SPIRAL MINIMAL SURFACES*

BY

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1. **Introduction.** The equations of any minimal surface S may be written†

$$
(1)\qquad
\begin{aligned}
x &= \frac{1}{2}\int (1-u^2)F(u)\,du + \frac{1}{2}\int (1-v^2)\Phi(v)\,dv = U_1 + V_1\\
y &= \frac{i}{2}\int (1+u^2)F(u)\,du - \frac{i}{2}\int (1+v^2)\Phi(v)\,dv = U_2 + V_2\\
z &= \int uF(u)\,du + \int v\,\Phi(v)\,dv = U_3 + V_3.
\end{aligned}
$$

If the surface is real F and Φ are conjugate functions, and for a real point with a real tangent plane u and v have conjugate values.‡ The direction cosines of the normal to S are

$$
(2)\qquad X = \frac{u+v}{uv+1}, \qquad Y = \frac{i(v-u)}{uv+1}, \qquad Z = \frac{uv-1}{uv+1}.
$$

The linear element ds is given by

$$
ds^2 = (uv+1)^2 F(u)\Phi(v)\,du\,dv.
$$

The differential equations of the lines of curvature and of the asymptotic lines of S are respectively

$$
F(u)\,du^2 - \Phi(v)\,dv^2 = 0, \qquad F(u)\,du^2 + \Phi(v)\,dv^2 = 0.
$$

The curves of any surface along each of which the tangent plane to the surface makes a constant angle with a fixed direction are called the *Minding parallels* of the surface with reference to that direction; the curves of the surface along each of which the normal is parallel to a plane containing the fixed direction are called the *Minding meridians*.§ We shall consider Minding

* Read before the American Mathematical Society, December 27, 1916.

† Darboux, *Théorie des surfaces*, vol. 1, 2d ed., p. 340. These equations are due to Enneper (1864) and Weierstrass (1866).

‡ Darboux, l. c., p. 343.

§ These curves were first studied by Minding, Journal für Mathematik, vol. 44 (1852), p. 66. Darboux, l. c., p. 367.

parallels and meridians of minimal surfaces with reference to the Z axis. If in (1) we set $u = re^{\phi i}$ and $v = re^{-\phi i}$, where for real points of a real surface r is real and positive and ϕ is real, it is evident from (2) that the curves (r), that is r constant, are the Minding parallels and the curves (ϕ) the Minding meridians. In the study of spiral minimal surfaces, the subject of this paper, the curves (r) play a part analogous to that taken by the curves (ϕ) in the discussion of minimal surfaces applicable to surfaces of revolution.

If F and Φ in (1) are replaced by $e^{\alpha i}F$ and $e^{-\alpha i}\Phi$ respectively, where α is a real constant, we obtain the equations of a family of minimal surfaces called *associate* to S.* Corresponding points of associate minimal surfaces are points given by the same parameter values u, v. Schwarz proved that the locus of points on a family of associate surfaces corresponding to given values of u, v is an ellipse whose center is the origin. These ellipses we call *Schwarz ellipses*. We have found the equations of the locus L of the vertices of the Schwarz ellipses for a given minimal surface with equations (1) to be†

$$
\begin{gathered}
x = U_1\sqrt[4]{\frac{\sum V_1^2}{\sum U_1^2}} + V_1\sqrt[4]{\frac{\sum U_1^2}{\sum V_1^2}}, \qquad y = U_2\sqrt[4]{\frac{\sum V_1^2}{\sum U_1^2}} + V_2\sqrt[4]{\frac{\sum U_1^2}{\sum V_1^2}}, \\
z = U_3\sqrt[4]{\frac{\sum V_1^2}{\sum U_1^2}} + V_3\sqrt[4]{\frac{\sum U_1^2}{\sum V_1^2}}.
\end{gathered} \tag{3}
$$

In (3) the first radical of each second member has the same determination; the second is the reciprocal and conjugate of the first for real points of a real surface S. The four vertices of each ellipse are given by the four determinations of the first radical. Evidently L consists generally of four nappes symmetrical in pairs with respect to the origin.

A *spiral surface* is most simply defined as follows:‡ A spiral surface is the locus of the different positions of any curve which is rotated about an axis and at the same time subjected to a homothetic transformation with respect to a point of the axis in such a way that the tangent to the locus described by any point of the curve makes a constant angle with the axis. The spiral surfaces include surfaces of revolution but we shall hereafter in speaking of a spiral surface suppose it not to be a surface of revolution. Evidently the locus of each point of the curve generating a spiral surface lies on a right cir-

* Associate minimal surfaces were first systematically considered by H. A. Schwarz, *Miscellen aus dem Gebiete der Minimalflächen*, Journal für Mathematik, vol. 80 (1875), p. 286, see also Darboux, l. c., p. 379.

† American Journal of Mathematics, vol. 40 (1918), p. 87.

