ON THE THEORY OF DEVELOPMENTS OF AN ABSTRACT CLASS IN RELATION TO THE CALCUL FONCTIONNEL*

BY

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In his *Introduction to a Form of General Analysis*† E. H. Moore has called attention to the great importance of developments Δ in analysis, and has used them in a general theory which includes the theories of continuous functions and convergent series. The authors of the present paper have made further studies of the properties of developments in relation to the theory of Moore.‡

It is the purpose of the present paper to develop the theory of developments along lines inaugurated by Fréchet and developed by him and other investigators. The methods of analysis suggested by this theory have led the authors to results in the Calcul Fonctionnel, some of which have been published previously.§

The theory of developments Δ is placed, in the present paper, into close relation with the theories of systems $(\mathfrak{L})$ and $(\mathfrak{B})$ of Fréchet|| and the topological space of Hausdorff.¶

We develop the general theory in terms of five completely independent properties of a development Δ which together suffice to make the developed class $\mathfrak{P}$ a compact metric space (cf. Hausdorff, loc. cit.).

The theory is applied to determine necessary and sufficient conditions that a topological space be a compact metric space. A further application is made to spaces s satisfying axiom systems Σ_1 or Σ_2 introduced by R. L.

* Presented to the Society, April, 1919, and in part under other titles in March, 1913, April, 1917.

† *The New Haven Mathematical Colloquium* (Yale University Press, New Haven, 1910), pp. 1–150.

‡ A. D. Pitcher, *Interrelations of eight fundamental properties of classes of functions*, Kansas University Science Bulletin, vol. 7, no. 1 (1914), pp. 1–67; and E. W. Chittenden, *Infinite developments and the composition property* $(K_{12}B_1)_*$ *in General Analysis*, Rendiconti di Palermo, vol. 39 (1915), pp. 81–108.

§ A. D. Pitcher and E. W. Chittenden, *On the foundations of the calcul fonctionnel of Fréchet*, these Transactions, vol. 19 (1918), pp. 66–78.

|| Cf. *Sur quelques points du calcul fonctionnel*, Rendiconti di Palermo, vol. 22 (1906), p. 1–64.

¶ *Grundzüge der Mengenlehre* (Veit and Co., Leipzig, 1914), p. 213.

Moore as bases for a theory of plane curves in non-metrical analysis situs.* It is shown that each such space s is the sum of an enumerable set of compact metric sets. A development Δ of s is defined such that the associated distance function δ is equivalent to an écart with respect to limit of a sequence.

1. Let $a = x_1 < x_2 < \cdots < x_m = b$ denote a set of points from an interval (a, b). The corresponding set of intervals (x_i, x_{i+1}), $i = 1, 2, 3, \cdots n-1$, is a division of the interval (a, b) of norm δ, where δ is the greatest value of $x_{i+1} - x_i$. Consider a sequence of divisions of the interval (a, b) with norms, $\delta_1, \delta_2, \cdots$ decreasing to the limit zero. In terms of such a sequence of divisions one may define the Riemann integral $\int_a^b f(x)\,dx$. Denote by $x_1^m, x_2^m, \cdots x_{n_m}^m$ the division points corresponding to the norm δ_m, and by $\mathfrak{P}^{mn}$ the interval (x_n^m, x_{n+1}^m). The system $\Delta \equiv ((\mathfrak{P}^{mn}))$ of all such intervals $\mathfrak{P}^{mn}$ is an example of a development of the interval (a, b).†

In general a development Δ of an abstract class $\mathfrak{P}$ is a sequence of systems Δ^m, called *stages*, each system Δ^m being composed of subclasses $\mathfrak{P}^{ml}$ of $\mathfrak{P}$, the index l having for each integer m a range $\mathfrak{L}^m$.‡ In the present paper we restrict ourselves to the cases in which the classes $\mathfrak{L}^m$ are composed of integers in the natural order and the classes $\mathfrak{P}^{ml}$ are existent classes for every value of the composite index ml in the system $((ml))$ of indices for the development Δ.

We make here the fundamental hypothesis that we are given a class $\mathfrak{P}$ and a development Δ of $\mathfrak{P}$.

2. M. Fréchet has recently introduced the following considerations: a class of elements is a class (V) if to every element p of the classt here is assigned a family of sets V_p called *neighborhoods*§ of p.

Denote by V_p^m the class of all elements of $\mathfrak{P}$ which belong to a class $\mathfrak{P}^{m'l}$ $(m' \geqq m)$ containing p. This class V_p^m will be called *the neighborhood of rank* m of p defined by the development Δ. Then $\mathfrak{P}$ is a class (V). If a class V contains V_p^m and is a proper subclass of V_p^{m-1} it will be called a neighborhood of p of rank m. Evidently such a neighborhood contains all neighborhoods of p of higher rank.

In terms of the neighborhoods V_p^m we define a distance function $\delta(p, q)$ as follows: if q belongs to V_p^m and not to V_p^{m+1} then $\delta(p, q) = 1/m$; if q

* *On the foundations of plane analysis situs*, these Transactions, vol. 17 (1916), pp. 131–164.

† Many examples of developments are given in the articles of E. H. Moore, Pitcher, and Chittenden cited above. The following example of a development of the class of positive integers ($\mathfrak{P}^{\mathrm{III}}$) may be of interest. Denote by $\mathfrak{P}^{m1}$ the class of integers $p \leqq m$, by $\mathfrak{P}^{m2}$ the class of integers $p > m$. Obviously the Cauchy condition for convergence of a sequence $\{a_n\}$ is equivalent to the statement: the oscillation of the sequence on $\mathfrak{P}^{m2}$ approaches zero with m.

‡ E. H. Moore (loc. cit.) assumes that the index l has for each m a finite range. Such developments will be called finite.

§ Comptes Rendus, vol. 165 (1917), p. 359. Fréchet uses the term "voisinages."

does not belong to V_p^m for any m, $\delta(p, q) = 1$; if q belongs to V_p^m for every m, then $\delta(p, q) = 0$; for every p, $\delta(p, p) = 0$.*

For the purposes of this paper we shall define limiting element as follows: *an element p is a limiting element of a class $\mathfrak{Q}$ if and only if every neighborhood of p contains an infinity of elements of $\mathfrak{Q}$.*†

By an obvious application of the Zermelo axiom of choice we obtain the following important proposition:

If p is a limiting element of a class $\mathfrak{Q}$, $\mathfrak{Q}$ contains a sequence $\{q_n\}$ of elements q_n, no two alike, such that $\mathrm{L}_n\, \delta(q_n, p) = 0$, that is, p is a limit of the sequence $\{q_n\}$.

3. The notion of fundamental sequence is helpful in studying developments. A fundamental sequence F is a sequence $\mathrm{F} \equiv \{\mathfrak{P}^{m_n l_n}\}$ of classes of Δ subject to the conditions: $m_n \geqq m_{n-1}$; $\mathrm{L}_n\, m_n = \infty$; *the classes;* $\mathfrak{P}^{m_1 l_1}, \mathfrak{P}^{m_2 l_2}, \cdots, \mathfrak{P}^{m_n l_n}$, *have at least one common element for every value of n.*

A fundamental sequence F of the development Δ is *closed* if there is an element common to all the classes $\mathfrak{P}^{m_n l_n}$. Otherwise F is said to be *open.* The class of all elements belonging to every $\mathfrak{P}^{m_n l_n}$ is called the core of the fundamental sequence and denoted by $\mathfrak{C}(\mathrm{F})$. If an element p is contained in the core of F, then F is said to belong to p and is denoted by F_p.

A sequence $S \equiv \{p_n\}$ is *connected* with a sequence $\{\mathfrak{P}^{m_n l_n}\}$ (which need not be a sequence F) if there exists a sequence $\mathfrak{P}^{\bar{m}_n \bar{l}_n}$ of the classes $\mathfrak{P}^{m_n l_n}$ such that $L_n\, \bar{m}_n = \infty$, and an integer n_0 such that for every $n \geqq n_0$, p_n is contained in $\mathfrak{P}^{\bar{m}_n \bar{l}_n}$.

THEOREM 1. *A necessary and sufficient condition that an element p be a limit of a sequence S is that S be connected with a fundamental sequence F_p.*

Since $\mathrm{L}_n\, \delta(p_n, p) = 0$ there is an integer n_e (for every positive number e) such that if $n \geqq n_e$ then $\delta(p_n, p) \leqq e$. Choose m so that $m + 1 > 1/e \geqq m$. Then p_n must belong to the neighborhood V_p^m. Hence there is a class $\mathfrak{P}^{m_n l_n}$ $(m_n \geqq m)$ such that $\mathfrak{P}^{m_n l_n}$ contains both p and p_n. Since p is common to the classes $\mathfrak{P}^{m_n l_n}$ so obtained they may be arranged to form a sequence F_p with which S is connected by definition.

To prove that the condition is also sufficient we need only observe that if s is connected with a sequence F_p, then for every m there is an n_m such that if $n \geqq n_m$ then $\mathfrak{P}^{m_n l_n}$ is contained in V_p^m. But there exists an integer k_m

* This definition is a special case of a definition of distance in terms of a K relation given by T. H. Hildebrandt, *A contribution to the foundations of Fréchet's calcul fonctionnel,* American Journal of Mathematics, vol. 34 (1912), p. 248.

† This definition is more restrictive than that of Fréchet (loc. cit.) and becomes equivalent to his only when every two neighborhoods of p have a neighborhood of p in common, p has an infinity of distinct neighborhoods, and the neighborhoods of p have no common element, except possibly p. Only the first of these conditions is satisfied by the neighborhoods $\mathfrak{V}_p^m$ in general.

such that p_κ $(k \geqq k_m)$ belongs to some class $\mathfrak{P}^{m_n l_n}$ $(n \geqq n_m)$. Therefore p_κ belongs to V_p^m and we have $\delta(p_\kappa, p) \leqq 1/m$.

