CERTAIN TYPES OF INVOLUTORIAL SPACE TRANSFORMATIONS*

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1. Statement of problem. Although there are many isolated examples of involutorial space transformations, the only type which has been systematically investigated is the monoidal one.† A (2, 1) correspondence between two spaces \((x)\) and \((x')\) may be expressed by three equations, algebraic and homogeneous in \(x_1, x_2, x_3, x_4\) and all linear and homogeneous in \(x'_1, x'_2, x'_3, x'_4\). With a point \(P_1\) in \((x)\) a unique point \(P'\) in \((x')\) is associated, but to \(P'\) correspond two points \(P_1\) and \(P_2\). Thus, when either \(P_1\) or \(P_2\) is given the other is uniquely defined. The pairs of points in \((x)\) thus determine an involutorial transformation \(I\) in the space \((x)\).

By this method it is possible to discuss various types of involutions and to develop certain properties common to all. Every space involution previously mentioned and having a surface of invariant points is included as a particular case in the present scheme.

2. Equations of the transformation. Let the defining equations be

\[(1a) \quad \sum a_i x'_i = 0,\]
\[(1b) \quad \sum b_i x'_i = 0,\]
\[(1c) \quad \sum c_i x'_i = 0,\]

in which \(a_i, b_i, c_i\) are polynomials in \(x_1, x_2, x_3, x_4\) of degrees \(n_1, n_2, n_3\) respectively. The surfaces in \((x)\) have points and curves in common which are together equivalent to \(n_1 n_2 n_3 - 2\) points.

The image of a plane \(\sum \lambda_i x_i = 0\) in \((x')\) is the surface

\[(2) \quad \sum |a_i b_2 c_3 \lambda_4| = 0\]

of order \(n_1 + n_2 + n_3 = n\). All the surfaces of the web pass through common basis curves \(\beta\), and may also have isolated basis points; the basis elements of any three are together equivalent to \(n - 2\) common points.

Two surfaces of the web \((2)\) intersect in a variable curve of definite order \(n'\); this curve \(c_{n'}\) is the image of a straight line in \((x')\). The image of a

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plane $\sum k_i x_i = 0$ in $(x)$ is a surface $s_n'$ of order $n'$ in $(x')$. The image in $(x')$ of a straight line in $(x)$ is a curve $c_n'$ of order $n$.

3. Surfaces of branch points and of coincidences. The locus of points in $(x')$ which have two coincident images in $(x)$ is the surface of branch points. It will be designated by $L'(x')$. The image of $L'(x')$ is the surface of coincidences $K(x)$, counted twice. The order of $L'$ is equal to the number of points in which it is met by a straight line $c_i'$, and also to the number of coincidences on the image curve $c_n'$. If the latter is of genus $\pi$, it follows from Zeuthen's formula that the order of $L'(x')$ is $2\pi + 2$.

4. Fundamental elements. The image of an isolated basis point in $(x)$ is a curve or a surface. The image of every point of a fundamental curve $\beta$ is a curve whose order is equal to the multiplicity of $\beta$ on the surfaces of the web $(2)$. These curves generate a surface $B'$ whose order is equal to the number of points in which $c_n'$ meets $\beta$. Similarly, there may be fundamental surfaces and curves in $(x)$, images of basis curves and points in $(x')$.

Another kind of basis element in $(x')$ may appear when $n_1 = n_2$, namely those points for which the two equations $\sum a_i x_i = 0$, $\sum b_i x_i = 0$ define the same surface in $(x)$.

The jacobian of the web $(2)$ consists of $K(x)$ and of fundamental elements.

5. The involution I. The image of a plane $s_1$ in $(x)$ is a surface $s_n'$. The image of $s_n'$, in $(x)$ is $s_1$ and a residual surface $s_N$, image of $s_1$ in the involution $I$.

The surface $s_1$ meets $K(x)$ in a plane curve $(s_1, K)$, through which $s_N$ passes. The surface $s_N$ also meets $s_1$ in a second curve $c_d$. The curve $(s_1, K)$ is the image of the curve of contact of $s_n'$, and $L'(x')$.

As in plane geometry* we shall make use of the following lemma:

Lemma. The necessary and sufficient condition that the image in $(x)$ of a locus in $(x')$ shall be composite is that the given locus shall touch $I'(x')$ at every common point.

The curve $c_d$ is the image of the double curve on $s_n'$.

The jacobian of the web $s_N$ includes all the fundamental surfaces of $I$. If $B'$ has $B$ for its image in $(x)$ residual to the basis curve $\beta$, then $B$ is a fundamental surface of the involution.

6. Example; two planes and a quadric surface. The simplest example of an involution from the present standpoint is furnished by $(1)$ when $a_i$, $b_i$ are linear in $(x)$, and $c_i$ is quadratic. The image of a plane $s_1'$ in $(x')$ is a quartic surface $s_4$ defined by $(2)$. It can be proved† that two surfaces $s_4$

* Sharpe and Snyder, Types of $(2, 2)$ correspondences between two planes, these Transactions, vol. 17 (1917), pp. 402-414. See p. 403.
intersect in a basis curve $\beta_{11}$ of order 11 and genus 14, common to all the quartics of the web, and in a variable quintic $c_5$ of genus 2, meeting $\beta_{11}$ in 18 points. The curve $c_5$ is the complete image of a line $c_1$ of $(x')$.

Let a line $c_1'$ be defined parametrically by

$$\tau x'_i = k_i + l_i \rho$$

in which $k$, $l$ are constants, and $\rho$ the parameter. By substituting these values for $x'_i$ in (1), and solving for $\rho$ we have equations of the form

$$\frac{\bar{a}}{a} = \frac{\bar{b}}{b} = \frac{\bar{c}}{c} = \frac{\bar{e}}{e} = \frac{\bar{c}}{c} = \rho,$$

in which $a, b, \bar{a}, \bar{b}$ are linear in $(x)$, and $\bar{e}, e$ are both quadratic. The surfaces $(\bar{a}c - \bar{a}e) = 0$, $(\bar{b}c - \bar{b}e) = 0$, $(\bar{b}a - \bar{b}a) = 0$ all contain the image, but the $c_5$ common to the first two has $c_4$ not on the third, hence the proper image is a $c_6$ on a quadric, hence of genus 2.

The image of a point on $\beta_{11}$ is a straight line, hence the image of the whole curve $\beta_{11}$ is a ruled surface $B_{18}$ of order 18.

The surface of coincidences $K_{12}$ is the jacobian of the web of quartics $s_4$; it is of order 12 and has $\beta_{11}$ for triple curve. $L_{6}$ is of order 6.

The image of a plane $s_1$ in $(x)$ is a quintic surface $s'_5$. The complete image of $s_4$ in $(x')$ consists of $s'_1$ taken twice, and of $B'$, hence we again find that $B'$ is of order 18. The complete image in $(x)$ of $s'_5$ consists of the plane $s_1$ and of a residual surface $s_{19}$ of order 19, having $\beta_{11}$ for five-fold curve. The surface $s_{19}$ meets $s_1$ in the plane curve $(s_1, K_{12})$, and in a residual curve $c_7$, image of the double curve of $s'_5$. The curve $c_7$ has 11 double points at the points in which $\beta_{11}$ meets $s_1$.