‡ Eisenhart, *Differential Geometry*, p. 151. Spiral surfaces were first studied by Maurice Lévy in a paper published in Comptes Rendus, vol. 87 (1878), p. 788. He gave them as an example of surfaces such that E, F, G, the coefficients in the square of the linear element, are homogeneous functions of the parameters of any degree other than -2. Lévy named them "surfaces pseudo-moulures logarithmiques."

cular cone whose axis is the axis of the surface, cuts all elements of the cone at the same angle, and is projected on a plane perpendicular to the axis in a logarithmic spiral. Such a curve we shall call simply a spiral.* The equations of any spiral surface may be written

$$(4) \qquad x = \rho e^v \cos(\omega + hv), \qquad y = \rho e^v \sin(\omega + hv), \qquad z = \zeta e^v,$$

where h is constant, ρ, ω, and ζ are functions of u alone, and the curves (u) are the spirals. Darboux has proved† that the orthogonal trajectories of the spirals of a spiral surface whose generating curve is given may be found by quadratures, and the square of the linear element written in the form

$$ds^2 = e^{2v} U^2 (du^2 + dv^2),$$

where U is a function of u alone. If U is given, h an arbitrary constant, and ζ, ρ, ω determined from

$$(5) \qquad h^2(\zeta^2 + \zeta'^2) = h^2 U^2 - U'^2, \qquad (h^2 + 1)\rho^2 = U^2 - \zeta^2,$$

$$(h^2 + 1)\rho^2 \omega' = -h\zeta\zeta' - \frac{1}{h} UU',$$

the surface (4) has the linear element in the form given, so that the solution of (5) determines a three-parameter family of spiral surfaces applicable to a given spiral surface.‡

Lie proved§ that all real minimal surfaces applicable to spiral surfaces are given by writing in (1)

$$(6) \qquad F(u) = cu^{m-2+ni}, \qquad \Phi(v) = \bar{c}v^{m-2-ni},$$

where c and $\bar{c}$ are conjugate constants and m and n are real constants. The number -2 is introduced in the exponents only to simplify future formulations. No essential restriction is imposed by taking the modulus of c as unity, so that we shall suppose $c = e^{\alpha i}$ where α is real. If in (6) $n = 0$ equations (1) and (6) give all minimal surfaces applicable to surfaces of revolution.‖ That these surfaces are applicable to surfaces of revolution if n is zero, and when n is not zero to spiral surfaces, appears from the linear element of (1) and (6) in terms of r, ϕ

$$ds^2 = e^{-2n\phi} r^{2m-4} (r^2 + 1)^2 (dr^2 + r^2 d\phi^2).$$

* These curves have been named "cylindro-conical helices." See *Encyclopädie der mathematischen Wissenschaften*, Bd. III, 3, Heft 2, 3, p. 252.

† L. c., pp. 148–150.

‡ Proved by Maurice Lévy, as stated by Darboux (l. c., p. 149). Of the three constants one is additive to ω and has no effect on the form of the surface, so that there are generally given by the integration of (5) a *two* parameter family of different surfaces.

§ Mathematische Annalen, vol. 15 (1879), p. 503. Darboux, l. c., p. 396.

‖ First proved by Schwarz, *Miscellen*, p. 296.

Evidently the Minding parallels (r) of (1) correspond to the spirals of the spiral surface, and, when n is zero, to the parallels of the surface of revolution. If $m = 0$, $n \neq 0$ equations (1) and (6) give all minimal surfaces which are spiral surfaces.*

A. Ribaucour proved† that the minimal surfaces associate to a minimal surface applicable to a surface of revolution are congruent to the given minimal surface except for $m = 0$, which value gives the minimal helicoids. Darboux states without proof that the minimal surfaces associate to a minimal surface applicable to a spiral surface are similar to the given surface. In the following section we prove these theorems, noting that exception must be made to Darboux's statement in the case of spiral minimal surfaces, and proving also a theorem, which we believe has not previously been stated, concerning the similarity to itself of the general surface given by (1) and (6). The remainder of this paper deals entirely with spiral minimal surfaces: we consider families of minimal surfaces associate to a surface of this kind, their Schwarz ellipses and the locus L of the vertices of these ellipses, the envelope of the family of associate surfaces, and the evolute surface; it is proved that on every spiral minimal surface there are an infinite number of plane spiral lines of curvature, the intersections of the surface with the plane part of L, an infinite number of spiral asymptotic lines, the intersections of the surface with the right conoids of L, and a single spiral geodesic; we show that on one member of a family of associate spiral minimal surfaces all the plane spiral lines of curvature except one are double curves; we discover certain symmetries of these surfaces, and finally obtain two characteristic properties of spiral minimal surfaces connected with the locus L and with the Minding parallels.

2. **Darboux's similarity theorem.** The first two of equations (1) may be combined and written

$$x + yi = -\int u^2 F(u)\,du + \int \Phi(v)\,dv.$$

For the minimal surfaces applicable to spiral surfaces we substitute from (6)

$$F(u) = e^{ai} u^{m-2+ni}, \qquad \Phi(v) = e^{-ai} v^{m-2-ni}, \qquad (n \neq 0).$$

When this surface is rotated about the Z axis through the angle β,

$$x' + y' i = e^{\beta i}(x + yi), \qquad z' = z,$$

where x', y', z' are the new coördinates of x, y, z. Sufficient conditions that this surface in the new position be similar to that surface for which

* First proved by Darboux, l. c., p. 359.