It is important to state the conditions under which no sequence s has more than one limit. We have, as can easily be verified

THEOREM 2. *A necessary and sufficient condition that limit of a sequence* S *be unique is that* Δ *have the following properties:*

if F *is a fundamental sequence the core of* F *contains at most one element;*

if a sequence S *is connected with two closed fundamental sequences* F_1, F_2 *then the cores of* F_1 *and* F_2 *have a common element.*

4. The development Δ is: *finite,* if for every integer m the number of classes $\mathfrak{P}^{ml}$ is finite; *complete,* if for every m and element p there is a class $\mathfrak{P}^{ml}$ containing p; *closed,* if every fundamental sequence F is closed.

THEOREM 3. *If the development* Δ *is complete there exists an enumerable subclass* $\mathfrak{R}$ *of* $\mathfrak{P}$ *such that* $\mathfrak{P} = \mathfrak{R} + \mathfrak{R}'$.

There exists a class $\mathfrak{R} \equiv [r^{ml}/ml]$ of elements of $\mathfrak{P}$ such that r^{ml} belongs to $\mathfrak{P}^{ml}$, and if $\mathfrak{P}^{ml}$, $\mathfrak{P}^{m'l'}$, (where ml and $m'l'$ are distinct) are not subclasses of $\mathfrak{R}$ then r^{ml} is distinct from $r^{m'l'}$.*

S nce Δ is complete there exists for every element p of $\mathfrak{P}$ a fundamental sequence $F_p \equiv \{\mathfrak{P}^{m_n l_n}\}$ of Δ. If p is not an element of $\mathfrak{R}$ the class $[r^{m_n l_n}/n]$ is infinite. The sequence $\{r^{m_n l_n}\}$ is connected with F_p and has the limit p (Theorem 1). It follows that every element p of $\mathfrak{P}$ belongs to $\mathfrak{R} + \mathfrak{R}'$ which was to be proved.

With Hildebrandt (cf. loc. cit., p. 278) we shall say that when $\mathfrak{P} = \mathfrak{R} + \mathfrak{R}'$, $\mathfrak{P}$ is *separable.* If $\mathfrak{P} = \mathfrak{R}'$, $\mathfrak{P}$ is separable in the sense of Fréchet.

LEMMA. *If the development* Δ *is finite and complete every infinite subclass of* $\mathfrak{P}$ *contains an infinite sequence* S *of distinct elements connected with a fundamental sequence* F.

Let $\mathfrak{Q}$ be any infinite subclass of $\mathfrak{P}$. Some class $\mathfrak{P}^{1l_1}$ must contain an infinite subclass $\mathfrak{Q}_1$ of $\mathfrak{Q}$. Likewise some class $\mathfrak{P}^{2l_2}$ must contain an infinite subclass $\mathfrak{Q}_2$ of $\mathfrak{Q}_1$, etc. The sequence $\{\mathfrak{P}^{ml_m}\}$ so obtained is evidently a fundamental sequence F. If the classes $\mathfrak{Q}_m$ have an infinite common subclass it contains the required sequence S. If not then infinitely many of the classes $Q_m - Q_{m+1}$ are non-null, say those with indices m'. Let $q_{m'}$ be an element of $\mathfrak{Q}_{m'} - \mathfrak{Q}_{m'+1}$. Then $S = \{q_{m'}\}$ is the required sequence.

THEOREM 4. *If the development* Δ *is finite, complete, and closed, the class* $\mathfrak{P}$ *is compact and separable.*

* This proposition depends upon the axiom of choice (Zermelo) and the principle of transfinite induction (Cantor). The proof is easy but tedious and is omitted. The argument is based upon the following principle of selection: An index ml precedes an index $m'l'$ if $m < m'$, or if $l < l'$ when $m = m'$. Then r^{ml} is any element of $\mathfrak{P}^{ml}$ which has not previously been selected. If there is no such element in $\mathfrak{P}^{ml}$ then r^{ml} may be any element of $\mathfrak{P}^{ml}$.

A system $R \equiv ((r^{ml}))$ is called by E. H. Moore (loc. cit.) a representative system of a development Δ.

This theorem is an immediate consequence of Theorem 3 and the lemma.

Let $\mu \equiv \mu(p)$ be any real single-valued function defined on $\mathfrak{P}$. Let $\mathfrak{A}_\mu$ be the class of all values $\mu(p)$ assumed by the function μ on $\mathfrak{P}$. If μ is uniformly continuous relative to $\delta(p, q)$ and if Δ is finite and complete it is easy to see that $\mathfrak{A}_\mu$ is bounded. Conversely if $\mathfrak{A}_\mu$ is bounded there exists a finite, complete development of $\mathfrak{P}$ such that μ is uniformly continuous with respect to the corresponding distance function δ. Hence

THEOREM 5. *A necessary and sufficient condition that $\mathfrak{A}_\mu$ be a bounded set is that $\mathfrak{P}$ admit a development Δ such that μ is uniformly continuous with respect to the corresponding distance function δ.*

COROLLARY. *A necessary and sufficient condition that $\mathfrak{A}_\mu$ be compact and closed is that $\mathfrak{P}$ admit a finite complete closed development Δ such that μ is uniformly continuous relative to the corresponding distance function δ.*

The corollary follows readily from Theorems 1 and 4.

5. Let $\mathfrak{Q}$ be a subclass of $\mathfrak{P}$ and $\mathfrak{Q}^{ml}$ be the divisor of $\mathfrak{Q}$ and $\mathfrak{P}^{ml}$. The system $((\mathfrak{Q}^{ml}))$, in which the indices ml of null classes $\mathfrak{Q}^{ml}$ are dropped, is a development $\Delta(\mathfrak{Q})$ obtained by reduction relative to $\mathfrak{Q}$ from the given development Δ.*

Evidently every fundamental sequence F of $\Delta(\mathfrak{Q})$ determines a fundamental sequence F of Δ. Hence any sequence S of elements of $\mathfrak{Q}$ which has a limit relative to $\Delta(\mathfrak{Q})$ has the same limit relative to Δ. Conversely, if a sequence S of $\mathfrak{Q}$ has a limit in $\mathfrak{Q}$ relative to Δ it has the same limit relative to $\Delta(\mathfrak{Q})$. The properties: finite, complete, are easily seen to be invariant under the reduction.

THEOREM 6. *If the development Δ is closed and $\mathfrak{Q}$ is closed then $\Delta(\mathfrak{Q})$ is closed.*

The assumption that $\Delta(\mathfrak{Q})$ contains an open fundamental sequence leads to the conclusion that $\mathfrak{Q}$ has a limiting element in $\mathfrak{P} - \mathfrak{Q}$ contrary to hypothesis.

THEOREM 7. *If $\Delta(\mathfrak{Q})$ is finite, complete, and closed, $\mathfrak{Q}$ is self-compact; that is, every infinite subclass of $\mathfrak{Q}$ has a limiting element in $\mathfrak{Q}$; furthermore $\mathfrak{Q}$ is separable.*

This theorem is a corollary of Theorem 4. The class $\mathfrak{Q}$ need not be closed.

6. An element p is *interior* to a set $\mathfrak{D}$ if $\mathfrak{D}$ contains a neighborhood of p. A class $\mathfrak{D}$ consisting entirely of interior elements will be called a domain.†

THEOREM 8. *If the development Δ is finite, complete, and closed and $\mathfrak{Q}$ is a closed set then any family $(\mathfrak{D})$ of domains whose sum contains $\mathfrak{Q}$, contains a finite subfamily $\mathfrak{D}_1, \mathfrak{D}_2, \cdots, \mathfrak{D}_n$ with the same property.*

* Cf. E. H. Moore, loc. cit., p. 91, also Pitcher, loc. cit., p. 24.

† Hausdorff, loc. cit., p. 215, calls attention to the importance of this concept. He uses the term "Gebiet" in this connection. R. L. Moore, loc. cit., p. 36, following the usage of Weierstrass assumes that a domain is also a connected set.

Let $\mathfrak{Q}$ be a closed set and $(\mathfrak{D})$ any family of domains, whose sum contains $\mathfrak{Q}$, for which the theorem fails. Then $\mathfrak{Q}$ is an infinite set. If $\mathfrak{Q}$ is divided into a finite number of parts $\mathfrak{Q}_1, \mathfrak{Q}_2, \mathfrak{Q}_3, \cdots \mathfrak{Q}_n$, there will be at least one part which will not be contained in any finite subfamily of $(\mathfrak{D})$. Since Δ is finite and complete there is a class $\mathfrak{P}^{1l_1}$ containing an infinite subclass $\mathfrak{Q}_1$ of $\mathfrak{Q}$ which is not contained in the sum class of any finite subfamily of $(\mathfrak{D})$. Likewise there will be a class $\mathfrak{P}^{2l_2}$ containing an infinite subclass $\mathfrak{Q}_2$ of $\mathfrak{Q}_1$ in the same relation to $(\mathfrak{D})$, etc. There exists, therefore, an infinite sequence $\{\mathfrak{Q}_n\}$ of infinite classes, each contained in its predecessor, and a corresponding sequence $\mathrm{F} \equiv \{\mathfrak{P}^{ml_m}\}$ which is evidently a fundamental sequence of Δ. Any element of the core of F is easily seen to be a limiting element of $\mathfrak{Q}$ and therefore, since $\mathfrak{Q}$ is closed by hypothesis, is an element of $\mathfrak{Q}$. Let q be an element of the core of F and $\mathfrak{D}_q$ be any domain of $(\mathfrak{D})$ containing q. Then $\mathfrak{D}_q$ contains a neighborhood V_q^m of q which must contain the elements of $\mathfrak{P}^{ml_m}$ and therefore contains $\mathfrak{Q}_m$. This contradicts the hypothesis that $\mathfrak{Q}_m$ is not contained in any finite subfamily of $(\mathfrak{D})$.*

THEOREM 9. *If the development $\Delta(\mathfrak{Q})$ derived by reduction relative to $\mathfrak{Q}$ is finite, complete, and closed, and $(\mathfrak{D})$ is any family of domains whose sum contains $\mathfrak{Q}$, there is a finite subfamily of $(\mathfrak{D})$ whose sum contains $\mathfrak{Q}$.*

This theorem is an evident generalization of Theorem 8. As remarked above the class $\mathfrak{Q}$ need not be closed relative to Δ. The domains $\mathfrak{D}$ need not be domains relative to $\mathfrak{P}$ and Δ.

