\). Then if \( \eta_0 \) is any solution of equation \((A)\), \( \eta_0 + \eta \) is also a solution of equation \((A)\). Further by proposition (4) of § 1, we have

\[
S_{01}(\eta_0 + \eta) = S_{01}(\eta_0) + S_1(\eta) = S_{01}(\eta_0),
\]

\[
S_{02}(\eta_0 + \eta) = S_{02}(\eta_0) + S_2(\eta).
\]

Applying Lemma I to the solutions \( \eta_0 \) and \( \eta_0 + \eta \), we have for every solution \( \hat{\eta} \) of the adjoint equation \((A')\)

\[
T_{01}(\hat{\eta}) + S_{02}(\eta_0) + S_2(\eta) + JT_{01}(\hat{\eta})S_{02}(\eta_0) + JT_{01}(\hat{\eta})S_2(\eta)
\]

\[
= T_{02}(\hat{\eta}) + S_{01}(\eta_0) + JT_{02}(\hat{\eta})S_{01}(\eta_0)
\]

\[
= T_{01}(\hat{\eta}) + S_{02}(\eta_0) + JT_{01}(\hat{\eta})S_{02}(\eta_0).
\]

Hence

\[
S_2(\eta) + JT_{01}(\hat{\eta})S_2(\eta) = 0.
\]

The proof for the lemma in the brackets runs parallel.

We are now in position to derive the following theorems.

**Theorem I.** If the system \((1)\)

\[
M_1(\eta) = 0, \quad S_{01}(\eta) = 0
\]

has a unique solution, then the adjoint system \((1')\)

\[
N_1(\hat{\eta}) = 0, \quad T_{01}(\hat{\eta}) = 0
\]

also has a unique solution and conversely.

Suppose \( \eta \) is the unique solution of system \((1)\), and let \( \hat{\eta}_0 \) be any solution of equation \((A')\), whose Fredholm determinant is not zero. Then by Lemma I we shall have for \( \eta \) and \( \hat{\eta}_0 \)

\[
S_{02}(\eta) + T_{01}(\hat{\eta}_0) + JT_{01}(\hat{\eta}_0)S_{02}(\eta) = T_{02}(\hat{\eta}_0).
\]

*Loc. cit., p. 87.
If we consider this as an integral equation in \( S_{02}(\eta) \), it has a unique solution, viz., the value of \( S_{02}(\eta) \). Hence the Fredholm determinant of \( T_{01}(\hat{\eta}_0) \) is not zero, which by Theorem I of § 1 is a necessary and sufficient condition for the existence of a unique solution of the system \((1')\).

**Theorem II.** If the system \((2)\)
\[
M_2(\eta) = 0, \quad S_1(\eta) = 0
\]
has \(n\)-fold compatibility, then the system \((2')\)
\[
N_2(\hat{\eta}) = 0, \quad T_1(\hat{\eta}) = 0
\]
has also \(n\)-fold compatibility, and conversely.

Let \( \mu_1, \ldots, \mu_n \) be a complete system of linearly independent solutions of the homogeneous equation
\[
\mu' + J S_{01}(\eta_0) \mu' = 0,
\]
\( \eta_0 \) being a solution of equation \((A')\) whose Fredholm determinant is not zero. Then by Theorem II of § 1
\[
\eta = \sum_{m=1}^{n} (\mu_m' + J \eta_0 \mu_m') \mu_m''
\]
\( \mu_m'' \) being arbitrary, is a general solution of system \((2)\). By Lemma II
\[
S_2(\eta) = \sum_{m=1}^{n} (\mu_m' + J S_{02}(\eta_0) \mu_m') \mu_m''
\]
will satisfy the equation
\[
\kappa + J T_{01}(\hat{\eta}_0) \kappa = 0,
\]
i. e., we shall have
\[
\sum_{m=1}^{n} (\mu_m' + J S_{02}(\eta) \mu_m' + J T_{01}(\hat{\eta}_0) (\mu_m' + J S_{02}(\eta) \mu_m')) \mu_m'' = 0.
\]
Since the \( \mu_m'' \) are arbitrary, we conclude that the equation
\[
\mu' + J T_{01}(\hat{\eta}_0) \mu' = 0,
\]
\( \hat{\eta}_0 \) being a solution of equation \((A')\) whose Fredholm determinant is not zero, has a set of \( n \) solutions
\[
\mu_1' + J S_{02}(\eta_0) \mu_1'.
\]
These solutions are linearly independent. For suppose they were not. Then there would exist \( n \) constants \( c_1, \ldots, c_n \), such that
\[
\sum_{m=1}^{n} c_m (\mu_m' + J S_{02}(\eta_0) \mu_m') = \sum_{m=1}^{n} c_m \mu_m' + J S_{02}(\eta_0) \sum_{m=1}^{n} c_m \mu_m' = 0.
\]
Let
\[
\sum_{m=1}^{n} c_m \mu_m' = \mu'.
\]
Then for every $\mu''$ the function