† "*Étude sur les élassoides ou surfaces à courbure moyenne nulle.*" Memoires Couronnés de l'Académie Royale de Belgique, vol. 44 (1882), chap. XX. Ribaucour disregards the cases $m = 0, \pm 1$; Darboux proves the theorem for all cases except $m = 0$ (l. c., pp. 395, 396).

$\alpha = 0$ in the original position with the origin as center, C as the ratio of similarity, and all constants of integration chosen as zero, are that u' and v' may be determined so that, with constant real values for β and C,

$$e^{\beta i}(-u'^2 F(u')\,du' + \Phi(v')\,dv') = C(-u^{m+ni}\,du + v^{m-2-ni}\,dv),$$
$$u' F(u')\,du' + v' \Phi(v')\,dv' = C(\ u^{m-1+ni}\,du + v^{m-1-ni}\,dv).$$

These equations are satisfied by constant real values of β and C by taking u' and v' as conjugate functions of u and v alone respectively if the same is true of the following four equations:

$$e^{(\alpha+\beta)i}\,u'^{m+ni}\,du' = Cu^{m+ni}\,du, \qquad e^{(\beta-\alpha)i}\,v'^{m-2-ni}\,dv' = Cv^{m-2-ni}\,dv,$$
$$e^{\alpha i}\,u'^{m-1+ni}\,du' = Cu^{m-1+ni}\,du, \qquad e^{-\alpha i}\,v'^{m-1-ni}\,dv' = Cv^{m-1-ni}\,dv.$$

From the last group of equations

$$u' = ue^{-\beta i}, \qquad v' = ve^{\beta i}.$$

Setting these values of u' and v' in the four equations, we have

$$C = e^{n\beta+(\alpha-m\beta)i}, \qquad C = e^{n\beta-(\alpha-m\beta)i},$$

so that we must have $\alpha - m\beta = 0$. If m is not zero the equations are satisfied by the constant real values,

$$\beta = \frac{\alpha}{m}, \qquad C = e^{n\alpha/m}. \tag{7}$$

If m is zero the proof fails. Equations (7) express the two theorems: (I) Minimal surfaces associate to a minimal surface applicable to a surface of revolution ($n = 0$) are congruent to that surface and may be made to coincide with it by a rotation about the Z axis, except in the case of the minimal helicoids ($m = 0$). That the associate minimal helicoids are not similar is well known.* (II) Minimal surfaces associate to a minimal surface applicable to a spiral surface ($n \neq 0$) are similar and are brought into similar position by a rotation about the Z axis, except in the case of the spiral minimal surfaces ($m = 0$). That the associates of a spiral minimal surface are not similar to that surface will appear in § 6 of this paper.

If we observe that the associate to any minimal surface given by $\alpha = 2k\pi$, where k is any integer, coincides with that surface, and that the associate $\alpha = (2k+1)\pi$ is symmetrical with respect to the origin to that surface, we have in (7) the proof of a theorem due to Ribaucour: A minimal surface applicable to a surface of revolution, not a minimal helicoid, is brought into coincidence with itself by rotation about the Z axis through the angle $2k\pi/m$; it appears also, though not stated by Ribaucour, that such a surface is brought

* See, for example, Scheffers, *Theorie der Flächen*, 2d ed., pp. 362, 363.

into coincidence with its symmetry with respect to the origin by rotation about the Z axis through the angle $(2k+1)\pi/m$. Finally it appears from (7) that a minimal surface applicable to a spiral surface ($m \neq 0$, $n \neq 0$) is similar to itself and to its symmetry with respect to the origin in an infinite number of positions if m is irrational, in a finite number if m is rational.

3. **Equations and linear element of $S_{n\alpha}$.** We denote by $S_{n\alpha}$ the spiral minimal surface given by substituting in (1)

$$F(u) = e^{\alpha i} u^{-2+ni}, \qquad \Phi(v) = e^{-\alpha i} v^{-2-ni}.$$

Performing the integrations indicated, taking all constants of integration as zero, then writing

$$u = re^{\phi i}, \qquad v = re^{-\phi i}, \qquad n = -\cot\beta,$$

we have the equations of $S_{n\alpha}$ in terms of the real parameters r, ϕ

$$\begin{aligned} x &= e^{-n\phi}\sin\beta\,[\,r\sin(\phi-\beta+\alpha+n\log r) \\ &\qquad\qquad + \frac{1}{r}\sin(\phi-\beta-\alpha-n\log r)\,], \\ y &= -e^{-n\phi}\sin\beta\,[\,r\cos(\phi-\beta+\alpha+n\log r) \\ &\qquad\qquad + \frac{1}{r}\cos(\phi-\beta-\alpha-n\log r)\,], \\ z &= -2e^{-n\phi}\tan\beta\sin(\alpha+n\log r). \end{aligned} \tag{8}$$

In these equations n is retained for the sake of abbreviation. If we let

$$n\log r = u, \qquad -n(\phi-\beta) = v, \qquad h = -\frac{1}{n}$$

and set

$$\begin{aligned} \rho\cos\omega &= 2e^{-n\beta}\sin\beta\sin(u+\alpha)\sinh\frac{u}{n}, \\ \rho\sin\omega &= -2e^{-n\beta}\sin\beta\cos(u+\alpha)\cosh\frac{u}{n}, \\ \zeta &= -2e^{-n\beta}\tan\beta\sin(u+\alpha), \end{aligned}$$

equations (8) take Darboux's form (4) thus proving that $S_{n\alpha}$ is actually a spiral surface and that the spirals are the Minding parallels (r).