A plane section of $s_4$ is of genus 3, thus $(s_1, s_4)$ is of genus 3; the image of $s_4$ is $s'_1$, such that to a point of $s_4$ corresponds a single point of $s'_1$; similarly, the image of $s_1$ is $s'_5$, hence the genus of $(s'_5, s'_1)$ is also 3. Hence the double curve of $s'_5$ is of order 3 and has $c_7$ for image in $(x)$.

The equation of $L_{6}$ is found by expressing the condition that the line $(1a)$, $(1b)$ in $(x)$ shall touch its associated quadric. The form of the equation shows that $L_{6}$ has four double points $P'_1, P'_2, P'_3, P'_4$, whose coördinates satisfy the system

$$\begin{vmatrix}
  a'_1 & a'_2 & a'_3 & a'_4 \\
  b'_1 & b'_2 & b'_3 & b'_4
\end{vmatrix} = 0.$$

The double curve of every quintic surface $s'_5$, image of a plane $s_1$ in $(x)$, passes through all the points $P'_i$.

The surface $L'_1$ has 35 other double points, the coördinates of which make all the first minors of the determinant in the equation of $L_{6}$ vanish. They are the
values of the parameters $x'_i$ for which the line $(1a), (1b)$ is a generator of its associated quadric. These points will be designated by $G'_i$.

The image $s'_i$ of a plane $s_i$ touches $L'_6$ along a curve $c'_6$ having double points at $P'_i$ and passing simply through $G'_i$. The image of $P'_i$ is a conic $p_i$ on $K_{12}$. The image of $G'_i$ is a straight line $g_i$ on $K_{12}$. Each conic $p_i$ meets $\beta_{11}$ in 8 points and each line $g_i$ meets $\beta_{11}$ in 4 points, hence 8 generators of $B'_{18}$ pass through each point $P'_i$ and 4 pass through each point $G'_i$.

A straight line $s_1$ meets $K_{12}$ in 12 points; its image $c'_1$ touches $L'_6$ in 12 points. The complete image in $(x)$ of $C_1$ is $c_1$ and a residual $c_{19}$ which meets $s_1$ in 12 points on $K_{12}$. A line $c_1$ meeting $\beta_{11}$ in $i$ points has for image a curve of order $4 - i$, touching $L'_6$ in $3 \cdot (4 - i)$ points, and $i$ generators of $B'_{18}$.

The surface $B'_{18}$ has for images in $(x)$ the curve $\beta_{11}$, and a surface $B_{72}$ having $\beta_{11}$ as 19-fold curve, each conic $p_i$ 8-fold and each line $g_i$ 4-fold.

In the involution $I$ each $s_4$ of the web goes into itself, and every point of $K_{12}$ is invariant. The image of $\beta_{11}$ is $B_{72}$, the jacobian of the web $s_{19}$. The image of any point of $p_i$ is the whole conic $p_i$, and of any point of $g_i$ is the whole line $g_i$.

Three surfaces of order 19 intersect in 6859 points. For the transformation considered the curve $\beta_{11}$ is equivalent to 6925 intersections but on account of the four intersections with $\beta_{11}$ each line $g_i$ counts for $-1$ intersections, while each conic $p_i$ counts for $-8$ intersections. Hence the number of variable intersections is $6859 - 6925 + 35 + 32 = 1$. This equation verifies the results found. We have therefore established the existence of the involution $I$ of order 19 having a fundamental five-fold curve $\beta_{11}$ of order 11 and genus 14, 35 fundamental simple lines which are quadrisections of $\beta_{11}$ and 4 fundamental double conics which are octasections of $\beta_{11}$.

7. Quadrics through a conic. If $(1b), (1c)$ represent systems of quadrics through the conic $\gamma_2 = x_4 = 0, \phi(x_1, x_2, x_3) = 0$ the equations are of the form

\[(1b) \quad x_4 b + \phi x'_1 = 0, \quad b = \sum b_{ik} x_i x'_k,\]
\[(1c) \quad x_4 c + \phi x'_2 = 0, \quad c = \sum c_{ik} x_i x'_k.\]

The quadric $(1b)$ intersects that defined by $(1c)$ in $\gamma_2$ and in a residual conic which is met by the plane $(1a)$ in two variable points. The equations (1) define an involution $I$ distinct from that already considered. Proceeding as before we find that a plane $s'_1$ has for image in $(x)$ a surface $s_{5}$ of order 5, having $\gamma_2$ for double curve. Also a straight line $c'_1$ goes into a curve $c_6$ of order 6 and genus 2, meeting $\gamma_2$ in 6 points. Two planes $s'_1$ meet in a line $c'_1'$; their image surfaces $s_5$ meet in $\gamma_2$ counted four times, in $c_6$, and in a basis curve $\beta_{11}$ of order 11 and genus 11. The curves $\beta_{11}, \gamma_2$ meet in 10 points. These facts
may be expressed by the symbols

\[ s' \sim s_b : \gamma_2^2 + \beta_{11}, \quad [\beta_{11}, \gamma_2] = 10, \]
\[ c'_i \sim c_b, \quad p = 2, \quad [c_b, \gamma_2] = 6, \quad [c_b, \beta_{11}] = 16. \]

To find the image of a plane \( s_1 = \Sigma k_i x_i = 0 \) we first replace (1b) by

(1b') \[ x'_1 c - x'_2 b = 0, \]

obtained from (1b) and (1c).

The equation (1b') is quadratic in \( (x') \), instead of linear, as considered in the preceding type. The equations, however, still define a \((1, 2)\) correspondence between the two spaces \((x'), (x)\), as may be seen as follows. Given a point \((x)\), of the two points in \((x')\), defined by (1), one is always in the fixed plane \( x'_2 = 0 \). If we find the image of \( s_1 \) by eliminating \( x_1, x_2, x_3, x_4 \) between (1) and \( x'_i, x'_2 = 0, x'_3 \) is a factor of the resultant. The other factor, equated to zero defines a surface of order 6, passing simply through the line \( \lambda' = x'_1 = 0, x'_2 = 0 \).

Hence \( s_1 \sim s'_6 : \lambda' \).

Similarly, by the method of Art. 6 we find \( L' \) to be of order 6, but that it does not contain \( \lambda' \).

The image of \( x_4 = 0 \) is a surface of order 6 containing \( \lambda' \) to multiplicity 2. From equations (1) we can see that the image of a point on \( x_4 = 0 \), not on \( \gamma_2 \) is a point of \( \lambda' \), hence we conclude \( \gamma_2 \sim \Gamma'_6 : \lambda'^2 \). The same result may be obtained by expressing the coördinates of a point of \( \gamma_2 \) in terms of a parameter \( \mu \), and eliminating \( \mu \) between (1a) and (1b'), when \( x_1, x_2, x_3 \) have been replaced by the quadratic functions of \( \mu \). Incidentally it also appears that the image in \((x')\) of a point on \( \gamma_2 \) is a conic in a plane through \( \lambda' \).