THEOREM 10. *If the development $\Delta(\mathfrak{Q})$ is finite, complete, and closed, and if the derived class of every subclass of $\mathfrak{Q}$ is closed, then every family $(\mathfrak{H})$ of classes*

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* The theorem fails for more general families $(\mathfrak{D})$ as is shown by the following example. Let $\mathfrak{P}$ denote the interval (0, 1). We define a development Δ of $\mathfrak{P}$. Let $\mathfrak{P}^{ml}$ be the interval

$$\left(\frac{l-1}{m}, \frac{l}{m}\right), \qquad (l = 1, 2, 3, \cdots m),$$

with the exception, each class which contains $1 - 1/m$ is to contain the element $1/m$ $(m > 2)$. This development is finite complete and closed, hence $\mathfrak{P}$ admits the property of Theorem 8. But each element of $\mathfrak{P}$ is interior to some one of the following set of classes:

$$\mathfrak{D}_0 \equiv (0, \tfrac{1}{2}); \quad \mathrm{D}_1 \equiv (\tfrac{1}{4}, \tfrac{3}{4}); \quad \mathrm{D}_2 \equiv (\tfrac{1}{2}, 1);$$

$$\mathfrak{D}_m \equiv \left[\frac{1}{m} - \frac{1}{2m} < p < \frac{1}{m} + \frac{1}{2m}\right] + \left[1 - \frac{1}{m} - \frac{1}{2m} < p < 1 - \frac{1}{m} + \frac{1}{2m}\right],$$

$$m = 3, 4, 5, \cdots,$$

$$\mathfrak{D}_\omega \equiv [\tfrac{1}{3}, \tfrac{1}{4}, \cdots] + [\tfrac{2}{3} \leqq p \leqq 1].$$

The elements $\frac{1}{3}, \frac{1}{4}, \cdots$ are not covered by any finite subfamily of $(\mathfrak{D})$ although for every m, $1/m$ is interior to $\mathfrak{D}_m$. That is the more general form of the Heine-Borel property does not hold. It will be noticed that the class $\mathfrak{D}_\omega$ is not a domain. Let $\mathfrak{Q}$ denote the set of irrational points of the interval $(0, \frac{1}{2})$. Then $\mathfrak{Q}' = (0, \frac{1}{2})$, while $\mathfrak{Q}''$ contains the element $p = 1$. That is in the system just defined, derived classes are not in general closed (cf. Theorem 10).

$\mathfrak{H}$, such that each element of $\mathfrak{Q}$ is INTERIOR *to some class $\mathfrak{H}$, contains a finite subfamily with the same property.**

Exactly as in the proof of Theorem 8 we obtain a sequence $\{\mathfrak{Q}_m\}$ and a fundamental sequence F of $\Delta(\mathfrak{Q})$. Let q be an element of the core of F. Then q is interior to some class $\mathfrak{H}_q$, which must as before, contain a class $\mathfrak{Q}_m$, for sufficiently large m. We shall prove that for some value of $m' > m$, $\mathfrak{Q}_{m'}$ is interior to $\mathfrak{H}_q$. If there is no such value of m', then for every $m' > m$, $\mathfrak{Q}_{m'}$ contains an element $q_{m'}$ not interior to $\mathfrak{H}_q$. The sequence $\{q_{m'} > m\}$ so obtained has the limit q. We may suppose the sequence $\{q_{m'}\}$ so chosen that it contains infinitely many distinct elements. It follows from a theorem of Hedrick (these Transactions, vol. 12 (1911), p. 286) on domains in which derived classes are closed that at least one of the elements $q_{m'}$ is interior to $\mathfrak{H}_q$. This is the desired contradiction.

As an immediate consequence of Theorem 10 we have

THEOREM 11. *If the development Δ is finite, complete, and closed, and if the derived class of every class is closed, then for every continuous function μ and small positive number e there exists a set of elements $p_1, p_2, p_3, \cdots, p_n$, and a corresponding set of integers $m_1, m_2, \cdots, m_n$ such that every element p is interior to some $\mathfrak{V}_{p_j}^{m_j}$ on which the oscillat'on of μ is less than e.*

7. The development Δ is *regular* if for every pair of elements (p_1, p_2) contained in a class $\mathfrak{P}^{ml}$ $(m > 1)$ there is a class $\mathfrak{P}^{m-1,\, l}$ which also contains the pair (p_1, p_2).

Two sequences $S_1 \equiv \{p_{1n}\}$, $S_2 \equiv \{p_{2n}\}$ are connected in case

$$L_n\, \delta(p_{1n}, p_{2n}) = 0.$$

A necessary and sufficient condition that two sequences S_1, S_2 be connected is that for every value of n there is a class $\mathfrak{P}^{m_n l_n}$ containing p_{1n} and p_{2n}, where $L_n\, m_n = \infty$.

THEOREM 12. *If the development Δ is finite, complete, and regular, then for every pair of connected sequences S_1, S_2 there exists a pair S_1', S_2' of connected subsequences of S_1, S_2, respectively, and a fundamental sequence F with which both S_1' and S_2' are connected.*

Since Δ is regular and S_1, S_2 are connected each pair (p_{1n}, p_{2n}) is contained in some class $\mathfrak{P}^{1l}$. Hence an infinity $\mathfrak{J}_1$ of these pairs must lie in a single class $\mathfrak{P}^{1l_1}$. Similarly (because $L_n\, m_n = \infty$ in the condition above) an infinite subclass $\mathfrak{J}_2$ of $\mathfrak{J}_1$ is contained in a single class $\mathfrak{P}^{2l_2}$, etc. We thus obtain a fundamental sequence $F \equiv \{\mathfrak{P}^{ml_m}\}$ of Δ which is the fundamental sequence of the theorem. Let (p_{1n_1}, p_{2n_1}) be the first element of $\mathfrak{J}_1$ which appears in the sequence $\{(p_{1n}, p_{2n})\}$. Let (p_{1n_2}, p_{2n_2}) be the first element of $\mathfrak{J}_2$ to follow (p_{1n_1}, p_{2n_1}) in the same sequence, etc. The sequences $S_1' \equiv \{p_{1n_\kappa}\}$, $S_2' \equiv \{p_{2n_\kappa}\}$ so obtained are connected with each other and with F.

* Theorems 9 and 10 are equivalent if limit is unique. Cf. Theorem 21.

If the fundamental sequence F is closed, S_1', S_2' have a common limit. Hence

THEOREM 13. *If the development Δ is finite, complete, closed, and regular the system $(\mathfrak{P};\ \Delta)$ is biextremal.**

From Theorem 13 and Theorems 11 and 12 of our previous paper, and the further fact that a continuous function on a closed set attains its bounds one readily sees that:

THEOREM 14. *If $\Delta(\mathfrak{Q})$ is finite, complete, closed, and regular, then any function μ continuous on $\mathfrak{Q}$, is uniformly continuous on $\mathfrak{Q}$, bounded and attains its bounds. Furthermore if $\mathfrak{Q}$ is a connected† set the function μ attains on $\mathfrak{Q}$ every value between its bounds.*

8. The theory of coherent systems $(\mathfrak{P};\ \delta)$ was developed in the previous paper. A system $(\mathfrak{P};\ \delta)$ is *coherent* if whenever two sequences S_1, S_2 are *connected* (cf. § 7) then every limit of S_1 is a limit of S_2. The following theorem is a consequence of Theorem 4 above and Theorem 6 of the previous paper.

THEOREM 15. *If the system $(\mathfrak{P},\ \delta)$ derived from the development Δ of the given class $\mathfrak{P}$ is coherent, the derived class of every class is closed; and if furthermore Δ is finite, complete, and closed, δ is equivalent with respect to limit to a distance function $\bar{\delta}^{2345}$.‡*

The relation of Theorem 15 to Theorem 10 should be noted.

THEOREM 16. *Under the hypothesis of Theorem* 15, *every continuous function μ is uniformly continuous, bounded, and attains its bounds.*

This theorem is to be compared with Theorem 5. We will show that μ is uniformly continuous. Under the contrary hypothesis there exists a positive number e_0 and for every n a pair of elements p_{1n}, p_{2n} such that

$$\delta(p_{1n},\ p_{2n}) \leqq 1/n,$$

while

$$|\mu(p_{1n}) - \mu(p_{2n})| > e_0.$$

The sequences $S_1 = \{p_{1n}\}$, $S_2 = \{p_{2n}\}$ are connected. If S_1 contains an element p repeated infinitely often, then p is a limit of S_2, which leads at once to a contradiction. If the elements of S_1 are not repeated then S_1 contains infinitely many elements and, since $\mathfrak{P}$ is compact, contains a subsequence S_1' with a limit p. Since the corresponding sequence S_2' must, because of the definition of coherence, have the limit p, the inequality assumed contradicts the hypothesis of continuity.

* Cf. Pitcher-Chittenden, loc. cit., Section 4. A system is *biextremal* in case there exists for every pair of connected sequences S_1, S_3 a pair S_1', S_2' of connected subsequences of S_1, S_3, respectively, which have a common limit.

† Cf. previous paper, p. 68, for definition of this term.

‡ The properties 2, 3, 4, 5 of a distance function δ are defined in Section 1. of the previous paper. If $\bar{\delta}$ has the further property δ^1; $\delta(p,\ q) = 0$ implies $p = q$, then $\bar{\delta}$ is a voisinage as defined in the thesis of Fréchet (loc. cit.).