The intersection of $S_{n\alpha}$ with the xy plane is given by

$$\alpha + n\log r = k\pi,$$

where k is any integer, and each such value of k gives a curve, as appears from the first two equations of (8), whose polar equation, coördinates R, Θ, is $R = e^{-n(\Theta+c)} = e^{(\Theta+c)\cot\beta}$, so that the surface cuts the plane in an infinite

number of congruent logarithmic spirals which cut all radii vectores at the angle β.

The linear element of (8) is given by

$$ds^2 = (r^2 + 1)^2 r^{-4} e^{-2n\phi} (dr^2 + r^2 d\phi^2).$$

Substituting the values u, v given after (8)

$$ds^2 = e^{2v} U^2 (du^2 + dv^2), \qquad U^2 = 4e^{-2n\beta} \tan^2 \beta \cosh^2 \frac{u}{n}.$$

We may determine the spiral minimal surfaces $S_{n\alpha}$ applicable to a given spiral minimal surface S_0 which are given by the integration of (5) as follows. Let the subscript apply to all quantities relating to S_0; from $U^2 = U_0^2$ follows $n^2 = n_0^2$; the first of equations (5) gives

$$\zeta^2 + \zeta'^2 = 4e^{-2n\beta} \tan^2 \beta = U_0^2 - U_0'^2/h^2.$$

In this equation we may replace U_0 by U, and are led to the condition $h^2 n^2 = h^2 n_0^2 = 1$. The solutions of (5) with $h = -1/n_0$ substituted in (4) give equations (8) where n is replaced by n_0, α is the constant introduced by the integration of the equation for ζ, and the constant additive to ω, which affects only the position of the surface, is taken as zero. The integration of (5) with $h = 1/n_0$ leads to the surfaces $S_{-n_0\alpha}$, surfaces not associate to S_0 but symmetrical with respect to the X axis to the associate surfaces, as will appear in § 9. Spiral surfaces, not minimal, applicable to S_0 would be given by the substitution in (4) of the solutions of (5) for values of h not equal to $\pm 1/n_0$, but it appears to be impossible to integrate the equation for ζ for such values.

4. **The locus L for $S_{n\alpha}$.** From equations (1) for S_{n0} we find

$$\sum V_1^2 / \sum U_1^2 = (uv)^{-2ni};$$

that part of L given by (3) where the first radical is taken as $(uv)^{-ni/2}$, which we call L_1, has the equations, obtained by substituting this value and writing as before $u = re^{\phi i}$, $v = re^{-\phi i}$,

$$\begin{aligned} x &= e^{-n\phi} \frac{r^2 + 1}{r} \sin \beta \sin (\phi - \beta), \\ y &= -e^{-n\phi} \frac{r^2 + 1}{r} \sin \beta \cos (\phi - \beta), \\ z &= 0. \end{aligned} \tag{9}$$

Points common to L_1 and $S_{n\alpha}$ are given by

$$(uv)^{-ni/2} = e^{\alpha i} \qquad \text{or} \qquad \alpha + n \log r = 2k\pi.$$

The equations of L_2, symmetrical with respect to the origin to L_1, are

obtained by changing the signs of the second members of (9). Points common to L_2 and $S_{n\alpha}$ are given by $\alpha + n \log r = (2k+1)\pi$. The curves common to L_1, L_2, and $S_{n\alpha}$ are the logarithmic spirals in which the latter cuts the xy plane.

Choosing the first radical in (3) as $i(uv)^{-ni/2}$ we obtain for L_3, part of the locus L, the equations

$$(10)\qquad x = e^{-n\phi}\frac{r^2-1}{r}\sin\beta\cos(\phi-\beta), \qquad y = e^{-n\phi}\frac{r^2-1}{r}\sin\beta\sin(\phi-\beta),$$
$$z = -2e^{-n\phi}\tan\beta.$$

Points common to L_3 and $S_{n\alpha}$ are given by

$$i(uv)^{-ni/2} = e^{\alpha i} \qquad \text{or} \qquad \alpha + n\log r = 2k\pi + \frac{\pi}{2}.$$

The equations of L_4, symmetrical to L_3, are found by reversing the signs of the second members of (10). Points common to L_4 and $S_{n\alpha}$ are given by $\alpha + n\log r = (2k+1)\pi + \pi/2$. Equations (9) and (10) may be obtained by substituting the appropriate values of r in (8) and regarding α as variable.

The surface L_3 is a spiral surface for (10) may be put in the form (4), and its spirals are the curves (r). From (10) it appears that every curve (ϕ) of L_3 is a straight line intersecting the Z axis and parallel to the xy plane, so that L_3 is a spiral right conoid.