The image of \( s_8 \) consists of \( s'_1 \) taken twice, \( \Gamma'_6 \) taken twice, and of \( B'_1 \), the image of \( \beta_{11} \). The ruled surface \( B'_6 \) contains \( \lambda' \) simply, so that we may write

\[ \beta_{11} \sim B'_6 : \lambda'. \]

To obtain the image of \( \lambda' \), we first consider a point \((0, 0, y_3', y_4')\) on it. From (1), the image is defined by the equations

\[ y'_3 a_3 + y'_4 a_4 = 0, \quad x'_4 (y'_3 b_3 + y'_4 b_4) = 0, \quad x'_4 (y'_3 c_3 + y'_4 c_4) = 0. \]

One image point is the intersection of the three planes

\[ y'_3 a_3 + y'_4 a_4 = 0, \quad y'_3 b_3 + y'_4 b_4 = 0, \quad y'_3 c_3 + y'_4 b_4 = 0, \]

and the other is the entire line

\[ x'_4 = 0, \quad y'_3 a_3 + y'_4 a_4 = 0. \]
Thus the point \( y' \) is fundamental as to one of its images, and regular as to the other. As the point \( y' \) describes \( \lambda' \), the image point describes a space cubic curve \( \alpha_3 \), and the associated line describes a plane pencil in \( x_4 = 0 \).

Thus we may write

\[ \lambda' \sim x_4 = 0 \quad \text{and} \quad \alpha_3. \]

Moreover, from the preceding results we conclude

\[ L' \sim K_{15} : \gamma^6_2 + \beta_{11}. \]

The same result is obtained from the jacobian of the web of \( s_6 \), images of the planes of \( (x') \). The jacobian consists of \( K_{15} \) and of the plane \( x_4 = 0 \).

The number of points \( G_i \) for which \((1a)\) \((1b')\) define a generator of the quadric \((1c)\) will be denoted by \( x \), and the number of points \( P_i \) for which \((1a)\) \((1b')\) define the same plane in \((x)\) by \( y \).

The image of \( s_6' : \lambda' \) in \((x)\) is \( s_1 \) and \( s_{28} : \gamma''_2 + \beta^6_1 + \alpha_3 \), image of \( s_1 \) in the involution \( I \). Since the image of \( c_1 \) in \( I \) is \( c_{28} \) the curve of intersection of two surfaces \( s_{28} \) consists of \( c_{28} \) and of fundamental curves. Each line \( g_i \) is simple on \( s_{28} \) and each conic \( p_i \) is double. We obtain

\[ x + 8y = 115. \]

The lines \( g_i \) and the conics \( p_i \) are simple on \( K_{15} \). Each surface \( s_{28} \) meets \( K_{15} \) in the plane curve \((s_1, K_{15})\) and in fundamental curves, hence

\[ x + 4y = 75. \]

Thus \( x = 35, y = 10 \).

The lines \( g_i \) meet \( \gamma_2 \) once and \( \beta_{11} \) in three points. The conics \( p_i \) meet \( \gamma_2 \) twice, and \( \beta_{11} \) in six points. In the involution \( I \) we may now write

\[
\begin{align*}
  s_1 & \sim s_{28} : \gamma^{11}_2 + \beta^{6}_{11} + 35g_i + 10p^9_i + \alpha_3, \\
  \gamma_2 & \sim \Gamma_{28} : \gamma^{11}_2 + \beta^{6}_{11} + 35g_i + 10p^9_i + \alpha_3, \\
  \beta_{11} & \sim B_{79} : \gamma^{31}_2 + \beta^{17}_{11} + 35g^2_i + 10p^5_i + \alpha_3, \\
  \alpha_3 & \sim s_1 : \gamma_2.
\end{align*}
\]

The jacobian of the involution, of order 108, consists of the plane \( x_4 = 0 \), of \( \Gamma_{28} \) and of \( B_{79} \).

The result may be stated as follows:

**Theorem:** There exists a space involution of order 28, the basis curves consisting of a conic \( \gamma_2 \) to multiplicity 11, a curve \( \beta_{11} \) (of order and genus 11 meeting \( \gamma_2 \) in 10 points) to multiplicity 6, thirty-five simple straight lines meeting \( \gamma_2 \) and having three points on \( \beta_{11} \), and ten double conics, meeting \( \gamma_2 \) twice and \( \beta_{11} \) six times.

8. **Quadrics and cubics.** The next case in order of simplicity is that in
which (1a) is linear, (1b) quadratic and (1c) cubic in \((x)\). If the quadrics have in common a line and a conic meeting it, and the cubics have the line double, and the conic simple, then the residual intersection is necessarily two skew lines belonging to the congruence of lines which meet the line and the conic. If the \((x)\) space is transformed birationally so that the lines of the congruence go into the lines of a bundle, then the involution \(I\) is transformed into an involution of the monoidal type already considered by Montesano.

If, however, the basis conic is replaced by two common generators, then the variable intersection is a proper conic, and a new involution results, which we proceed to discuss. Let \(\gamma = x_1 = 0, x_3 = 0\) and \(\delta = x_2 = 0, x_4 = 0\) be the two skew lines meeting \(\alpha = x_1 = 0, x_2 = 0\).

The equations then take the forms

\[(1b) \quad x'_1x_1x_2 + x'_2x_1x_4 + x'_3x_2x_3 = 0,\]
\[(1c) \quad x_1x_2(c'_1x_1 + c'_2x_2) + x_1x_4(c'_3x_1 + c'_4x_2) + x_2x_3(c'_5x_1 + c'_6x_2) = 0,\]

in which \(c'_i\) is linear in \((x')\).

Proceeding as in the previous cases we find

\[s'_1 \sim s_6 : \alpha^3 + \gamma^2 + \delta^2 + \beta_{12}, \quad [\alpha, \beta_{12}] = 6, \quad [\gamma, \beta_{12}] = [\delta, \beta_{12}] = 5, \]
\[c'_1 \sim c_7, \quad p = 2, \quad [c_7, \alpha] = 4, \quad [c_7, \gamma] = [c_7, \delta] = 3, \quad [c_7, \beta_{12}] = 16.\]

There are two fundamental lines in \((x')\): \(\lambda' = x'_3 = 0, c'_6 = 0\), whose image is \(x_1 = 0, \) and \(\mu' = x'_4 = 0, c'_3 = 0\), whose image is \(x_2 = 0\). There is also a fundamental point \(Q' = (0, 0, 0, 1)\) whose image is a plane cubic curve \(q_3\) in \(a_4 = 0, [\alpha, q_3] = 2, [\beta_{12}, q_3] = 8, [\gamma, q_3] = 1, [\delta, q_3] = 1\). We also find \(L'_7 : Q'^2\) and consequently

\[K_{18} : \alpha^9 + \gamma^6 + \delta^8 + \beta_{12}^3 + q_3.\]

Moreover

\[s_1 \sim s'_1 : \lambda' + \mu' + Q'^3, \quad \alpha \sim A'_1 : \lambda' + \mu' + Q'^2, \]
\[\gamma \sim \Gamma'_1 : \lambda' + Q', \quad \delta \sim \Delta'_1 : \mu' + Q', \quad \beta_{12} \sim B'_1 = \lambda' + \mu' + Q'^3.\]

In the involution \(I\)

\[s_1 \sim s_{29} : \alpha^{19} + \gamma^{13} + \delta^{13} + \beta_{12}^3 + q_3, \]
\[\alpha \sim A_{22} : \alpha^{11} + \gamma^7 + \delta^8 + \beta_{12}^2 + q_3^2, \]
\[\gamma \sim \Gamma_{17} : \alpha^8 + \gamma^6 + \delta^6 + \beta_{12}^2 + q_3, \]
\[\delta \sim \Delta_{17} : \alpha^8 + \gamma^6 + \delta^6 + \beta_{12} + q_3, \]
\[\beta_{12} \sim B_{84} : \alpha^{46} + \gamma^{31} + \delta^{31} + \beta_{12}^3 + q_3^3.\]

The jacobian consists of \(A_{22}, \Gamma_{17}, \Delta_{17}, B_{84}\) and the two planes \(x_1 = 0, x_2 = 0\). Further, it is found that \(x = 24, y = 18\). Ten of the lines \(g_i\) are bisecants of
\( \beta_{12} \) and meet \( \gamma \) and \( \delta \); the other 14 are bisecants of \( \beta_{12} \) and meet \( \alpha \). The 18 conics \( p_1 \) meet \( \beta_{12} \) in 5 points, and have one point on \( \alpha, \gamma, \delta \). 