THEOREM 17. *Every coherent system derived from a system* $(\mathfrak{P}; \Delta)$ *implies a coherent system such that to every element and limit relation in the first there corresponds a unique element and limit relation in the second, and in the second system limit is unique.**

Denote by $\mathfrak{C}_p$ the class of all elements q satisfying the equation $\delta(p, q) = 0$. If two classes $\mathfrak{C}_p$ and $\mathfrak{C}_q$ have a common element they are identical. For the identical sequences corresponding to the elements p and q, respectively, are connected with the identical sequence of this common element, which has the limit q. It follows from the definition of coherence that $\delta(p, q) = 0$.

Let $\mathfrak{S} \equiv [s]$ be a definite subclass of $\mathfrak{P}$ such that for every s there is a class $\mathfrak{C}$ containing s, and for every class $\mathfrak{C}$ there is a unique element s of $\mathfrak{S}$ contained in $\mathfrak{C}$ (Zermelo Axiom).

Let $\mathfrak{W} \equiv [w]$ denote any class in 1–1 correspondence with the class of all classes $\mathfrak{C}$. We define $\delta(w_1, w_2) = \delta(s_1, s_2)$, where s_1, s_2 correspond to w_1, w_2, respectively. To each element of p corresponds a unique pair of elements s, w. If w_n corresponds to p_n and $\mathrm{L}_n\, \delta(p_n, p) = 0$, then $\mathrm{L}_n\, \delta(w_n, w) = 0$. In the system $(\mathfrak{W}; \delta)$ limit is unique. For every sequence $\{w_n\}$ with limit w, there is a unique sequence $\{s_n\}$ with corresponding limits s. Hence, if $\{w_n\}$ has limits w', w'', the sequence $\{s_n\}$ has corresponding limits s', s''. But s', s'' must belong to the same class $\mathfrak{C}$. Therefore $s' = s''$, $w' = w''$, which was to be proved. A similar argument shows that $(\mathfrak{W}; \delta)$ is a coherent system.

COROLLARY: *If* μ *is continuous on* $\mathfrak{P}$ *and* $\mathfrak{W}$ *corresponds to* $\mathfrak{P}$, *the function* $\mu(w) = \mu(s)$ *is continuous on* $\mathfrak{W}$ *and assumes every value on* $\mathfrak{W}$ *that it assumes on* $\mathfrak{P}$.†

9. Denote by $\mathfrak{R}_p^m$ the class of all elements q such that p and q are contained in some class $\mathfrak{P}^{m'l'}$ for every $m' \leqq m$. Evidently $\mathfrak{R}_p^m$ contains $\mathfrak{R}_p^{m+1}$.

THEOREM 18. *If the development* Δ *is finite, complete, and closed, and if limit of a sequence is unique, then every class* $\mathfrak{R}_p^m$ *contains an infinite subclass of every class* $\mathfrak{Q}$ *which has* p *for a limiting element.*

COROLLARY. *The class* $\mathfrak{R}_p^m$ *contains all but a finite number of elements of any sequence with the limit* p.

The class $\mathfrak{Q}$ contains an infinite sequence $\mathrm{S} = \{q_n\}$ with the limit p, containing an infinity of distinct elements. Let $\mathfrak{S}$ denote the class of elements in the sequence S. By means of an argument applied several times before we obtain a sequence $\{\mathfrak{S}_m\}$ of infinite subclasses of $\mathfrak{S}$ each contained in its predecessor and in every class $\mathfrak{P}^{m'l_{m'}}$ $(m' \leqq m)$ of a fundamental sequence F $(m = 1, 2, 3, \cdots)$. Since F is closed and limit is unique, the core of F consists of the element p. Therefore the infinite class $\mathfrak{S}_m$ belongs to $\mathfrak{R}_p^m$ for every value of m.

* Theorem 17 holds for any coherent system $(\mathfrak{P}; \delta)$ such that for every element p of $\mathfrak{P}$, $\delta(p, p) = 0$.

† Every continuous function on $\mathfrak{P}$ is a constant on $\mathfrak{C}p$ for every p.

THEOREM 19. *Under the hypothesis of Theorem* 18 *the class* $\mathfrak{P}$ *admits a definition of a distance function* ρ, *which is equivalent to* δ *with respect to limit of a sequence, such that the system* $(\mathfrak{P}; \rho)$ *is coherent.**

We define the function $\rho(p, q)$ as follows: if q belongs to R_p^m and does not belong to $\mathfrak{R}_p^{m+1}$ then $\rho(p, q) = 1/m$; if $q = p$, or if q belongs to $\mathfrak{R}_p^m$ for every integer m, then $\rho(p, q) = 0$. Evidently $\delta(p, q) \leqq \rho(p, q)$. Hence every limit in terms of ρ implies a limit in terms of δ. Conversely, suppose that $L_n \delta(p_n, p) = 0$. We must prove that $L_n \rho(p_n, p) = 0$. From the corollary above, all but the first k_m of the elements p_n belong to $\mathfrak{R}_p^m$. Hence if $n > k_m$, $\rho(p_n, p) \leqq 1/m$, which implies $L_n \rho(p_n, p) = 0$. This establishes the equivalence of ρ and δ with respect to limit. We shall now prove that the system $(\mathfrak{P}, \rho)$ is coherent. We have to prove that if $S_1 \equiv \{p_n\}$, $S_2 \equiv \{q_n\}$ satisfy the condition

$$L_n \rho(p_n, q_n) = 0,$$

then any limit of S_1 is a limit of S_2. Suppose that p is the limit of S_1 and is not a limit of S_2. Then S_2 must contain a subsequence $S_2' \equiv \{q_n'\}$ such that for some fixed integer $\overline{m}$, and for every integer n, q_n' does not belong to $\mathfrak{R}_p^{\overline{m}}$. If p_n' is the element of S_1 which corresponds to q_n', we have $L_n \rho(p_n', q_n') = 0$. Hence an infinity $\mathfrak{J}_1$ of the pairs (p_n', q_n') lie in some class $\mathfrak{P}^{1l_1}$, an infinity of these $\mathfrak{J}_2$ lie in a class $\mathfrak{P}^{2l_2}$, etc. The fundamental sequence F of Δ so obtained is closed by hypothesis and hence its core consists of the element p. Therefore p is a limit of a subsequence of S_2', which is the desired contradiction.

THEOREM 20. *Under the hypothesis of Theorem* 18 *the distance function* ρ *is equivalent (with respect to limit of a sequence) to an écart.*

Cf. Theorem 19 above and Theorem 7 of the previous paper. As an immediate consequence of preceding theorems we have:

THEOREM 21. *If the development* Δ *is finite, complete, and closed, and if limit of a sequence is unique the class* $\mathfrak{P}$ *is a compact metric space.*†

10. An element p is *singly developed* by a development Δ if for every integer m there exists an integer $m' > m$ such that some class $\mathfrak{P}^{m'l'}$ consists of the single element p. A development Δ is *asingular* in case no element of $\mathfrak{P}$ is singly developed.

THEOREM 22. *If the development* Δ *is finite, complete, asingular and every fundamental sequence of* Δ *has a singular core, then* $\mathfrak{P}$ *is compact, perfect, and separable.*

We will show that $\mathfrak{P} = \mathfrak{R}'$ where $\mathfrak{R}$ is the enumerable class introduced in the proof of Theorem 3. Let $F = \{\mathfrak{P}^{m_n l_n}\}$ be a fundamental sequence belonging to p. Since Δ is asingular every class $\mathfrak{P}^{m_n l_n}$ of F contains at least

* If Δ is regular the functions δ and ρ are identical.

† Cf. Hausdorff, loc. cit., p. 211 et seq., for consequences of this theorem.

one element distinct from p. The class of such elements is infinite, since the core of F is singular. Hence from the definition of $\mathfrak{R} \equiv ((r^{ml}))$, the sequence $\mathfrak{S} = \{r^{m_n l_n}\}$ contains infinitely many distinct elements. Since $\mathfrak{S}$ has the limit p, p is in $\mathfrak{R}'$ which was to be proved.

THEOREM 23. *A necessary and sufficient condition that a class $\mathfrak{P}$ in a system* ($\mathfrak{L}$) *of Fréchet be a compact class* "(V) NORMALE"* *is that $\mathfrak{P}$ admit a finite, complete, asingular, closed development Δ, such that limit in Δ is equivalent to limit in the system* ($\mathfrak{L}$).

The condition is sufficient. In the system ($\mathfrak{L}$) limit is unique. Hence from Theorem 22, $\mathfrak{P}$ is compact, separable, and perfect. From Theorem 21 it follows that $\mathfrak{P}$ admits a generalization of the theorem of Cauchy.

We show that the condition is necessary by defining a development Δ with the desired properties. Let (p, q) denote the voisinage of elements p and q. Denote by V_p^m the neighborhood of p of rank m. Then p is interior to V_p^m. From the Heine-Borel theorem, which applies to this case, we have, for every positive integer m, a finite set:

$$r_1^m, \; r_2^m, \; \cdots \; r_l^m,$$

of elements such that every element of $\mathfrak{P}$ is interior to at least one of the classes $V_{r_l^m}^m$. Denote by $\mathfrak{P}^{ml}$ the derived class of $V_{r_l^m}^m$. The development $\Delta \equiv ((\mathfrak{P}^{ml}))$ is obviously finite, complete, and asingular. The proofs of closure and equivalence can be carried through easily by reference to the properties of voisinage and fundamental sequences.

11. In the foregoing sections the principal results have been stated for closed developments. In the present section we extend these results to a class of open developments by the introduction of ideal elements defined in terms of fundamental sequences.

Two fundamental sequences F_1, F_2 of Δ are connected if there exists an infinite sequence of elements of $\mathfrak{P}$ connected with both F_1 and F_2. It is evident that *if* F_1 *and* F_2 *are fundamental subsequences formed from the classes* $\mathfrak{P}^{m_n l_n}$ *of a fundamental sequence* F *they are connected.*

Two fundamental sequences F and F′ are *connected* of *order* n if there exists a chain, $F_1, F_2, \cdots F_{n-1}$, of $n-1$ fundamental sequences such that F_1 is connected with F, F_2 is connected with F_1, etc., F_{n-1} being connected with F′.