The distances b of r, ϕ of L_1 from the origin and a of r, ϕ of L_3 from the origin are given respectively by

$$b^2 = e^{-2n\phi}\left(\frac{r^2+1}{r}\right)^2\sin^2\beta, \qquad a^2 = e^{-2n\phi}\left[\left(\frac{r^2-1}{r}\right)^2\sin^2\beta + 4\tan^2\beta\right],$$

and for all r, ϕ we have $a^2 > b^2$, so that a and b are respectively the semi-major and minor axes of Schwarz's ellipse r, ϕ; then the xy plane (L_1, L_2) is the locus of the extremities of the minor axes, and the two conoids (L_3, L_4) form the locus of the ends of the major axes. Since $a^2 - b^2$ is independent of r all ellipses corresponding to a constant ϕ have the same focal distance; the eccentricity is independent of ϕ so that all ellipses corresponding to points of a spiral (r) have the same eccentricity. The maximum eccentricity is $\sin\beta$ and is given by $r = 1$; all ellipses, $r = 1$, have their major axes on the Z axis and touch S_{n0} at the extremities of their minor axes. When r approaches zero or becomes infinite the eccentricity approaches zero, corresponding points of $S_{n\alpha}$ receding indefinitely from the origin.

5. **Surfaces related to $S_{n\alpha}$.** The *envelope* of a family of minimal surfaces associate to (1) consists of two surfaces symmetrical with respect to the origin

given by*

$$x = U_1 H + V_1 H^{-1}, \qquad y = U_2 H + V_2 H^{-1}, \qquad z = U_3 H + V_3 H^{-1},$$

where

$$H = \pm \sqrt{\frac{(u+v)V_1 + i(v-u)V_2 + (uv-1)V_3}{(u+v)U_1 + i(v-u)U_2 + (uv-1)U_3}}.$$

For E_1 and E_2, the two parts of the envelope of a family of associate spiral minimal surfaces, corresponding to the upper and lower signs respectively in H,

$$H = \pm e^{(\xi - n \log r)i}, \qquad \tan \xi = \frac{r^2 - 1}{r^2 + 1} \tan \beta.$$

The equations of E_1 are also given by writing $\xi - n \log r$ in place of α in (8). That E_1 is a spiral surface, whose spirals are the curves (r), may be shown by reducing its equations to the form (4); the expressions for $\rho \cos \omega$, $\rho \sin \omega$, and ζ for E_1 are obtained by replacing α by $\xi - u$ in the equations following (8). Points common to $S_{n\alpha}$ and E_1, $S_{n\alpha}$ and E_2 are given respectively by

$$\xi = \alpha + n \log r + 2k\pi, \qquad \xi = \alpha + n \log r + (2k+1)\pi.$$

Each of these equations has just one positive root r for every integral value of k, so that every surface $S_{n\alpha}$ is tangent to each surface of the envelope along an infinite number of spirals. The solution of the first equation for $\alpha = k = 0$ is $r = 1$, so that S_{n0} is tangent to E_1 along this curve, which will appear as a curve of particular interest.

The coördinates of the points of the two nappes of the *evolute surface* of $S_{n\alpha}$ corresponding to r, ϕ of the latter are

$$x \pm RX = x \pm e^{-n\phi} \frac{r^2 + 1}{r} \cos \phi,$$

$$y \pm RY = y \pm e^{-n\phi} \frac{r^2 + 1}{r} \sin \phi,$$

$$z \pm RZ = z \pm e^{-n\phi} \frac{r^4 - 1}{2r^2},$$

where x, y, z are the coördinates of r, ϕ on $S_{n\alpha}$ given by (8), X, Y, Z the direction cosines of the normal, and R the absolute value of either principal radius of curvature. The only interest in this evolute surface consists in the fact that it is a spiral surface, whose spirals are the curves (r), as may be seen by reducing its equations to the form (4).

In connection with the spiral surfaces related to $S_{n\alpha}$ the following considerations are of interest: the right circular cone whose axis is the Z axis

* My paper, American Journal of Mathematics, *l. c.*

and whose elements make the angle γ with this axis is

$$x^2 + y^2 = z^2 \tan^2 \gamma .$$

The spiral which cuts the elements of this cone under the angle λ is

$$x = e^{v \sin \gamma \cot \lambda} \sin \gamma \cos (c + v), \qquad y = e^{v \sin \gamma \cot \lambda} \sin \gamma \sin (c + v),$$
$$z = e^{v \sin \gamma \cot \lambda} \cos \gamma .$$

Equations (4) of any spiral surface may be written, setting $v = -n\phi$ and $h = -1/n$,

$$(4') \quad x = e^{-n\phi} \rho \cos (\omega + \phi), \qquad y = e^{-n\phi} \rho \sin (\omega + \phi), \qquad z = e^{-n\phi} \zeta .$$

The curve (r) of (4′) lies on the cone $x^2 + y^2 = \rho^2 z^2/\zeta^2$, so that $\tan \gamma = \rho/\zeta$, which gives, for $r = 1$, $\tan \gamma = \cos \beta \cot \alpha$. We find

$$\tan \lambda = \sin \gamma \tan \beta ,$$

so that λ is determined by $n = -\cot \beta$ and γ. Since for fixed n (4′) includes the surfaces $S_{n\alpha}$, L_3, E_1, and the evolute surface of $S_{n\alpha}$, it follows that every right circular cone whose axis is the Z axis cuts all these surfaces in congruent spirals.