9. Generalization, and a basis of classification. The next case in order of simplicity is given by the equations

\[
(1b) \quad H_1 \sum b_{ik} x_i x'_k + H_2 \sum c_{ik} x_i x'_k = 0, \\
(1c) \quad H_1 x'_1 + H_2 x'_2 = 0,
\]

in which \( H_i = 0 \) is a general quadric surface. But more generally we may take for \((1b), (1c)\) any two systems of surfaces which intersect in a variable conic in \( (x) \), the same equations defining surfaces which intersect in a variable line in \( (x') \). The equations may be combined to produce two linear equations in \( (x') \), or to produce one linear equation and one quadratic equation in \( (x) \).

In the cases discussed in Articles 7 and 8, the equations which define the variable conic in \( (x) \) explicitly determine in \( (x') \) a variable line and an extraneous curve lying on an extraneous surface, which appears as a component of \( L' \) and of the image of a plane of \( (x) \).

With the exception of such cases, the variable line is defined by two equations of one of the following forms in \( (x') \).

I. Both equations linear.
II. One equation linear, the other nonlinear.
III. Neither equation linear.

Of these, I has been completely discussed in Art. 6. In II, the linear equation defines a pencil of planes, and the other a system of surfaces of order \( n \), having the axis of the pencil to multiplicity \( n - 1 \). Two cases arise, according as the linear equation in \( (x') \) is linear or quadratic in \( (x) \). In III, the lines belong to a congruence of lines of order one; they consist of the lines meeting a rational curve of order \( n \) and its \((n - 1)\)-fold secant, or of the bisecants of a space cubic curve.

10. Surfaces intersecting in two skew lines. The systems \((1b) (1c)\) may be surfaces intersecting in two variable skew lines. When these lines belong to a rational congruence, a birational point transformation can be found which transforms the congruence into a bundle. The involution is therefore of the monoidal type already considered by Montesano. When the lines do not belong to a congruence, they may be defined by a pair of planes and a system of surfaces of order \( n \), having the line of intersection to multiplicity \( n - 1 \). The equations have the form

\[
(1c) \quad x'_1 x_1^2 + x'_2 x_1 x_2 + x'_3 x_2^2 = 0, \\
(1b) \quad \sum x'_i b_i = 0,
\]
in which \( b_i = 0 \) is a surface of order \( n \), having \( x_1 = 0, x_2 = 0 \) to multiplicity \( n - 1 \).

The surface \( L' \) is a quadric cone, hence \( K \) is a rational surface. It belongs to a linear system of rational surfaces, of which the images of the planes of \( (x') \) is a partial system. Every surface of the system is transformed into itself by the involution \( I \). Within the complete system is a web having a basis point in common. If \( s_1 = 0, s_2 = 0, s_3 = 0 \) define three of these surfaces which also pass through the image of the basis point in \( I \), and \( s_4 = 0 \) is the fourth independent surface of the web, then the transformation

\[
y_1 = s_1, \quad y_2 = s_2, \quad y_3 = s_3, \quad y_4 = s_4
\]

transforms the involution into one of monoidal type, in which the lines of the bundle with vertex at \((0, 0, 0, 1)\) remain invariant. We now resume the discussion of Types II and III.

11. Type II. Pencil of quadrics in \((x')\). The defining equations have the form

\[
\sum a_i x'_i = \sum a'_i x_i = 0,
\]

\[
\sum b_i x_i = 0 \quad \text{or} \quad \sum b_i x'_i = 0,
\]

\[
c_1 x'_1 + c_2 x'_2 = 0,
\]

in which \( b'_i = 0 \) is a surface of order \( n \), having \( \lambda' = x'_1 = 0, x'_2 = 0 \) for line of multiplicity \( n - 1 \). The surfaces \( b_i = 0 \) are of order \( 2n - 1 \), having \( \gamma_4 \), the curve common to the quadrics \( c_1 = 0, c_2 = 0 \) to multiplicity \( n - 1 \).

The image of a plane \( s'_1 \), by (2) is found to be \( s'_{2n+2} \), a surface of order \( 2n + 2 \), having \( \gamma_4 \) as an \( n \)-fold curve, and by the method of Art. 6 the image of a straight line \( c'_1 \) is a curve \( c_{2n+3} \) of order \( 2n + 3 \) and genus \( n + 1 \). The basis curve \( \beta_{6n+1} \) is of order \( 6n + 1 \) and genus \( 9n - 3 \). The curve \( c_{6n+3} \) meets \( \gamma_4 \) in \( 4n + 4 \) points, and \( \beta_{6n+1} \) in \( 6n + 4 \) points. The curves \( \beta_{6n+1} \) and \( \gamma_4 \) meet in \( 12n - 4 \) points. Since the image of \( c_{6n+3} \) is \( c'_1 \) in \((x')\), it follows that the image \( B' \) of \( \beta_{6n+1} \) is of order \( 6n + 4 \), and \( \Gamma' \), the image of \( \gamma_4 \), is of order \( 4n + 4 \).

The image of a plane \( s_1 \) is a surface \( s'_{2n+3} \) of order \( 2n + 3 \), having the line \( \lambda' \) to multiplicity \( 2n - 1 \). A line \( c_1 \) has for image a curve \( c'_{2n+2} \).

From (2) it follows at once that the image of a point \((0, 0, y'_3, y'_4)\) on \( \lambda' \) is the plane curve

\[
a_3 y'_3 + a_4 y'_4 = 0, \quad b_3 y'_3 + b_4 y'_4 = 0
\]

of order \( 2n - 1 \) and having four points of multiplicity \( n - 1 \) on \( \gamma_4 \). The image of \( \lambda' \) is the surface

\[
A_{2n} = a_3 b_4 - a_4 b_3 = 0.
\]

It contains \( \beta_{6n+1} \) simply, and \( \gamma_4 \) to multiplicity \( n - 1 \). The quadric
\[ c_1 x'_1 + c_2 x'_2 = 0 \] meets \( \beta_{6n+1} \) in 6 points not on \( \gamma_4 \), hence the image plane \( c_1 x_1 + c_2 x_2 = 0 \) meets \( B_{6n+4} \) in 6 lines, so that \( \lambda' \) is of multiplicity \( 6n - 2 \) on \( B'_{6n+4} \).

The equation of \( \Gamma' \), the image of \( \gamma_4 \) may be obtained by eliminating \( (x) \) from the equations
\[
c_1 = 0, \quad c_2 = 0, \quad \sum a_i x'_i = 0, \quad \sum b_i x'_i = 0.
\]
The result is a surface \( \Gamma' \) of order \( 4(n + 1) \), containing \( \lambda' \) to multiplicity \( 4(n - 1) \).