Every fundamental sequence F determines a *family* $\mathfrak{F}$ of fundamental sequences, the family of all fundamental sequences connected with F of order n for some value of n. *It is evident that if* F_1 *and* F_2 *are connected of order* n *they determine the same family* $\mathfrak{F}'$.

The classes $\mathfrak{P}^{ml}$ which belong to the fundamental sequences of a family $\mathfrak{F}$

* A "class (V) normale" is separable, perfect, admits a definition of voisinage and a generalization of the theorem of Cauchy.

form a definite *set* of classes of Δ which we shall also denote by $\mathfrak{F}$, speaking of the *set* $\mathfrak{F}$ of classes of Δ associated with a fundamental sequence F.

If an element p of $\mathfrak{P}$ belongs to the core of a fundamental sequence F of Δ the associated *family* $\mathfrak{F}$ or *set* $\mathfrak{F}$ will be denoted by $\mathfrak{F}_p$. Let $\mathfrak{F}_0$ denote the class of all elements p with which a given family $\mathfrak{F}$ is associated. If $\mathfrak{F}_0$ is a null-class every fundamental sequence of $\mathfrak{F}$ is open.

Let $[\mathfrak{F}]_*$ be the class of all families $\mathfrak{F}$ of Δ which consist entirely of open sequences, and let $\mathfrak{E}_*$ be any set of elements which has no element in common with $\mathfrak{P}$ and is in 1–1 correspondence with $[\mathfrak{F}]_*$. Let p_* be the element of $\mathfrak{E}_*$ which corresponds to a family $\mathfrak{F}$ of $[\mathfrak{F}]_*$. Denote by $\mathfrak{F}_{*0}$: the class $\mathfrak{F}_0$, when $\mathfrak{F}_0$ is not null; and the class of the single element p_*, when $\mathfrak{F}_0$ is a null-class.

We proceed to define a development Δ_* of the class $\mathfrak{P}_* \equiv \mathfrak{P} + \mathfrak{E}_*$ which we term the $*$-extension of Δ. For every class $\mathfrak{P}^{ml}$ there is a definite class $\mathfrak{T}^{ml}$ (which may be a null-class) of sets $\mathfrak{F}$ which contain $\mathfrak{P}^{ml}$. Let $\mathfrak{E}_*^{ml}$ denote the class of all elements contained in some class $\mathfrak{F}_{*0}$ derived from a set $\mathfrak{F}$ of $\mathfrak{T}^{ml}$. The development Δ_* of $\mathfrak{P}_*$ is the system $((\mathfrak{P}_*^{ml}))$ where $\mathfrak{P}_*^{ml} \equiv \mathfrak{P}^{ml} + \mathfrak{E}_*^{ml}(ml)$.

During the remaining part of the discussion in this section of the paper we shall denote by $\{\mathfrak{P}_n\}$ *a sequence of classes of* Δ *such that* $\mathfrak{P}_n = \mathfrak{P}^{m_n l_n}$, *where* $L_n m_n = \infty$ *and* $m_n \leqq m_{n+1}(n)$.

If $F \equiv \{\mathfrak{P}_n\}$ *is a fundamental sequence of* Δ, $F_* \equiv \{\mathfrak{P}_{*n}\}$ *is a closed fundamental sequence of* Δ_*. *The sequence* F_* is called the $*$-extension of F.

Every limit relation of the system $(\mathfrak{P}; \Delta)$ is present in the system $(\mathfrak{P}_*; \Delta_*)$.

We give an example showing that the development Δ_* is not always closed. Let $\mathfrak{P}$ be the class of all real numbers; $\mathfrak{L}^m \equiv [-m, -m+1, \cdots, -2, -1, 1, 2, 3, \cdots]$; $\mathfrak{P}^{ml} \equiv [\text{all } p(>-l) \text{ of the form } ae^{-l}$, where a is a rational number and $e = 2.71828\cdots]$ $(l = -m, -m+1, \cdots, -2, -1)$; $\mathfrak{P}^{ml} \equiv [l - 1/m \leqq p < l]$ $(l = 1, 2, 3, \cdots)$; $\mathfrak{E}_* \equiv [1, 2, 3, \cdots]$. The sequence $\{\mathfrak{P}_*^{m,-m}\}$ is an open fundamental sequence of Δ_*.

Let $\{\mathfrak{T}_n\}$ denote a sequence of sets of families $\mathfrak{F}$ of Δ such that $\mathfrak{T}_n$ contains $\mathfrak{T}_{n+1}(n)$. Let $H \equiv \{\mathfrak{P}_n\}$ satisfy the condition: $\mathfrak{P}_n$ belongs to some sequence of every family $\mathfrak{F}$ of $\mathfrak{T}_n(n)$. The development Δ is *closeable* if every such sequence H of Δ is a fundamental sequence.

The following chain of propositions follow from the hypothesis: *the development* Δ *is closeable.*

Every sequence $\{\mathfrak{P}_n\}$ *formed from the classes of a single set* $\mathfrak{F}$ *of* Δ *is a fundamental sequence.*† This follows from the definition of the closeable property on setting $\mathfrak{T}_n = \mathfrak{F}(n)$. This proposition has the corollary: *any sequence*

† Hence, if Δ is finite the classes of any *set* $\mathfrak{F}$ associated with a fundamental sequence F form a fundamental sequence of Δ of which F is a subsequence.

$\{\mathfrak{P}_n\}$ formed from the classes of two connected fundamental sequences is a fundamental sequence.

If a fundamental sequence is connected with two fundamental sequences they are connected. For the three sequences can be combined to form a single fundamental sequence, and any pair of fundamental subsequences of a fundamental sequence is connected. Hence, *if two fundamental sequences are connected of order n, they are connected of the first order, and every family $\mathfrak{F}$ of fundamental sequences of Δ consists of a fundamental sequence* F *and all sequences connected with* F.

If $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are classes of a set $\mathfrak{F}$ of Δ they have a common element. For $\mathfrak{P}_1$ and $\mathfrak{P}_2$ form part of a sequence $\{\mathfrak{P}_n\}$ of classes of $\mathfrak{F}$ and this sequence must be a fundamental sequence.

If H_* is a fundamental sequence of Δ_*, H_* is the $*$-extension of a fundamental sequence H *of* Δ. Suppose that $\mathrm{H}_* \equiv \{\mathfrak{P}_{*n}\}$, and that $\mathrm{H} = \{\mathfrak{P}_n\}$ is not a fundamental sequence of Δ. Let n_0 be an integer such that $\mathfrak{P}_1, \mathfrak{P}_2, \cdots, \mathfrak{P}_{n_0}$, have no common element. Since $\mathfrak{P}_{*1}, \mathfrak{P}_{*2}, \mathfrak{P}_{*3}, \cdots, \mathfrak{P}_{*n_0}$, have a common element p_*, there is an integer $i \leqq n_0$ such that $\mathfrak{P}_i$ does not contain p_*. The element p_* is either an element of some class $\mathfrak{F}_0$ or else an element of $\mathfrak{E}_*$. In either case p_* determines a family $\mathfrak{F}p_*$, and every class $\mathfrak{P}_1, \mathfrak{P}_2, \cdots, \mathfrak{P}_{n_0}$ belongs to the corresponding set $\mathfrak{F}_{p*}$ of classes of Δ. Let $\mathfrak{H}_{*n}$ be the class of all such elements p_* which are common to $\mathfrak{P}_{*1}, \cdots, \mathfrak{P}_{*n}$. Let $\mathfrak{T}_n$ be the class of all families $\mathfrak{F}$ which correspond to elements of $\mathfrak{H}_{*n}$. Then $\mathfrak{T}_n$ contains $\mathfrak{T}_{n+1}$ and $\mathfrak{P}_n$ belongs to every set $\mathfrak{F}$ of $\mathfrak{T}_n$. Since Δ is closeable this contradicts the hypothesis that H is not a fundamental sequence of Δ.

From the definition of $*$-extension and the proposition just proved we obtain the theorem:

THEOREM 24. *If the development Δ is closeable the development Δ_* of $\mathfrak{P}_*$ is closed.*

THEOREM 25. *If Δ is closeable Δ_* is closeable.*

Suppose $\{\mathfrak{T}_{*n}\}$ is a sequence of sets of families $\mathfrak{F}_*$ of fundamental sequences of Δ_*. Then $\mathfrak{T}_{*n}$ is the extension of a set $\mathfrak{T}_n$ of families of Δ, and if $\mathfrak{P}_*^{ml}$ belongs to every family of $\mathfrak{T}_{*n}$ then $\mathfrak{P}^{ml}$ belongs to every family of $\mathfrak{T}_n$.† Hence, if $\{\mathfrak{P}_{*n}\}$ is a sequence of classes of Δ_* such that $\mathfrak{P}_{*n}$ belongs to every family of $\mathfrak{T}_{*n}(n)$, then the class $\mathfrak{P}_n$ corresponding to $\mathfrak{P}_{*n}$ belongs to every family of $\mathfrak{T}_n$. If $\mathfrak{T}_{*n}$ contains $\mathfrak{T}_{*n+1}$, $\mathfrak{T}_n$ contains $\mathfrak{T}_{n+1}(n)$; and it follows from the closeable property of Δ that $\{\mathfrak{P}_n\}$ is a fundamental sequence of Δ. This implies that $\{\mathfrak{P}_{*n}\}$ is a fundamental sequence of Δ_*, which was to be proved.