6. **Special spirals on $S_{n\alpha}$.** The differential equation of the *lines of curvature* of (1) becomes for $S_{n\alpha}$

$$e^{\alpha i} u^{-2+ni} du^2 - e^{-\alpha i} v^{-2-ni} dv^2 = 0 .$$

This equation integrated gives

$$e^{\alpha i} (uv)^{ni/2} \pm 1 = cv^{ni/2} .$$

The only spirals (r) among the lines of curvature are given by $c = 0$. For them we have, belonging respectively to the two families of lines of curvature,

$$\alpha + n \log r = 2k\pi, \qquad \alpha + n \log r = (2k + 1)\pi .$$

These are the curves of intersection of $S_{n\alpha}$ with the xy plane, and are logarithmic spirals; they are moreover the curves common to $S_{n\alpha}$ and L_1, $S_{n\alpha}$ and L_2 respectively. For $\alpha = k = 0$ the first equation gives $r = 1$, which is therefore a line of curvature of S_{n0}.

The differential equation of the *asymptotic lines* of $S_{n\alpha}$ is

$$e^{\alpha i} u^{-2+ni} du^2 + e^{-\alpha i} v^{-2-ni} dv^2 = 0 ,$$

giving the integral

$$e^{\alpha i} (uv)^{ni/2} \pm i = cv^{ni/2} .$$

The only spirals among the asymptotic lines are given by $c = 0$; for these,

belonging respectively to the two families of asymptotic lines,

$$\alpha + n \log r = 2k\pi + \frac{\pi}{2}, \qquad \alpha + n \log r = (2k+1)\pi + \frac{\pi}{2}.$$

These are the curves common to $S_{n\alpha}$ and L_3, $S_{n\alpha}$ and L_4 respectively. The first of these equations gives $r = 1$ for $\alpha - \pi/2 = k = 0$. This line on $S_{n\pi/2}$ is the Z axis. Since on every surface the Minding parallels with reference to the Z axis and the curves of steepest ascent form a conjugate system,* the asymptotic spirals are curves of steepest ascent and are the only such spirals.

The properties of the Minding parallels (r) of $S_{n\alpha}$ are analogous to properties of the Minding meridians (ϕ) of minimal surfaces applicable to surfaces of revolution.† We note further that $S_{n\alpha}$ has no line of curvature or asymptotic line (ϕ) while minimal surfaces applicable to surfaces of revolution, with the exception of the catenoid and its adjoint surface, the right helicoid, have no line of curvature or asymptotic line (r).

From the linear element of $S_{n\alpha}$, given in § 3, it appears‡ that the only *spiral geodesic* of this surface is $r = 1$. This spiral being plane for $\alpha = 0$, S_{n0} has symmetry with respect to its plane, the xy plane; since, for $\alpha = \pi/2$, this spiral is the Z axis $S_{n\pi/2}$ has symmetry with respect to this axis. On any surface similar to $S_{n\alpha}$ there will correspond to the spiral geodesic, $r = 1$, a spiral geodesic on a cone whose semi-vertical angle γ is given by

$$\tan \gamma = \cos \beta \cot \alpha.$$

On the associate surface $S_{n\alpha'}$ the only spiral geodesic lies on a cone of semi-vertical angle γ' given by $\tan \gamma' = \cos \beta \cot \alpha'$. Then $S_{n\alpha}$ and $S_{n\alpha'}$ are not similar if $\alpha' - \alpha \neq k\pi$. There is no geodesic (ϕ) of $S_{n\alpha}$.

7. **Double spirals of S_{n0}.** Writing $\alpha = 0$ in (8) the equations of S_{n0} are

$$x = e^{-n\phi} \sin \beta \left[r \sin (\phi - \beta + n \log r) + \frac{1}{r} \sin (\phi - \beta - n \log r) \right],$$

$$y = - e^{-n\phi} \sin \beta \left[r \cos (\phi - \beta + n \log r) + \frac{1}{r} \cos (\phi - \beta - n \log r) \right],$$

$$z = - 2e^{-n\phi} \tan \beta \sin (n \log r).$$

If in these equations r is replaced by $1/r$, x and y are unchanged and the sign of z is reversed. If we consider those values of r satisfying the equation, $n \log r = k\pi$, we have the logarithmic spiral lines of curvature in which S_{n0}

* My paper, Annals of Mathematics, ser. 2, vol. 19, p. 4.

† My paper, Annals of Mathematics, l. c.

‡ Eisenhart, *Differential Geometry*, pp. 134, 266, 267.

intersects the xy plane; excepting the value $r = 1$, it appears that each of these spirals is given by two different reciprocal values of r. The direction cosines of the normal to the surface, given in § 1, are in terms of r, ϕ

$$X = \frac{2r \cos \phi}{r^2 + 1}, \qquad Y = \frac{2r \sin \phi}{r^2 + 1}, \qquad Z = \frac{r^2 - 1}{r^2 + 1}.$$

Changing r to $1/r$, X and Y are unaltered and the sign of Z is reversed. It follows that along each spiral line of curvature of S_{n0}, except $r = 1$, two branches of the surface intersect; moreover they intersect at a constant angle along each curve, the angle depending on r alone and approaching the limit π as r increases indefinitely from unity.