12. Surfaces of coincidences and of branch points. From Art. 3 it follows that \( L'_{2n+4} \) is of order \( 2n + 4 \), and by Art. 6 it has \( \lambda' \) to multiplicity \( 2n \).

Let \( x \) be the number of double points \( G_i \) and \( y \) the number of double points \( P'_i \). The surface of coincidences \( K \) is the jacobian of the web \( s_{2n+2} \), after removing the component \( \Lambda_{2n} \). It is of order \( 6n + 4 \), contains \( \gamma_4 \) to multiplicity \( 3n \), and \( \beta_{6n+1} \) to multiplicity \( 2 \). It also contains \( y \) conics \( p_i \) and \( x \) lines \( g_i \). A plane section of \( K_{6n+4} \) has for image the curve of contact of \( s_{2n+3} \) and \( L'_{2n+4} \). It is of order \( 8n + 6 \).

13. The involution. The image of \( s_1 \) is \( s'_{2n+3} \), having \( \lambda' \) to multiplicity \( 2n - 1 \); its image in \( (x) \) consists of \( s_1 \) and of a surface \( s_{12n+5} \) of order \( 12n + 5 \) having \( \gamma_4 \) to multiplicity \( 6n - 1 \), and \( \beta_{6n+1} \) as a 4-fold curve. A plane \( s_1 \) meets its image \( s_{12n+5} \) in \( (s_1, K_{6n+4}) \) and a residual curve \( c_{6n+1} \) of order \( 6n + 1 \). This curve has four points of order \( 3n - 1 \) on \( \gamma_4 \) and \( 6n + 1 \) points of order \( 2 \) on \( \beta_{6n+1} \). It is of genus \( 9n - 5 \). Its image is the double curve \( \delta_{2n} \) of order \( 3n \) on \( s'_{2n+3} \). It passes through each point \( P'_i \). The image of \( y_4 \) in \( I \) is a surface \( \Gamma_{24n+8} \) of order \( 24n + 8 \), containing \( \gamma_4 \) to multiplicity \( 12n - 3 \), and \( \beta_{6n+1} \) to multiplicity \( 8 \). The image of \( \beta_{6n+1} \) is a surface \( B_{24n+2} \) of order \( 24n + 8 \). It contains \( \gamma_4 \) to multiplicity \( 12n - 2 \) and \( \beta_{6n+1} \) to multiplicity \( 7 \). The jacobian of \( I \) consists of \( \Gamma \) and \( B \).

To determine \( x \) and \( y \), it is known that two surfaces \( s_{12n+5} \) intersect in a curve \( c_{12n+5} \) of order \( 12n + 5 \) and in fundamental elements consisting of \( \gamma_4 \), \( \beta_{6n+1} \), \( y \) conics \( p_i \), double on each, and \( x \) lines \( g_i \), simple on each. Hence
\[
x + 8y = 60n.
\]
Similarly, the intersection of \( s_{12n+5} \) with \( K_{6n+4} \) consists of \( (s_1, K_{5n+4}) \) and of fundamental elements
\[
x + 4y = 36n + 8.
\]
Hence
\[
x = 12n + 16, \quad y = 6n - 2.
\]
Each line \( g_i \) is a bisecant of \( \gamma_4 \) and of \( \beta_{6n+1} \). Each conic \( p_i \) meets \( \gamma_4 \) and \( \beta_{6n+1} \) in four points.

Three surfaces of order \( 12n + 5 \) intersect in \( (12n + 5)^2 \) points. The
equivalence of $\gamma_4$ is $1728n^3 + 2160n^2 - 864n + 76$ and of $\beta_{6n+1}$ is $3456n^2 - 672n + 496$. The decrease in equivalence for the $12n - 4$ intersections of $\beta_{6n+1}$ and $\gamma_4$ is $3456n^2 - 2496n + 448$. Each line $g_i$ decreases the equivalence by 1 and each conic $p_i$ by 8. The total equivalence is therefore $(12n + 5)^3 - 1$ as required.

There exists then an involution of order $12n + 5$ having a fundamental quartic $\gamma_4$ of genus 1 to multiplicity $6n - 1$ and a fundamental curve $\beta_{6n+1}$ of order $6n + 1$, genus $9n - 3$ to multiplicity 4, meeting $\gamma_4$ in $12n - 4$ points. It has also $12n + 16$ simple fundamental lines meeting $\gamma_4$, $\beta_{6n+1}$ each twice, and $6n - 2$ fundamental double conics meeting $\gamma_4$, $\beta_{6n+1}$ each in 4 points.

14. Type II$_2$. Pencil of planes in $(x)$. The equations are

\begin{align*}
(1a) & \quad \sum a'_i \; x_i = \sum a_i \; x'_i = 0, \\
(1b) & \quad x'_1 \; x_1 + x'_2 \; x_2 = 0, \\
(1c) & \quad \sum c_i \; x'_i = 0, \quad \text{or} \quad (1c') \quad \sum c_i \; H_i = 0,
\end{align*}

in which $c_i$ is of order $n + 1$ in $(x)$, and of order $n - 1$ in $x_1, x_2$; $c'_i$ is of order $n$ in $(x')$ and of order $n - 1$ in $x'_1, x'_2$. $H_i$ is quadratic in $(x)$.

Let $\gamma = x_1 = 0, x_2 = 0$ and $\lambda'$ be defined as before.

We may now write

\begin{align*}
\gamma \sim s_{n+3}; & \quad [\beta_{5n+5}, \gamma] = 5n - 2. \quad p = 12n + 4. \\
c' \sim c_{n+4}; & \quad \text{genus } n + 1. \quad [c_{n+4}, \gamma] = n + 2. \\
s_1 \sim s_{n+4}; & \quad \lambda'^n. \\
c_1 \sim c'_{n+3}; & \quad [c'_{n+3}, \lambda'] = n + 2. \\
\lambda' \sim \Lambda_{n+2}; & \quad [\gamma_{n+1} + \beta_{5n+5}] + [\gamma_{n+1} + \beta_{5n+5}]. \quad \text{Point on } \lambda' \sim \Lambda_{n+1} \text{ with } n - 1 \text{ fold point on } \gamma. \\
x \sim \Gamma'_{n+2}; & \quad \lambda'^{n+1}. \quad \text{Point on } \gamma \sim \gamma' \text{ with } n - 1 \text{ fold point on } \lambda'. \\
\beta_{5n+5} \sim B_{5n+5}; & \quad \lambda'^{5n+3}.
\end{align*}

There are $15n + 10$ fundamental lines $g_i$ which meet $\gamma$ and are trisecants of $\beta_{5n+5}$ and two fundamental conics $p_i$ which are bisecants of $\gamma$ and hexasecants of $\beta_{5n+5}$. Moreover, we have $K_{3n+6} : \gamma^{3n} + \beta_{5n+5}^3 \sim L'_{4n+4} : \lambda'^{2n}$.