If Δ is closeable and limit is unique in the system $(\mathfrak{P}; \Delta)$ then every funda-

† This is an immediate consequence of the proposition: if $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are classes of a set $\mathfrak{F}$ they have a common element (together with the definition of Δ_*).

mental sequence of Δ_* has a singular core. Let F_* be a fundamental sequence of Δ_* whose core contains two elements p_*, p'_*. Then p_* and p'_* are associated with the same family F of fundamental sequences of Δ and are, therefore, both elements of $\mathfrak{P}$. It follows from the preceding propositions that p_* and p'_* belong to connected closed fundamental sequences of Δ, and since limit is unique in the system $(\mathfrak{P}; \Delta)$ the cores of these sequences have a common element. As the cores are singular they coincide. This is the desired contradiction. We have

THEOREM 26. *If limit is unique in the system $(\mathfrak{P}; \Delta)$ and if Δ is closeable then limit is unique in the system $(\mathfrak{P}_*, \Delta_*)$ and Δ_* has the closeable property.*

THEOREM 27. *A necessary and sufficient condition that Δ_* be closed is that every sequence $H \equiv \{\mathfrak{P}_n\}$ of Δ with the property:*

(A) there exists a sequence $\{\mathfrak{T}_n\}$ of classes of sets $\mathfrak{F}$ of Δ such that there is no set $\mathfrak{F}$ common to the classes $\mathfrak{T}_n$, $\mathfrak{T}_n$ contains $\mathfrak{T}_{n+1}$, and every set $\mathfrak{F}$ of $\mathfrak{T}_n$ contains $\mathfrak{P}_n$ $(n = 1, 2, 3, \cdots)$;

has the further property:

(B) there exists a family $\mathfrak{F}$ of Δ such that the corresponding set $\mathfrak{F}$ of classes $\mathfrak{P}^{ml}$ contains every class $\mathfrak{P}_n$ of H.

If a sequence H has property (B), then from the definition of $*$-extension H_* is closed. It is easy to see that any open fundamental sequence H_* of Δ_* must correspond to a sequence H of Δ with property (A). This leads at once to a contradiction. Hence the condition is sufficient.

The condition is necessary. The $*$-extension H_* of any sequence H with property (A) is a fundamental sequence of Δ. Since Δ_* is closed, H_* is closed. Therefore H belongs to the set F_{p*} where p_* is any element of the core of H_*. Therefore H has property (B), which was to be proved.

THEOREM 28. *If limit is unique in the system $(\mathfrak{P}; \Delta)$ and the development Δ is finite, complete, and closeable, the class $\mathfrak{P}_*$ in the system $(\mathfrak{P}_*, \Delta_*)$ is a compact, metric space.*

THEOREM 29. *If the development Δ is finite, complete, regular, and closeable; the development Δ_* is also finite, complete, regular, and closeable; and the system $(\mathfrak{P}_*; \Delta_*)$ is biextremal. Any function μ continuous on $\mathfrak{P}_*$ is uniformly continuous on $\mathfrak{P}_*$ (and therefore on $\mathfrak{P}$), bounded and attains its bounds.*

Theorems 28 and 29 are consequences of Theorems 13, 14, 21 and the results of the present section.

12 Two developments Δ_1, Δ_2 of the class $\mathfrak{P}$ will, for the purposes of this paper, be said to be *equivalent* if they are equivalent with respect to limit of a sequence.†

† Cf. E. W. Chittenden, On the Equivalence of Relations Kq_1q_2m, American Journal of Mathematics, vol. 39 (1917), p. 266. The necessary and sufficient condition for equivalence of developments is contained in Theorem I of that paper. Two elements p, q are said to be in the relation Kq_1q_2m (determined by Δ) if they both belong to the neighborhood $\mathfrak{B}_p^m$.

It is important to notice that the following operations on a development Δ lead to an equivalent and sometimes simpler development Δ_0.

(1) Any finite number of stages of Δ may be removed. A development which under this operation becomes equivalent to a finite, complete, etc., development is said to be ultimately finite, ultimately complete, etc. All the previous theorems admit of immediate extension to such developments.

(2) If $\mathfrak{P} = \mathfrak{R} + \mathfrak{R}'$, where $\mathfrak{R} = ((r^{ml}))$ is the representative system of a development obtained in § 4, and the derived class of every class is closed; then the development $\Delta_0 \equiv ((\mathfrak{P}_0^{ml}))$, where $\mathfrak{P}_0^{ml} \equiv V_{r^{ml}}^m$, is equivalent to Δ.

(3) If Δ_1, Δ_2 are equivalent, the development Δ whose mth stage contains all the classes of the respective mth stages of Δ_1 and Δ_2 is equivalent to both Δ_1 and Δ_2.

(4) From a given development Δ we may obtain an equivalent *regular* development Δ_0 as follows: if $\mathfrak{P}^{m'l'}$ $(m' > m)$ contains a pair of elements p, q not contained in any class $\mathfrak{P}^{ml}$, adjoin to stage Δ^m of Δ the class $\mathfrak{P}^{m'l'}$. The development Δ_0 thus obtained will in general be infinite. In case Δ_0 is finite and complete the reduction is evidently of importance.

The principle of reduction introduced in § 5 has an important application to incomplete or infinite developments. Denote by $\mathfrak{P}_{-m}$ the class of all elements p which belong to some class $\mathfrak{P}^{m'l}$ for every $m' \geqq m$. Evidently $\mathfrak{P}_{-m}$ is contained in $\mathfrak{P}_{-(m+1)}$.* If Δ is finite the reduction of Δ relative to $\mathfrak{P}_{-m}$ is finite and complete. Hence if Δ is finite and closed the reduced development is finite, complete, and closed. In this case the class $\mathfrak{P}_{-m}$ is self-compact.

The following is an example of a finite closed development relative to which every class $\mathfrak{P}_{-m}$ is compact. Let $\mathfrak{P}$ be the class $[1 \leqq x]$. Let a^{ml} be a set of numbers such that

$$a^{m1} = 1, \qquad 0 < a^{m,\,l+1} - a^{ml} < \frac{1}{2^{m-1}}, \qquad a^{m,\,m2^m} = m + 1.$$

Then $\mathfrak{P}^{ml} \equiv [a^{ml} \leqq x \leqq a^{m,\,l+1}]$ $(l = 1, 2, \cdots, m2^m)$ defines a development of the type indicated. The class $\mathfrak{P}_{-m}$ is the interval $[1 \leqq x \leqq m + 1]$. Such a development can be used to define the improper integral $\int_1^\infty f(x)\,dx$. If the index l^m is not limited as above the development is infinite. Denote by $\mathfrak{Q}_m$ the class of all points contained in classes $\mathfrak{P}^{ml}$ $(l = 1, 2, \cdots l^m)$ where l^m is chosen arbitrarily (in the present case we could choose $l^m = m2^m$, and then $\mathfrak{Q}_m = [1 \leqq x \leqq m + 1]$ as before). Then the reduction relative to $\mathfrak{Q}_m$ is finite, complete, and closed, and $\mathfrak{P}$ is again representable as the sum of a set of compact classes. These examples serve to illustrate the general theory.

The following type of incomplete development is equivalent to a complete development. Suppose that there is a sequence $\{m_\kappa\}$ of integers m_κ $(> m_{\kappa-1})$ such that every element of $\mathfrak{P}$ is contained in some class $\mathfrak{P}^{ml}$, $m_\kappa \leqq m < m_{\kappa+1}$,

* The notation is due to E. H. Moore. Cf. Pitcher, loc. cit., pp. 40–42.

for every k. The equivalent complete development Δ_0 is obtained by uniting the stages m_κ to $m_{\kappa+1}$ of Δ, forming stage k of Δ_0.

13. We have developed the theory of developments Δ in terms of five fundamental properties: (1) finite; (2) complete; (3) closed; (4) every fundamental sequence has at most one element in its core; (5) if two closed fundamental sequences are connected with a sequence $\{p_n\}$ their cores have a common element.* We shall show that these five properties are completely independent in the sense of E. H. Moore.† The proof of complete independence requires the exhibition of $2^5 = 32$ examples, representing all possible combinations of the five properties and their negatives.

THEOREM 30. *The five fundamental properties of a development Δ are completely independent.*

The 32 examples are defined in terms of a basis system $(\mathfrak{P}_0, \Delta_0)$ possessing the five properties in question.

The class $\mathfrak{P}_0$ is the interval $[0 \leqq x \leqq 1]$. Each class $\mathfrak{P}^{ml}$ is an interval $[l/2^m \leqq x \leqq (l+1)/2^m]$. $(l^m = 0, 1, 2, \cdots, 2^m - 1)$.

We next define five independent operations on $(\mathfrak{P}_0, \Delta_0)$ each of which removes a specified property of Δ_0.

(1) Adjoin to $\mathfrak{P}$ the points of the open interval $[4 \leqq x]$ and to Δ the classes $\mathfrak{P}^{ml} \equiv [l/2^m \leqq x \leqq (l+1)/2^m]$, $l = (4 \cdot 2^m, 4 \cdot 2^m + 1, \cdots)$.

(2) Remove from Δ all classes $\mathfrak{P}^{ml}$ of index $l = 2^m - 1$.

(3) Remove from $\mathfrak{P}_0$ the point $x = \frac{1}{2}$.

(4) Adjoin to $\mathfrak{P}_0$ the point $x = 2$ and add this point to the classes $\mathfrak{P}^{ml}(m, l = 0)$.

(5) Adjoin to $\mathfrak{P}_0$ the point $x = 3$ and add to Δ the classes $\mathfrak{P}^{ml} \equiv (3, 1/m)$, the index l being assigned for each m the value $l = 2^m$.

A development Δ, of a class $\mathfrak{P}$, having any combination of the five properties is obtainable from $\mathfrak{P}_0$ and Δ_0 by a proper combination of the operations listed above. Suppose for example it is desired to construct a development Δ with Properties 1, 5 only. The application of operations 2, 3, 4 leads to a class $\mathfrak{P} \equiv [0 \leqq x < \frac{1}{2}$, or $\frac{1}{2} < x \leqq 1$, or $x = 2$, or $x = 3]$, and a development Δ where $\mathfrak{P}^{ml} = [l/2^m \leqq x \leqq (l+1)/m]$ when $l = (1, 2, \cdots, 2^m - 1)$ omitting the values $l = 2^{m-1} - 1$, $l = 2^{m-1}$; $\mathfrak{P}^{ml} = (1/2 - 1/2^m \leqq p < 1/2)$ if $l = 2^{m-1} - 1$, $\mathfrak{P}^{ml} \equiv [1/2 < p \leqq 1/2 + 1/2^m]$ if $l = 2^{m-1}$; $\mathfrak{P}^{ml} \equiv [1/m, 3]$ if $l = 2^m$; and $\mathfrak{P}^{ml} \equiv [0 \leqq x \leqq 1/2^m$ or $x = 2]$ if $l = 0$. That this development does not possess Properties 2, 3, 4 is easily verified.

