TRANSFORMATIONS OF SURFACES APPLICABLE TO A QUADRIC*

BY

LUTHER PFAHLER EISENHART

If a conjugate system of curves, or *net*, N, on a surface S and a congruence G of straight lines are so related that the developables of G meet S in N, the net and congruence are said to be *conjugate*. Two nets conjugate to the same congruence are said to be in the relation of a *transformation* T, if the nets are not parallel. In a previous paper† the author developed a general theory of these transformations T. When two surfaces, S and $\bar{S}$, are applicable, there is a unique net on S which is deformed into a net on $\bar{S}$. Let N and $\bar{N}$ denote these nets. Peterson‡ showed that if a net N' parallel to N is known, a net $\bar{N}'$ parallel to $\bar{N}$ can be found by quadratures such that N' and $\bar{N}'$ are applicable. In a former paper§ the author showed that when two such parallel nets N' and $\bar{N}'$ are known, two new applicable nets N_1 and $\bar{N}_1$ can be found by a quadrature such that N_1 and $\bar{N}_1$ are T transforms of N and $\bar{N}$ respectively.

In the present paper these general results are applied to the case where $\bar{N}$ is a net on a quadric, that is when N lies upon a surface applicable to a quadric. We consider first the case where the quadric is a general central quadric not of revolution and find readily the T transforms of N into nets of the same kind as described in the following theorems:

THEOREM A. *If N is a net applicable to a central quadric, not of revolution, Q, there can be found by the solution of a Riccati equation and quadratures three sets of ∞^2 T transforms N_1 which are applicable to Q; these transforms are conjugate to ∞^1 congruences G, there being ∞^1 transforms conjugate to each congruence G; the lines of these congruences G through a point of N form a quadric cone; the tangent planes at points of a line of G to the nets N_1 conjugate to it envelope a quadric cone and the points on Q corresponding to these points of the nets N_1 on a line of G lie on a conic.*

THEOREM B. *If N is a net applicable to Q, there exist an infinity of sets*

* Presented to the Society, September 3, 1919.

† These Transactions, vol. 18 (1917), pp. 97–124. This paper will be referred to as M_1.

‡ *Ueber Curven und Flächen*, Moskau and Leipzig, 1868, p. 106.

§ These Transactions, vol. 19 (1918), pp. 167–185. This paper will be referred to as M_2.

of ∞^2 T transforms N_1 of N; these transforms are conjugate to ∞^2 congruences G; their determination requires the solution of a certain completely integrable system of eight equations.

The existence of these transformations was established in a different manner, and at great length, by Guichard in his Mémoire sur la deformation des quadrics* awarded one half of the Grand Prize of the French Academy of Sciences in 1909. However, his method did not reveal the relations between the nets $\bar{N}$ and $\bar{N}_1$ on the quadric to which N and the nets N_1 are applicable; consequently the property mentioned in the last part of Theorem A is new. Furthermore, Guichard did not show that the nets N_1 in Theorem B are T transforms of N. This fact is of especial interest because of the relation between these transformations and the *transformations B_k* of surfaces applicable to a quadric discovered by Bianchi,† awarded the other half of the Grand Prize. Subsequently Bianchi showed‡ that if N_1 and N_1' are two suitable B_k transforms of a net N applicable to a quadric, then N_1 and N_1' are in relation of a transformation T.

The transformations of Guichard will be referred to as transformations G_k, k being a constant which is the same for all the transformations of a set. Three particular values of k determine the three sets of transformations described in Theorem A. In Section 5 we establish the following theorem of permutability of these transformations:

THEOREM C. *If N_1 and N_2 are transforms of a net N applicable to a net $\bar{N}$ on a central quadric Q by means of transformations G_{k_1} and G_{k_2}, where $k_2 \neq k_1$, there can be found without quadratures a net N_{12} applicable to Q which is in the relation of transformations G'_{k_2} and G'_{k_1} with N_1 and N_2 respectively.*

In certain cases it is possible also to apply this theorem when $k_2 = k_1$.

Guichard limited his consideration to the case where Q is a central quadric not of revolution. In Section 6 we consider the case of nets applicable to nets on a central quadric of revolution. We find that Theorem B holds equally true for this case, and that Theorem A applies with the exception that there is only one set of these transformations. However, we find that in addition such a net admits two sets of ∞^2 transformations T for which the congruences are normal. We consider also the case where Q is a sphere, real or imaginary, and discover that Theorem B holds in this case, that N and N_1 are composed of lines of curvature on surfaces of constant gaussian curvature, and that they are the transformations established by Bianchi.§

In Section 7 we establish transformations T of nets applicable to a paraboloid P. When P is not a surface of revolution, Theorems A and B apply, with the

*Mémoires à l'Académie, ser. 2, vol. 34 (1909).

† Ibid.

‡ Rendiconti dei Lincei, ser. 5, vol. 20 (1911), p. 145.

§ *Lezioni di geometria differenziale*, second edition, Chapter 20.

difference that there are only two sets of transformations of the former kind. When P is a surface of revolution, Theorem B applies but not Theorem A. Also these are transformations with normal congruences analogous to those of a net applicable to a net on a central quadric of revolution.