15. The involution I. From the preceding results it follows that in the involution

\begin{align*}
\gamma \sim \Gamma_{4n+8}; & \quad \gamma^{an+1} + \beta_{5n+5}^{2} + (15n + 10) g_i + 2p_i^2. \\
c_1 \sim c_{n+9}; & \quad (4n + 8) \text{ points on } \gamma \text{ and } (12n + 24) \text{ points on } \beta_{5n+5}.
\end{align*}

The fundamental elements are the line $\gamma$ and the basis curve $\beta_{5n+5}$

\begin{align*}
\gamma \sim \Gamma_{4n+8}; & \quad \gamma^{an} + \beta_{5n+6}, \\
\beta_{5n+5} \sim B_{12n+4}; & \quad \gamma^{12n+3} + \beta_{5n+5}^3.
\end{align*}
A point of $\beta_{5n+5}$ has for image a cubic curve lying in a plane through $\gamma$. Each such plane is invariant, hence the section made on $B_{12n+4}$ by every plane through $\gamma$ consists of 7 cubic curves. The jacobian of the involution consists of $\Gamma_{4n+2} + B_{12n+24}$.

The equivalence for the intersection of three surfaces, images of planes in the involution, is expressed as follows:

$$(4n + 9)^3 = 64n^3 + 432n^2 + 972n + 729.$$  
For $\gamma^{4n+1} - (64n^3 + 432n^2 + 204n + 25)$.  
For $\beta_{3n+5} - (540n^2 + 567n + 729)$.  
For $5n - 2$ intersections  
$$+ 540n^2 - 210n.$$  
For $\sum g_i = 15n + 10$.  
For $\sum p_i = 16$, making a total of 1, as it should.

For every positive integer $n$ we can now state the following

**Theorem:** There exists an involution of order $4n + 9$, the basis curves consisting of a line $\gamma$ to multiplicity $4n + 1$, a curve $\beta$ of order $5n + 5$, genus $12n - 4$, meeting $\gamma$ in $5n - 2$ points, to multiplicity 3, of two conics in planes through $\gamma$ and meeting $\beta$ in 6 points, to multiplicity 2, and finally of $15n + 10$ simple lines which meet $\gamma$ and are trisecants of $\beta$.

16. Type III. Basis curve of odd order. Pencil of planes. Two cases appear, according as a pencil of planes or of quadrics in $(x)$ is used as one of the defining systems. For the first one, the equations are

$$(1a) \quad \sum a_i x'_i = \sum a'_i x_i = 0,$$
$$(1b') \quad x_1 b'_1 + x_2 b'_2 = 0,$$
$$(1c') \quad b'_i (H_1 x'_1 + H_2 x'_2) + b'_i (H_3 x'_1 + H_4 x'_2) = 0,$$

in which $b'_i = 0$ is a surface of order $n$ in $(x')$, having $\lambda' = x'_1 = 0$, $x'_2 = 0$ for $(n - 1)$-fold line, and $H_i = 0$ is a quadric in $(x')$. By means of $(1b')$, $(1c')$ we may write

$$(1b) \quad b_1 x'_1 + b_2 x'_2 = 0,$$

in which $b_1 = H_1 x_2 - H_3 x_1$, $b_2 = H_2 x_2 - H_4 x_1$, and

$$(1c) \quad \sum c_i x'_i = 0,$$

wherein $c_i$ is of order $n - 1$ in $b_1$, $b_2$, and linear in $x_1$, $x_2$. The equation $b_i = 0$ defines a cubic surface containing $\mu = x_1 = 0$, $x_2 = 0$ and $\gamma_2$, a curve of order 8, genus 7, meeting $\mu$ in 4 points. The surfaces $c_i = 0$ are of order $3n - 2$, having $\mu$ for $n$-fold line, and $\gamma_8$ to multiplicity $n - 1$. 

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The surfaces \( b_i = 0, \ b'_i = 0 \), each of order \( n \), intersect in \( \lambda' \), \( (n - 1) \)-fold on each; and in a rational curve \( \theta'_{2n-1} \) of order \( 2n - 1 \), which meets \( \lambda' \) in \( 2n - 2 \) points.

This case offers no new difficulties. The scheme of its characteristic features may be written as follows

\[ s'_i \sim s_{3n+3} : \mu^{n+1} + \gamma_i^2 + \beta_{7n}, \quad \gamma_i, \ p = 7 \ [\gamma_i, \ \mu] = 4, \]
\[ c'_i \sim c_{3n+3} : p = 2n + 1, [c_{3n+3}, \mu] = n + 3; [c_{3n+3}, \gamma_i] = 8n + 4; \]
\[ [c_{3n+3}, \beta_7] = 7n + 1. \]

\[ \beta_{7n} \sim B'_{7n+1} : \lambda'^{7n-4} + \theta_{2n-1}^{2n}. \] Five generators concurrent on \( \theta'_{2n-1} \).

\[ s_1 \sim s_{3n+3} : \lambda'^{5n-2} + \theta_{2n-1}^3 p = 0, \ [\theta'_{2n-1}, \lambda'] = 2n - 2. \]
\[ c_1 \sim c_{3n+2}; [c_{3n+2}, \lambda'] = 3n - 1; [c_{3n+2}, \theta'_{2n-1}] = 6n - 2. \]

\[ \mu \sim M'_{n+3} : \lambda'^n + \theta_{2n-1}^n. \] Point on \( \mu \sim \) plane \( \mu'_{n+1} \).

\[ \gamma_8 \sim \Gamma'_{8n+4} : \lambda'^{8n-1} + \theta'_{2n-1}. \] Point on \( \gamma_8 \sim \) plane \( \gamma'_n \).

\[ \lambda' \sim \Lambda_{3n-1} : \mu^n + \gamma'^{-1} + \beta_{7n}. \] Point on \( \lambda' \sim \) plane \( \lambda_{3n-2} \).

\[ \theta'_{2n-1} \sim \Theta_{6n-2} : \mu^{2n-1} + \gamma'^{2n-1} + \beta_{7n}. \] Point on \( \theta'_{2n-1} \sim \) plane \( \theta_3; [\theta_3, \ \mu] = 1. \)

\[ K_{3n+7} : \mu^{n+4} + \gamma'^{n+1} + \beta_{7n} \sim L'_{4n+4} : \lambda'^{4n-2} + \theta'_{2n-1}. \]

\[ x = 10n + 20, \]
\[ y = n + 1. \]

In the involution I we now have

\[ s_1 \sim s_{6n+9} : \mu^{2n+6} + \gamma_{5}^{2n+1} + \beta_{7n}^2, \]
\[ \mu \sim M_{6n+8} : \mu^{2n+5} + \gamma_5^{2n+1} + \beta_{7n}^3, \]
\[ \gamma_8 \sim \Gamma_{12n+16} : \mu^{4n+12} + \gamma_8^{4n+1} + \beta_{7n}^4, \]
\[ \beta_{7n} \sim B_{6n+8} : \mu^{2n+6} + \gamma_8^{2n+1} + \beta_{7n}. \]

The jacobian of the system consists of \( M_{6n+8}, \ \Gamma_{12n+16}, \) and \( B_{6n+8} \).

17. Type III₂. Basis curve of odd order. Pencil of quadrics. In this case, (1a) is as in the last preceding type, but the others are

\[ (1b) \quad b'_1 (x_1 x'_1 + x_2 x'_2) + b'_2 (x_3 x'_1 + x_4 x'_2) = 0, \]
\[ (1c') \quad H_1 b'_1 + H_2 b'_2 = 0, \]

in which \( b'_i, H_i \) have the same meaning as before.