8. **Further properties of spirals on $S_{n\alpha}$.** In the first paper* published dealing with minimal surfaces applicable to surfaces of revolution E. Bour showed that every Minding meridian (ϕ) of the surface cuts at the same angle all members of either family of lines of curvature and all level curves (z). The Minding parallels (r) have the same property on $S_{n\alpha}$, as may be shown directly; we find, for example, that the angle of (r) and (z) is $\alpha + n \log r$, and is therefore constant with r.

The equation of the tangent plane to $S_{n\alpha}$ at r, ϕ is

$$x \cos \phi + y \sin \phi + z \frac{r^2 - 1}{2r} = P(r) e^{-n\phi},$$

$$P(r) = -\frac{r^2 + 1}{r} \sin^2 \beta \sec \xi \cos (\alpha + n \log r - \xi),$$

where ξ is the angle introduced in § 5. The helicoidal developable tangent to $S_{n\alpha}$ along (r) therefore intersects the xy plane in the envelope of the lines, $x \cos \phi + y \sin \phi = P(r) e^{-n\phi}$, which is a logarithmic spiral congruent to the plane lines of curvature (r). It may also be shown from the equation of the tangent plane that the planes tangent to the surface along (ϕ) envelop a cylinder whose elements are parallel to the xy plane, and that the cylinders corresponding to different values of ϕ are similar and are brought into similar position by rotation about the Z-axis through the angle ϕ. Bour stated in the paper cited that in the case of a minimal surface applicable to a surface of revolution the developables tangent to the surface along a Minding meridian (ϕ) are cylinders whose elements are parallel to the xy plane; we have proved that this is a property of every non-developable surface.† Ribaucour proved in the *mémoire* cited that in the case of minimal surfaces applicable to surfaces

* "*Théorie de la déformation des surfaces,*" Journal de l'école polytechnique, cahier 39 (1862).

† Annals of Mathematics, l. c.

of revolution these cylinders are similar. Stübler* determined all minimal surfaces which are envelopes of similar cylinders which may be brought into similar position by rotation about the Z-axis, finding that such surfaces are given by three different forms of $F(u)$ in (1), each depending on several parameters, these surfaces including those applicable to surfaces of revolution and spiral minimal surfaces.

9. **Symmetries of $S_{n\alpha}$.** From the equations (8) of $S_{n\alpha}$ it appears that if α and r are replaced by $-\alpha$ and $1/r$ respectively the values of x and y are unchanged and the sign of z is reversed; it follows that $S_{n\alpha}$ and $S_{n-\alpha}$ have symmetry with respect to the xy plane, symmetrical points being given by r, ϕ and $1/r, \phi$. If in (8) n is changed to $-n$, and consequently β to $-\beta$, if then r, ϕ are replaced by $1/r, -\phi$ respectively, x is unchanged, and the signs of both y and z are reversed; then $S_{n\alpha}$ and $S_{-n\alpha}$ are symmetrical with respect to the X axis. It follows that $S_{n\alpha}$ and $S_{-n-\alpha}$ are symmetrical with respect to the xz plane. If the isothermal parameters $\log r, \phi$ are used as coördinates symmetrical points in all three cases are given by changing the signs of one or of both coördinates.

10. **Two properties characteristic of $S_{n\alpha}$.** We prove a characteristic property of spiral minimal surfaces together with a similar characteristic property of minimal surfaces applicable to surfaces of revolution given by the following theorem, a converse of theorems given in § 8. If every Minding parallel (r) of a real minimal surface is an isogonal trajectory of either family of lines of curvature or of the level curves of the surface the minimal surface is a spiral surface or a helicoid; if every Minding meridian (ϕ) of a real minimal surface is an isogonal trajectory of either of the families named the minimal surface is applicable to a surface of revolution.

To prove the first part of this theorem we observe that the condition that all curves (r) be isogonal trajectories of either of the families named may be expressed by the equation

$$\frac{u^2 F(u)}{v^2 \Phi(v)} = f(uv),$$

where F and Φ are the conjugate functions in (1) and f is unknown. If this equation is differentiated first with respect to u and then with respect to v, and the two values of f' equated, we have

$$\frac{2F(u) + uF'(u)}{F(u)} = -\frac{2\Phi(v) + v\Phi'(v)}{\Phi(v)} = ni,$$

where n is a real constant. These give

$$F(u) = cu^{-2+ni}, \qquad \Phi(v) = \bar{c}v^{-2-ni},$$

* Mathematische Annalen, vol. 75 (1914).

so that the minimal surface is a spiral surface, or, if $n = 0$, a helicoid. The condition that each curve (ϕ) of (1) be an isogonal trajectory of any one of the families named is

$$\frac{u^2 F(u)}{v^2 \Phi(v)} = f\left(\frac{u}{v}\right),$$

which leads in a similar way to

$$F(u) = cu^{m-2}, \qquad \Phi(v) = \bar{c}v^{m-2},$$

where m is a real constant. The only real minimal surfaces such that both families of curves (r) and (ϕ) have the isogonal property are the helicoids. It is easily shown that it is only on the minimal helicoids that either of these families, (r) and (ϕ), forms an isogonal *system* with any of the three other families named, for only in this case is the angle of intersection constant.