$$\eta = \mu' \mu'' + J\eta_0 \mu' \mu''$$

will satisfy both of the conditions

$$S_1(\eta) = 0, \quad S_2(\eta) = 0.$$

But from our assumption relative to $S_1$ and $S_2$ it follows that

$$\eta(x_1) = 0, \quad \eta(x_2) = 0,$$

from which we have $\mu' = 0$, which is contrary to our assumption that the $\mu'_m$ are linearly independent. It follows therefore that the expressions

$$\mu'_m + JS_{\eta_0}(\eta_0)\mu'_m$$

are linearly independent.

Now if the linear equation

$$\mu' + JT_{\eta_0}(\eta_0) \mu' = 0$$

has at least $n$ linearly independent solutions, then the adjoint linear integral equation

$$\mu'' + J\mu'' T_{\eta_0}(\eta_0) = 0$$

will also have at least $n$ linearly independent solutions, so that by Theorem II of § 1, we know that the system (2') has at least $n$-fold compatibility.

By following through a similar line of reasoning, we show that if the system (2') has $m$-fold compatibility, the system (2) has at least $m$-fold compatibility. But from this we conclude that $m = n$, which is the desired result.

Since 0-fold compatibility of system (2) yields a unique solution for the system (1), Theorem I may be regarded as a corollary to Theorem II.

We note further that in the course of the proof we have extended Lemma II to

**Lemma IIa.** Every solution $\kappa$ of the equation

$$\kappa + JT_{\eta_0}(\eta_0) \kappa = 0, \quad [\kappa + J\kappa S_{\eta_0}(\eta_0) = 0]$$

when $\eta_0 [\eta_0]$ is a solution of equation (A') [(A)] whose Fredholm determinant is not zero, is expressible in the form $S_2(\eta) [T_2(\eta)]$, where $\eta [\eta]$ is a solution of the system (2) [(2')].

**Theorem III.** A necessary and sufficient condition for the existence of a solution of the non-homogeneous system (3) [(3')] is that

$$M_2(\eta) = \alpha_0, \quad S_1(\eta) = \sigma_0, \quad [N_2(\eta) = \alpha_0, \quad T_1(\eta) = \tau_0]$$

is that

$$\int_{x_1}^{x_2} J\eta_0 \alpha_0 = -JT_2(\eta) \sigma_0, \quad \left[\int_{x_1}^{x_2} J\alpha_0 \eta = -J\tau_0 S_2(\eta)\right]$$

for every solution $\eta [\eta]$ of the adjoint homogeneous system (2') [(2)].
Suppose \( \eta \) is a solution of the system (3). Then if we let \( \hat{\eta} \) be any solution of the adjoint system \((2')\) and apply Green's Theorem \( G_2^* \)

\[
\int_{x_1}^{x_2} J(\hat{\eta}M_2(\eta) + N_2(\hat{\eta})\eta) = J\hat{\eta}(x_2)\eta(x_2) - J\hat{\eta}(x_1)\eta(x_1)
= JT_1(\hat{\eta})S_2(\eta) - JT_2(\hat{\eta})S_1(\eta),
\]

we get at once the condition of the theorem, i.e., the condition is necessary.

On the other hand, the condition is sufficient. For if \( \eta_1 \) be a particular solution of equation \((C)\), and \( \hat{\eta} \) be a solution of the system \((2')\), we get by the Green's Theorem \( G_2 \)

\[
\int_{x_1}^{x_2} J\hat{\eta}\alpha_0 = -JT_2(\hat{\eta})S_1(\eta_1).
\]

Applying the conditions of our theorem, we have for any particular solution \( \eta_1 \) of equation \((C)\) and every solution \( \hat{\eta} \) of system \((2')\)

\[
JT_2(\hat{\eta})(S_1(\eta_1) - \sigma_0) = 0.
\]

By Lemma IIa however, every solution of the equation

\[
\kappa + J\kappa S_{01}(\eta_0) = 0
\]

is expressible in the form \( T_2(\hat{\eta}) \), i.e., we have for every such \( \kappa \)

\[
J\kappa(S_1(\eta_1) - \sigma_0) = 0.
\]

By Theorem III of § 1, this is sufficient for the existence of a solution of system (3).

Ann Arbor, Mich.,
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* Cf. I, loc. cit., p. 86.