We may write

\[ b_1 x'_1 + b_2 x'_2 = 0, \]
\[ b_1 \equiv x_1 H_2 - x_3 H_1, \quad b_2 \equiv x_2 H_2 - x_4 H_1, \]

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and by substituting in \((1c')\), obtain
\[(1c) \quad \sum c_i x_i' = 0,\]
wherein \(c_i = 0\) is a surface in \((x)\), of order \(3n - 1\), having \(\mu_4 = (H_1, H_2)\) to
multiplicity \(n\), and \(\gamma_5\), the residual intersection of \(b_1 = 0, b_2 = 0\) to multi-
plicity \(n - 1\). The quartic \(\mu_4\) is of genus 1 and meets \(\gamma_5\) in 8 points, and \(\gamma_5\)
is of genus 2. The other fundamental lines are defined as before. We now have
\[
\begin{align*}
\delta' &\sim s_{3n+3} : \mu_4^{n+1} + \gamma_5^n + \beta_{7n+1}, \\
\epsilon' &\sim c_{3n+4} : [c_{3n+4}, \mu_4] = 4n + 8, [c_{3n+4}, \gamma_5] = 5n + 2, \quad [c_{3n+4}, \beta_{7n+1}] = 7n + 2. \\
\delta &\sim s_{3n+4} : \lambda^{3n-1} + \theta_{2n-1}^3. \\
\epsilon &\sim c_{3n+3} + [c_{3n+3}, \lambda'] = 3n, [c_{3n+3}, \theta_{2n-1}'] = 6n - 2. \\
\lambda &\sim \Lambda_{3n} : \mu_4^n + \gamma_5^{n-1} + \beta_{7n+1}. \quad \text{Point on } \lambda' \sim \text{plane } \lambda_{3n-1}. \\
\mu_4 &\sim M_{4n+8} : \lambda'^{4n} + \theta_{2n-1}^n. \quad \text{Point on } \mu_4 \sim \text{plane } \mu_{4n+1} \text{ with } n\text{-fold point on } \lambda'. \\
\theta_{2n-1} &\sim \Theta_{5n-2} : \mu_4^{2n-1} + \gamma_5^{2n-1} + \beta_{7n+1}. \quad \text{Point on } \theta_{2n-1} \sim \text{plane } \theta_3. \\
\gamma_5 &\sim \Gamma_{5n+2} : \lambda'^{5n-5} + \theta_{2n-1}^3. \quad \text{Point on } \gamma_5 \sim \text{plane } \gamma_n. \\
\beta_{7n+1} &\sim B_{7n+2} : \lambda'^{7n-3} + \theta_{2n-1}^5. \quad \text{All generators in a plane through } \lambda' \text{ concurrent.} \\
K_{3n+16} &\sim \mu_4^{n+1} + \gamma_5^{n+1} + \beta_{7n+1} \sim L_{4n+4} : \lambda'^{4n-2} + \theta_{2n-1}^4. \\
x &\sim 10n + 20. \quad [g_i, \mu_4] = 2, \quad [g_i, \gamma_5] = 1, \quad [g_i, \beta_{7n+1}] = 1. \\
y &\sim n + 4. \quad [p_i, \mu_4] = 4, \quad [p_i, \gamma_5] = 2, \quad [p_i, \beta_{7n+1}] = 2.
\end{align*}
\]

The features of the involution I are as follows:
\[
\begin{align*}
\delta_1 &\sim s_{6n+17} : \mu_4^{2n+7} + \gamma_5^{2n+2} + \beta_{7n+1}^2, \\
\mu_4 &\sim M_{12n+32} : \mu_4^{4n+13} + \gamma_5^{4n+4} + \beta_{7n+1}^4, \\
\gamma_5 &\sim \Gamma_{6n+16} : \mu_4^{2n+7} + \gamma_5^{2n+1} + \beta_{7n+1}^2, \\
\beta_{7n+1} &\sim B_{6n+16} : \mu_4^{2n+7} + \gamma_5^{2n+2} + \beta_{7n+1}. \\
\end{align*}
\]

The jacobian of the involution consists of \(M_{12n+32}, \Gamma_{6n+16},\) and \(B_{6n+16}\).

18. **Type III. Basis curve of even order.** Equation \((1a)\) has the same
form, while the others are
\[
\begin{align*}
(1b) \quad (x_1 x_1' + x_2 x_2') b_2' + x_3 b_3' &= 0, \\
(1c) \quad (H_1 x_1' + H_2 x_2') b_2' + H_3 b_3' &= 0,
\end{align*}
\]
in which $H_i$ is quadratic in $(x)$; $b'_2$ is of order $n - 1$ in $(x')$ and of order $n - 2$ in $x'_1, x'_2$; $b'_3$ is of order $n$ in $(x')$, and of order $n - 1$ in $x'_1, x'_2$. The surfaces $x'_1 b'_2 = 0, x'_i b'_2 = 0, b'_i = 0$ are all of order $n$, all have $\lambda'$ to multiplicity $n - 1$ and also pass through a rational curve $\theta_{2n-2}$. This curve meets $\lambda'$ in $2n - 3$ points.

The surfaces $b_1 \equiv x_2 H_3 - x_3 H_2 = 0, b_2 = 0, b_3 = 0$ are cubics passing through a curve $\gamma_7$ of order 7 and genus 5. By means of (1b) and (1c) we may write

\begin{align*}
(1b') && b_2 x'_1 - b_1 x'_2 &= 0, \\
(1c') && \sum c_i x'_i &= 0,
\end{align*}

in which $c_i = 0$ is a surface of order $3n - 3$, having $\gamma_7$ to multiplicity $n - 1$, and the residual conic $\mu_2$ of $b_1 = 0, b_2 = 0$.

Proceeding as in the former cases we now find

\begin{align*}
\delta'_1 &\sim \delta_{3n+1} : \mu_2^{n-3} + \gamma_7^n + \beta_{7n-3}. \\
c'_1 &\sim c_{3n+2}, [c_{3n+2}, \mu_2] = 2(n - 1); [c_{3n+2}, \gamma_7] = 7n + 6; \\
&[c_{3n+2}, \beta_{7n-3}] = 7n - 2. \\
\lambda' &\sim \Lambda_{3n-2} : \mu_2^{n-2} + \gamma_7^{n-1} + \beta_{7n-3}. \quad \text{Point on } \lambda' \sim \text{plane } \Lambda_{3n-3}. \\
\delta_1 &\sim \delta'_{3n+2} : \lambda^{3n-3} + \beta_2^{3(n-1)}. \\
c_1 &\sim c_{3n+1}, [c_{3n+1}, \lambda'] = 3n - 2. \\
\theta'_{2n-2} &\sim \Theta_{6n-5} : \mu_2^{2n-2} + \gamma_7^{2n-2} + \beta_{7n-3}. \quad \text{Point on } \theta'_{2n-2} \sim \text{plane } \Theta_3. \\
\mu_2 &\sim M'_{2(n-1)} : \lambda^{2(n-2)} + \beta_{2n-2}. \quad \text{Point on } \mu_2 \sim \text{plane } \mu_{n-1}^2. \\
\gamma_7 &\sim \Gamma'_{7n+6} : \lambda^{7n-1} + \beta_{2n-2}. \quad \text{Point on } \gamma_7 \sim \gamma'_n; [\gamma'_n, \lambda'] = 1. \\
\beta_{7n-3} &\sim B'_{7n-2} : \lambda^{7(n-1)} + \beta_{2n-2}^5. \\
K_{3n+7} : \mu^{n-2} + \gamma_7^{2n+2} &\sim L'_{4n+4}. \\
x &= 10n + 25. \\
y &= n + 3.
\end{align*}