It appeared in § 4 that for spiral minimal surfaces part of the locus L is a plane containing the center of the Schwarz ellipses;* we have elsewhere proved that the same thing is true for minimal surfaces applicable to surfaces of revolution if m is different from zero or plus or minus one. We now prove that this property is characteristic of these two classes of real minimal surfaces.

Since the equations of any minimal surface in any position are given by (1) we may without restriction suppose the equations of a minimal surface for which part of L is a plane containing the center of the Schwarz ellipses to have the form (1) and this plane part of L to be the xy plane, $z = 0$. From (3) it follows that, for all u, v,

$$z = U_3 \sqrt[4]{\frac{\sum V_1^2}{\sum U_1^2}} + V_3 \sqrt[4]{\frac{\sum U_1^2}{\sum V_1^2}} = 0,$$

for some determination of the first radical. This gives one of the identities

$$\frac{U_3}{\sqrt{\sum U_1^2}} = \frac{V_3}{\sqrt{\sum V_1^2}} = c \qquad \text{or} \qquad \frac{U_3}{\sqrt{\sum U_1^2}} = -\frac{V_3}{\sqrt{\sum V_1^2}} = c,$$

where $\sqrt{\sum U_1^2}$ and $\sqrt{\sum V_1^2}$ are conjugate and c is a constant real in the first case and pure imaginary in the second. We consider the equation

$$U_3 / \sqrt{\sum U_1^2} = c,$$

and write

$$\sqrt{\sum U_1^2} = \rho, \qquad U_1 = \lambda\rho, \qquad U_2 = \mu\rho, \qquad U_3 = c\rho.$$

From the last equations

$$\lambda^2 + \mu^2 + c^2 = 1, \qquad \lambda\lambda' + \mu\mu' = 0,$$

$$U_1' = \lambda\rho' + \lambda'\rho, \qquad U_2' = \mu\rho' + \mu'\rho, \qquad U_3' = c\rho'.$$

*American Journal of Mathematics, l. c.

From the fact that $u+v$, $i(v-u)$, $uv-1$ are proportional to the direction cosines of the normal to (1) it follows that for all u, v

$$(u+v)U_1' + i(v-u)U_2' + (uv-1)U_3' = 0,$$

which may be replaced by the two equations

$$u(U_1' - iU_2') - U_3' = 0, \qquad U_1' + iU_2' + uU_3' = 0.$$

The last two equations give $\sum U_1'^2 = 0$, from which, substituting the values given for U_1', U_2', U_3'

$$\rho'^2 + \rho^2(\lambda'^2 + \mu'^2) = 0, \qquad \rho = Ce^{\pm i\int\sqrt{\lambda'^2+\mu'^2}du}.$$

Substituting the values of U_1', U_2', U_3' in the two linear equations connecting these quantities, then replacing in these ρ' by $\pm \rho i\sqrt{\lambda'^2+\mu'^2}$, and cancelling ρ, which cannot be identically zero unless (1) is a plane,* we find

$$\frac{\lambda' + i\mu'}{\mp\sqrt{\lambda'^2+\mu'^2}} = i(\lambda + \mu i + cu), \qquad \frac{\lambda' - i\mu'}{\mp\sqrt{\lambda'^2+\mu'^2}} = i\left(\lambda - \mu i - \frac{c}{u}\right).$$

Multiplying the last two equations,

$$(\lambda + i\mu + cu)\left(\lambda - i\mu - \frac{c}{u}\right) + 1 = 0.$$

From this and $\lambda^2 + \mu^2 + c^2 = 1$ we find

$$\lambda + i\mu = \frac{1 - c^2 \pm \sqrt{1-c^2}\,u}{c}, \qquad \lambda - i\mu = -\frac{1 - c^2 \mp \sqrt{1-c^2}}{cu}.$$

Differentiating and multiplying the results, $\lambda'^2 + \mu'^2 = (c^2-1)/u^2$, and

$$\rho = Ku^{\pm\sqrt{1-c^2}}, \qquad U_3 = c\rho = cKu^{\pm\sqrt{1-c^2}}.$$

If $c = 0$ the surface (1) is the plane, $z = 0$; if $c = \pm 1$ the surface is the plane $z =$ constant; for all other values of c the function $F(u)$ of (1) is given by

$$F(u) = \frac{1}{u}U_3' = cKu^{-2\pm\sqrt{1-c^2}}.$$

If c is pure imaginary we have $F(u) = Au^{m-2}$, where m is a real constant numerically greater than one; if c is real and numerically less than one, $F(u)$ has the same form and m is real and numerically less than one. Such values of c give all minimal surfaces applicable to surfaces of revolution, except that

* My paper, American Journal of Mathematics, l.c.

the values $m = 0, \pm 1$ are excluded, and distinguish two classes of these surfaces which have widely different properties.* Finally, if c is real and numerically greater than one, $F(u) = Au^{-2+ni}$, where n is a real constant, and the surface (1) is a spiral surface.

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* My paper, Annals of Mathematics, l. c.