14. F. Hausdorff‡ has introduced the concept of *topological* space. Such a

* The properties (4), (5), are together equivalent to the statement, limit of a sequence is unique. Cf. § 3, Theorem 2.

† Loc. cit., § 48.

‡ Loc. cit., p. 213.

space consists of a class $\mathfrak{E}$ of points x which correspond to subsets $\mathfrak{U}_x$ of $\mathfrak{E}$ called regions* and are subject to the following conditions:

(A) Each point x implies at least one region $\mathfrak{U}_x$; every region $\mathfrak{U}_x$ contains the point x;

(B) If $\mathfrak{U}_x, V_x$ are two regions belonging to the same point x, then there is a region W_x which is contained in both $\mathfrak{U}_x$ and V_x;

(C) If a point y is contained in $\mathfrak{U}_x$ there is a region $\mathfrak{U}_y$ which is a subset of $\mathfrak{U}_x$;

(D) For every two distinct points x, y there are two regions $\mathfrak{U}_x, \mathfrak{U}_y$ without a common point.

Let $\delta(x, y)$ be a symmetric distance function defined on a class $\mathfrak{E}$. We define regions $\mathfrak{U}_x$ as follows: for every positive number a the class of all points y such that $\delta(x, y) < a$ is a region $\mathfrak{U}_x$ of x. With this definition of region *a necessary and sufficient condition that* $\mathfrak{E}$ *be a topological* space is that:

(1) *If* $\delta(x, y) < a$ *there exist a positive number* b† *such that whenever* $\delta(y, z) < b$ *then* $\delta(x, z) < a$.

(2) *If* x *and* y *are any two points there is a positive number* a *such that* $\delta(x, z) < a$ *implies* $\delta(y, z) > a$.

From the above one easily deduces the conditions under which a development Δ determines a topological space.

The properties (1), (2) of δ stated above do not imply the property of coherence as is shown by the following example. Let $\mathfrak{E}$ denote the points: $0, 1/n\,(n)$ together with points $x = 1+1/n\,(n)$. The function

$$\delta(x, y) = |x - y|,$$

except when $x = 1/n$, $y = 1 + 1/n$, when $\delta(x, y) = 1/n$. The space $\mathfrak{E}$ is compact and topological but not coherent.

The topological space defined by a symmetric distance function δ is not general but satisfies the first enumerability axiom of Hausdorff in the sense that the system $(\mathfrak{U}_x)$ is equivalent to a system (V_x) satisfying this axiom.‡

We shall now state the conditions under which a topological space becomes equivalent to a metric space. In place of axiom (A) we shall employ the stronger axiom:

(A_ω) There exists for each point x a fixed sequence $\{\mathfrak{U}_x^m\}$ of regions of x such that $\mathfrak{U}_x^m$ contains x and $\mathfrak{U}_x^{m+1}$.§

A region $\mathfrak{U}_x$ is of rank m if it is contained in a region $\mathfrak{U}_y^m$ for some point y and is not contained in any region $\mathfrak{U}_y^{m+1}$.

(G) If $\mathfrak{U}_1, \mathfrak{U}_2, \mathfrak{U}_3, \cdots \mathfrak{U}_m \cdots$ is any sequence of regions such that $\mathfrak{U}_m$ is

* Hausdorff uses the term "Umgebung."

† The number b is dependent on x and y.

‡ Cf. Hausdorff, loc. cit., p. 263.

§ This is virtually the assumption of a type of development. The further assumptions make it possible to replace the infinite development by a finite one.

of rank m (at least) and there is a point x common to the classes $\mathfrak{U}_m^0$† then there is for every region $\mathfrak{U}_x$ of x an m_0 such that for every $m \geqq m_0$, $\mathfrak{U}_m$ is contained in $\mathfrak{U}_x$.

(H) If $(\mathfrak{U})$ is any family of domains such that $\mathfrak{E}$ is the aggregate of the classes $\mathfrak{U}$ then there is a finite subfamily $\mathfrak{U}_1, \mathfrak{U}_2, \cdots \mathfrak{U}_n$ of $(\mathfrak{U})$ whose aggregate is $\mathfrak{E}$.

From axioms (A_ω), (G), it follows that *for every region* $\mathfrak{U}_x$ *there is an* m *such that* $\mathfrak{U}_x^m$ *belongs to* $\mathfrak{U}_x$. Every region $\mathfrak{U}_x$ is a domain. It follows readily from axiom (H) that $\mathfrak{E}$, and every closed subset of $\mathfrak{E}$, is compact and admits the Heine-Borel theorem.

From axioms (A_ω) and (H) we have for every positive integer m a finite set of regions;

$$\mathfrak{U}^{m1}, \mathfrak{U}^{m2}, \mathfrak{U}^{m3}, \cdots, \mathfrak{U}^{ml_m};$$

such that

$$\mathfrak{E} = \sum_{l=1}^{l=lm} \mathfrak{U}^{ml}.$$

The development $\Delta \equiv ((\mathfrak{P}^{ml}))$, *where* $\mathfrak{P}^{ml}$ is the set of α-points of $\mathfrak{U}^{ml}$ is a finite complete development of $\mathfrak{E}$. We shall show that Δ is closed and equivalent to the system $(\mathfrak{U}_x)$ with respect to limit of a sequence. Let $F = \{\mathfrak{P}^{m_n l_n}\}$ be any fundamental sequence of Δ. There is a closed compact class $\mathfrak{F}_n$ common to $\mathfrak{P}^{m_1 l_1}, \cdots \mathfrak{P}^{m_n l_n}$. Since $\mathfrak{F}_n$ contains $\mathfrak{F}_{n+1}$ there must be a point x common to the $\mathfrak{F}_n$. Hence F is closed. Suppose a sequence $\{x_n\}$ is connected with a fundamental sequence F. Without loss of generality we may suppose x_n contained in $\mathfrak{P}^{m_n l_n}$. Then relative to Δ, the sequence has a limit x. Let y be a limit of the sequence in $\mathfrak{E}$. Since every class $\mathfrak{P}^{m_n l_n}$ contains x it follows (Axiom (G)) that for sufficiently large values of m the corresponding regions $\mathfrak{U}^{m_n l_n}$ are contained in a given region $\mathfrak{U}_x$ of x. Since y must be at least an α-point of $\mathfrak{U}^{m_n l_n}$, it follows that y is an α-point of every region $\mathfrak{U}_x$, that is $y = x$, which was to be proved. That any limit in $\mathfrak{E}$ is a limit relative to Δ is evident. Consequently limit is unique relative to Δ and the two systems of limits are the same.

It follows from the conclusions just reached and Theorem 21 that the set $\mathfrak{E}$ is a compact metric space.

THEOREM 31. *A necessary and sufficient condition that a topological space be a compact metric space is that the axioms* (A_ω), (G), (H) *be satisfied.*

The extension of this theory to non-compact spaces will be the topic of a separate discussion.

15. We shall apply the results of the preceding article to spaces $\mathfrak{S}$ satisfying

† $\mathfrak{U}_m^0$ is the class of all the α-points of $\mathfrak{U}_m$. An α-point (Hausdorff) is a point x such that every $\mathfrak{U}_x$ contains a point of $\mathfrak{U}_m$.

the axiom systems Σ_1, Σ_2 of R. L. Moore.* In the systems Σ_1, Σ_2 the undefined elements are points p and classes R of points called regions. A region $\mathfrak{R}$ will be a region $\mathfrak{R}_p$ in the sense of § 14 if $\mathfrak{R}$ contains p. Then $\mathfrak{S}$ is easily shown to be a topological space. We shall establish the further result

THEOREM 32. *A space $\mathfrak{S}$ satisfying axiom systems Σ_1 or Σ_2 is a topological space satisfying the additional axioms* (A_ω), (G). *Any limited closed subset of $\mathfrak{S}$ satisfies axiom* (H).

The axiom systems Σ_1, Σ_2 each consist of eight axioms and differ only in the 6th and 7th. We shall refer to the axioms of Moore as numbered by him. The proof of the above theorem makes use of Axioms 1, 3, 4, 5 only. That Axiom (A_ω) holds is a consequence of Axiom 1 and Theorems 5 and 6 of the article cited. We shall show that Axiom (G) is satisfied. Suppose

$$\mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{R}_3, \cdots, \mathfrak{R}_m, \cdots$$

is any set of regions of rank increasing with m (for convenience suppose $\mathfrak{R}_m$ is of rank m), such that a point p is common to the classes R'_m.† We have to show that if $\mathfrak{R}$ is any region containing p there exists an m_0 such that for every $m \geqq m_0$, $\mathfrak{R}_m$ is contained in $\mathfrak{R}$. Since $\mathfrak{R}_m$ is of rank m there is a sequence $\{\mathfrak{K}_{n_m}\}$ of the fundamental regions postulated in Axiom 1 such that $\mathfrak{K}_{n_m}$ contains $\mathfrak{R}_m^{(m)}$ and $L_m n_m = \infty$. Let $\mathfrak{R}_0$ be a region containing p such that for every positive integer m there is an integer $m' > m$ such that $\mathfrak{K}_{n_{m'}}$, is not contained in $\mathfrak{R}_0$. Let q be any element of $\mathfrak{R}_0$ distinct from p (the existence of q is a consequence of Axiom 3 and Theorem 4, loc. cit.). From the second part of Axiom 1 (loc. cit.) there is a positive integer k such that, if n exceeds k and $\mathfrak{K}_n$ contains p, then $\mathfrak{K}'_n$ is a subset of $\mathfrak{R}_0 - q$. Therefore for every m such that $n_m > k$, $\mathfrak{K}'_{n_m}$ is a subset of $\mathfrak{R}_0$. It follows that every region containing p contains $\mathfrak{K}'_{n_m}$ and therefore $\mathfrak{R}'_m$, for sufficiently large values of m. The proposition is proved.