The associated involution has the characteristics

\begin{align*}
\delta_1 &\sim \delta_{6n+10} : \mu_2^{2(n-1)} + \gamma_7^{2n+3} + \beta_{7n-3}, \\
\gamma_7 &\sim \Gamma_{13n+27} : \mu_2^{6(n-1)} + \gamma_7^{6n+8} + \beta_7^{6}, \\
\beta_{7n-3} &\sim B_{6n+9} : \mu_2^{2(n-1)} + \gamma_7^{2n+3} + \beta_{7n-3}. \\
\end{align*}

19. Basis curve a space cubic curve. The equations have the form (1a) as
before and the other two are

\begin{align*}
(1b) & \quad b_1 x'_1 + b_2 x'_2 + b_3 x'_3 = 0, \\
(1c) & \quad b_1 x'_1 + b_2 x'_3 + b_3 x'_4 = 0,
\end{align*}

in which \( b_i \) is defined in Art. 18. The lines in \((x')\) defined by \((1b), (1c)\) are bisecants of the cubic curve common to the quadrics

\[
x'_1 x'_3 - x'_2^2 = 0, \quad x'_1 x'_4 - x'_1 x'_3 = 0, \quad x'_2 x'_4 - x'_3^2 = 0.
\]

It will be denoted by \(\theta'_5\). The characteristics of the \((1, 2)\) transformation are

\[
s'_1 \sim s_7 : \gamma_7^2 + \beta_{13}; \quad [\beta_{13}, \gamma_7] = 32. \quad \beta_{13} \text{ is of genus } 11.
\]

\[
c'_1 \sim c_8; \quad [c_8, \gamma_7] = 20; \quad [c_8, \beta_{13}] = 14. \quad p = 4.
\]

\[
s_1 \sim s_8 : \theta'_5^2.
\]

\[
c_1 \sim c'_7; \quad [c'_7, \theta'_5] = 11.
\]

\[
\gamma_7 \sim \Gamma_{20} : \theta'_5. \quad \text{Point on } \gamma_7 \sim \gamma'_2.
\]

\[
\beta_{13} \sim B'_{14} : \theta'_5^7.
\]

\[
\theta'_5 \sim \Theta_{11} : \gamma_7^2 + \beta_{13}^7. \quad \text{Point on } \theta'_5 \sim \theta_5.
\]

\[
K_{16} : \gamma_7^2 + \beta_{13} \sim L'_{10} : \theta'_5^4.
\]

\[
x = 35.
\]

\[
y = 4.
\]

The characteristics of the involution are

\[
s_1 \sim s_{22} : \gamma_7^2 + \beta_{13}^2,
\]

\[
\gamma_7 \sim \Gamma_{63} : \gamma_7^{30} + \beta_{13}^8,
\]

\[
\beta_{13} \sim B_{21} : \gamma_7^7 + \beta_{13}.
\]

We may therefore state the following

**Theorem:** There exists an involution of order 22, having for fundamental curves a curve \(\gamma\) of order 7 and genus 5 sevenfold, a curve \(\beta\) of order 13 and genus 11, two fold. The fundamental curves \(\gamma\) and \(\beta\) meet in 32 points. In addition there are 35 simple basis lines and 4 double conics. The lines meet \(\gamma\) in 3 points, and \(\beta\) in one. The conics meet \(\gamma\) in 6 points and \(\beta\) in two.

21. **Reduction of a congruence to a bundle.** In the various cases defined by lines belonging to a rational congruence in \((x')\) we now make the following transformation:

Let

\[
y'_1 = x'_1 b'_1, \quad y'_2 = x'_2 b'_1, \quad y'_3 = x'_1 b'_2
\]
if the basis curve is of odd order;

\[ y'_1 + x'_1 b'_1, \quad y'_2 = x'_2 b'_2, \quad y'_3 = b'_3 \]

if the basis curve is of even order, and

\[ y'_1 = x'_1 x'_4 - x'_2 x'_3, \quad y'_2 = x'_1 x'_2 - x'_2^2, \quad y'_3 = x'_2 x'_4 - x'_3^2 \]

if the basis curve is a space cubic.

The transformation

\[ z'_1 = y'_1 x'_4, \quad z'_2 = y'_2 x'_4, \quad z'_3 = y'_3 x'_4, \quad z'_4 = \sum a_{ik} x'_i y'_k \]

is birational and transforms the congruence into a bundle.

The equations of III, become

\[ x_1 z'_1 + x_2 z'_3 = 0, \]

\[ H_1 z'_1^2 + H_2 z'_2 z'_3 + H_3 z'_1 z'_3 + H_4 z'_2 z'_3 = 0, \]

\[ \sum x_i M'_k = 0, \]

in which \( M'_k = 0 \) is a monoid with vertex at \((0, 0, 0, 1)\).

In the equations of III, the coefficients \( x_i, H_i \) in the first two should be interchanged, the third remaining as before. In the two remaining types, the first two equations are replaced by

\[ x_1 z'_1 + x_2 z'_2 + x_3 z'_3 = 0, \]

\[ H_1 z'_1 + H_2 z'_2 + H_3 z'_3 = 0, \]

and the third has the same form.

22. Three general types of \((1, 2)\) correspondences. Let \( f_1, f_2, f_3 \) be three independent cremona functions of \( y'_1, y'_2, y'_3 \). The equation

\[ (1a) \quad \sum a_{ik} x'_i x'_k = 0 \]

and either

\[ x_1 f'_1 + x_2 f'_2 = 0, \quad \sum H_i F'_i = 0, \]

or

\[ \sum x_i F'_i = 0, \quad H_1 f'_1 + H_2 f'_2 = 0, \]

or

\[ x_1 f'_1 + x_2 f'_2 + x_3 f'_3 = 0, \quad H_1 f'_1 + H_2 f'_2 + H_3 f'_3 = 0, \]

in which \( F'_i \) is of order \( k \) in \( f'_1, f'_2, f'_3 \) and linear in \( f'_4 \) are general types of which the various forms of Type III are special cases. The transformation

\[ z'_1 = f'_1 x'_i, \quad z'_2 = x'_4 f'_2, \quad z'_3 = x'_4 f'_3, \quad z'_4 = \sum a_{ik} x'_i f'_k \]

reduces them to forms similar to those of the preceding articles. All such correspondences are therefore included as special cases of the type in which a point in \((x')\) is determined by the intersection of a line of a bundle with a monoid.

This discussion completes the classification of those involutions in which one of the equations (1) defining the (1, 2) correspondence between \((x)\) and \((x')\) is bilinear.

23. **Forms containing general quadrics.** When one equation (1) represents a general quadric in \((x)\), the other two must define a general line. As in Art. 9 these equations may be both linear, one linear, or neither linear. The first two cases have already been treated. In the third case the lines belong to a congruence of order one, which can be mapped upon a bundle, thus reducing the involution to the type already discussed by Montesano.

24. **Forms having basis points or curves.** Another paper is completed in which the quadric surfaces have basis curves or points in common; certain other types will be treated, in which all of the defining equations are of degree higher than the second, thus necessarily having basis elements.

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