From Theorem 12 (loc. cit.) it follows that Axiom (H) is satisfied by limited closed sets. From Theorem 31 above it follows that every limited closed set is metric. From Axiom 1 (loc. cit.) it follows that $\mathfrak{S}$ is the sum of an enumerable infinity of metric sets.

On the basis of the complete axiom systems Σ_1, Σ_2 we define an infinite development Δ of $\mathfrak{S}$ for which the corresponding distance function is equivalent to an écart on every limited subset of $\mathfrak{S}$, while limit relative to Δ is equivalent to limit relative to the system of regions R.

From Axiom 1 (loc. cit.) there exists an infinite set of regions $\mathfrak{K}_1$, $\mathfrak{K}_2$, $\mathfrak{K}_3$, $\cdots$ $\mathfrak{K}_n$, $\cdots$ such that if p is a point there is a positive integer n such that $\mathfrak{K}_n$

* *On the foundations of plane analysis situs*, these Transactions, vol. XVII (1916), pp. 131–164.

† $\mathfrak{S}$ is perfect, consequently the derived set coincides with the set of α-points.

contains p. Set $\mathfrak{Q}_1 = \mathfrak{K}_1'$. Each point of $\mathfrak{Q}_1$ may be enclosed in a simple closed curve (Theorem 36, loc. cit.). Hence there is a simple closed curve J_1 containing $\mathfrak{Q}_1$ in its interior (Theorem 42, loc. cit.). Denote by $\mathfrak{P}_1$ the set of all points of J_1 and its interior $\mathfrak{J}_1$. Let n_2 be the least index greater than 1 such that $\mathfrak{K}_{n_2}$ is not a subclass of $\mathfrak{P}_1$. Since $\mathfrak{S}$ is connected (Theorem 22, loc. cit.) there is a simple chain connecting $\mathfrak{P}_1$ and $\mathfrak{K}_{n_2}$ of which $\mathfrak{K}_{n_2}$ is the last and possibly only link. Suppose each point of $\mathfrak{P}_1$ and the derived set of this chain enclosed in a simple closed curve. A finite number of such curves is effective. Hence, as before, we have a closed curve J_2 whose interior $\mathfrak{J}_2$ contains $\mathfrak{P}_1$ and $\mathfrak{K}_n$ $(n \leqq n_2)$. Let $\mathfrak{P}_2 = J_2 + \mathfrak{J}_2$. The process leads to the result stated in the following theorem.

THEOREM 33. *The space* $\mathfrak{S} = \sum_{n=1}^{n=\infty} \mathfrak{P}_n$ *where* $\mathfrak{P}_n$ *is a connected closed limited set bounded by a simple closed curve and* $\mathfrak{P}_n$ *is interior to* $\mathfrak{P}_{n+1}$.

In terms of the sequences $\{\mathfrak{K}_n\}$* and $\{\mathfrak{P}_n\}$ we define a development Δ of $\mathfrak{S}$ as follows. Set $n_1 = 1$. Let n_2 be the least value of $n\,(>1)$ such that $\mathfrak{K}_n$ contains a point of $\mathfrak{P}_1$ not in $\mathfrak{K}_1$. Let n_3 be the least value of $n\,(>n_2)$ such that $\mathfrak{K}_n$ contains a point of $\mathfrak{P}_1$ not contained in any region $\mathfrak{K}_{n'}\,(n' \leqq n_2)$. In general, denote by n_{j+1} the least value of $n\,(>n_j)$ such that $\mathfrak{K}_n$ contains a point of $\mathfrak{P}_1$ not in any region of index less than n_{j+1}. Since $\mathfrak{P}_1$ is limited there is a greatest value j_1 of j for which there is an existent class $\mathfrak{K}_{n_j}$ (Heine-Borel theorem). Denote by $\mathfrak{Q}_1$ the class $\sum_{j=1}^{j=j_1} \mathfrak{K}_{n_j}$. $\mathfrak{P}_1$ is interior to $\mathfrak{Q}_1'$. Let n_{j_1+1} be the least value of n such that $\mathfrak{K}_n$ contains a point of $\mathfrak{P}_2$ not in $\mathfrak{Q}_1$ but no point of $\mathfrak{P}_1$, n_{j_1+2} the least value of n such that $\mathfrak{K}_n$ contains a point of $\mathfrak{P}_2$ not in $\mathfrak{K}_{n_{j'}}\,(j' \leqq j_1 + 1)$† but no point of $\mathfrak{P}_1$. As before the indices $j_1 + i$ have a greatest value j_2. There exists, therefore, a sequence of indices n_j and a sequence of indices j_i such that $\mathfrak{P}_i$ is interior to $\mathfrak{Q}_i = \sum_{j=1}^{j=j_i} \mathfrak{K}_{n_j}$ and no region $\mathfrak{K}_{n_j}$ of index $j > j_i$ contains a point of $\mathfrak{P}_i$. Set $\mathfrak{P}^{1j} \equiv \mathfrak{K}_{n_j}'\,(j = 1, 2, 3, \cdots)$. Let $(\mathfrak{K}_n^1)$ denote the infinite sequence of regions comprising all regions of $(\mathfrak{K}_n)$ which are proper subregions of regions $\mathfrak{K}_{n_j}$. The order of arrangement of the sequence $(\mathfrak{K}_n)$ will be preserved in the sequence $(\mathfrak{K}_n^1)$. The sequence $(\mathfrak{K}_n^1)$ is easily seen to be a fundamental sequence of regions satisfying the conditions of Axiom 1. From the sequences $\{\mathfrak{P}^{1j}\}$, $\{\mathfrak{K}_n^1\}$ we deduce, precisely as before, a sequence $\{\mathfrak{P}^{2j}\}$ and a fundamental set of regions $\{\mathfrak{K}_n^2\}$. The classes $\mathfrak{P}^{2j}$ are closed subclasses of classes $\mathfrak{P}^{1j}$ and at most a finite number of the $\mathfrak{P}^{2j}$ contain points of a given class $\mathfrak{P}^{1j}$. The continuation of the process leads to a system $\Delta \equiv ((\mathfrak{P}^{mj}))$ which is a complete development of S. Since the classes $\mathfrak{P}^{mj}$ are compact and closed, it is easy to see that Δ is closed. Let $\mathfrak{Q}$ be any limited subclass of S. Then $\mathfrak{Q}$ is contained in some

* We shall assume for convenience that no region $\mathfrak{K}_n$ contains a region $\mathfrak{K}_{n'}$, where $n' < n$. This assumption involves no loss in generality under the hypothesis of Axiom 1.

† The existence of the indices n_{j_1+i} follows from Axiom 1.

class $\mathfrak{P}_{\bar{n}}$. At most a finite number $(j_{\bar{n}})$ of the classes $\mathfrak{P}^{1j}$ contain points of any one class $\mathfrak{P}^{1j}$ and therefore at most a finite number of classes $\mathfrak{P}^{2j}$ contain points of $\mathfrak{P}_{\bar{n}}$. Evidently the reduction of Δ relative to $\mathfrak{P}_{\bar{n}}$ (and therefore to $\mathfrak{Q}$) is finite and complete. If $\mathfrak{Q}$ is closed, $\Delta(\mathfrak{Q})$ is also closed (Theorem 6 above). The development Δ is also regular.

We shall prove that limit as defined by Δ is equivalent to limit as defined by region. Let p be a limit of the sequence $\{p_n\}$ in terms of region. Then every region $\mathfrak{R}$ containing p contains all but a finite number of the elements of the sequence $\{p_n\}$. Consequently since there is a fundamental sequence $\mathrm{F} \equiv \{\mathfrak{P}^{ml_m}\}$ of Δ such that p is interior to every $\mathfrak{P}^{ml_m}$ it follows that the sequence $\{p_n\}$ is connected with F. Conversely, if a sequence $\{p_n\}$ is connected with a fundamental sequence F_p, we will show that all but a finite number of the elements p_n are contained in any region $\mathfrak{R}$ containing p. Let q be an element of $\mathfrak{R}$ distinct from p. Then from the second part of Axiom 1 there is an n_0 such that for every $n \geqq n_0$, any region $\mathfrak{R}_n$ containing p is, together with its derivative, contained in $\mathfrak{R} - q$. As every class of stage m is a sublcass of stage $m - 1$ it follows that if $m > n_0$ then every class of $\{\mathfrak{R}_n^m\}$ carries in $\{\mathfrak{R}_n\}$ an index $n' > n_0$. Hence, for $m > n_0$, every class $\mathfrak{P}^{ml}$ containing p is contained in $\mathfrak{R}$. Since the sequence $\{p_n\}$ is connected with F_p it is at once evident that the sequence has the limit p in terms of region.

It follows from the results just obtained, because of Theorem 21 above, that every closed limited subset of $\mathfrak{S}$ is a compact metric space and that $\mathfrak{S}$ is therefore the sum of an enumerable infinity of compact metric spaces. As the proof that $\mathfrak{S}$ is in fact a metric space does not involve a direct application of developments Δ it is reserved for a separate paper. We summarize the results of this section in the following theorem:

THEOREM 34. *A space $\mathfrak{S}$ satisfying axiom systems Σ_1 or Σ_2 has the form $\mathfrak{S} = \sum_{n=1}^{\infty} \mathfrak{P}_n$ where $\mathfrak{P}_n$ is a connected, closed, limited set bounded by a simple closed curve, and $\mathfrak{P}_n$ is interior to $\mathfrak{P}_{n+1}$. The space* S *admits a regular, complete, closed development Δ whose reduction relative to any limited closed set $\mathfrak{Q}$ is finite, complete, and closed. Furthermore, if $\mathfrak{Q}$ is perfect, $\mathfrak{Q}$ is a set of the type "(E) normale" of Frechet.* Limit of a sequence is the same whether defined in terms of the development Δ or of regions $\mathfrak{R}$.*

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* Cf. Theorem 23. The development Δ is obviously asingular.