1. Transformations T of applicable nets

If N is a net, the cartesian coördinates, x, y, z, of the net satisfy an equation of the form

$$\frac{\partial^2\theta}{\partial u\partial v} = A\frac{\partial\theta}{\partial u} + B\frac{\partial\theta}{\partial v}. \tag{1}$$

A net N' is said to be parallel to N when its tangents are parallel to the corresponding tangents to N. If x', y', z' are the coördinates of N', then

$$\begin{aligned} &\frac{\partial x'}{\partial u} = h\frac{\partial x}{\partial u}, \qquad \frac{\partial y'}{\partial u} = h\frac{\partial y}{\partial u}, \qquad \frac{\partial z'}{\partial u} = h\frac{\partial z}{\partial u}, \\ &\frac{\partial x'}{\partial v} = l\frac{\partial v}{\partial x}, \qquad \frac{\partial y'}{\partial v} = l\frac{\partial y}{\partial v}, \qquad \frac{\partial z'}{\partial v} = l\frac{\partial z}{\partial v}, \end{aligned} \tag{2}$$

where h and l are a pair of solutions of

$$\frac{\partial h}{\partial v} = (l - h)A, \qquad \frac{\partial l}{\partial u} = (h - l)B. \tag{3}$$

Conversely, every pair of solutions of (3) determines a parallel net N'.

We call equation (1) the *point equation of* N. The coördinates of N' satisfy a similar equation. Moreover, to each solution θ of (1) there corresponds a solution θ' of the point equation of N'. It is determined by

$$\frac{\partial\theta'}{\partial u} = h\frac{\partial\theta}{\partial u}, \qquad \frac{\partial\theta'}{\partial v} = l\frac{\partial\theta}{\partial v}. \tag{4}$$

If θ and θ' are any pair of corresponding solutions of the point equations of N and N', the functions x_1, y_1, z_1, defined by equations of the form

$$x_1 = x - \frac{\theta}{\theta'}x', \tag{5}$$

are the cartesian coördinates of a net N_1 which is a T transform of N.* Conversely, the most general transformation T is defined in this way.

By means of the above formulas we establish the following relations:

$$\begin{aligned} x_1 - \frac{\theta}{\theta'\dfrac{\partial}{\partial u}\left(\dfrac{\theta}{\theta'}\right)}\frac{\partial x_1}{\partial u} &= x - \frac{\theta}{\dfrac{\partial\theta}{\partial u}}\frac{\partial x}{\partial u}, \\ x_1 - \frac{\theta}{\theta'\dfrac{\partial}{\partial v}\left(\dfrac{\theta}{\theta'}\right)}\frac{\partial x_1}{\partial v} &= x - \frac{\theta}{\dfrac{\partial\theta}{\partial v}}\frac{\partial x}{\partial v}. \end{aligned}$$

* M_1, p. 109.

Consequently the tangents to the curves v = const. or u = const. at corresponding points of N and N_1 meet in the points, P_1, P_2, whose coördinates are of the above forms. If we take another transformation T given by (5) with θ replaced by another solution θ_1 of (1), we find that the tangent plane to the transform meets the corresponding tangent planes to N and N_1 in the point whose coördinates are of the form

$$\xi = x - \frac{\left(\theta\frac{\partial\theta_1}{\partial v} - \theta_1\frac{\partial\theta}{\partial v}\right)\frac{\partial x}{\partial u} - \left(\theta\frac{\partial\theta_1}{\partial u} - \theta_1\frac{\partial\theta}{\partial u}\right)\frac{\partial x}{\partial v}}{\frac{\partial\theta}{\partial u}\frac{\partial\theta_1}{\partial v} - \frac{\partial\theta}{\partial v}\frac{\partial\theta_1}{\partial u}}.$$

From the form of this expression it follows that this point lies also on the tangent plane to the transform of N whose coördinates are of the form obtained by replacing θ in (5) by $\theta + c\theta_1$, where c is any constant, and θ' by a solution of the corresponding equations (4). Hence the corresponding tangent planes of the transforms obtained by varying c envelop a cone.

In order that corresponding points, M and M_1, of N and N_1 be equidistant from the corresponding point P_1, and also from P_2, defined above, we must have

$$\sum x'^2\frac{\partial\theta}{\partial u} = 2\theta'\sum x'\frac{\partial x}{\partial u}, \qquad \sum x'^2\frac{\partial\theta}{\partial v} = 2\theta'\sum x'\frac{\partial x}{\partial v}.$$

In consequence of (2) and (4) these equations are equivalent to

$$\theta' = c\sum x'^2,$$

where c is a constant. In this case the spheres with centers at P_1 and P_2 passing through M and M_1 intersect in a circle C orthogonal to N and N_1. Hence the circles C form a cyclic system,* and consequently N and N_1 lie on the sheets of the envelope of a two-parameter family of spheres.† Moreover, since $\sum x'^2$ is a solution of the point equation of N', the nets N', N, and N_1 consist of the lines of curvature of the surfaces on which they lie. Hence N and N_1 are all in the relation of a transformation of Ribaucour.‡

If $\bar{N}$ is a net applicable to N, its coördinates, $\bar{x}$, $\bar{y}$, $\bar{z}$, satisfy (1). Moreover, the net $\bar{N}'$, whose coördinates are given by quadratures of the form

$$\frac{\partial\bar{x}'}{\partial u} = h\frac{\partial\bar{x}}{\partial u}, \qquad \frac{\partial\bar{x}'}{\partial v} = l\frac{\partial\bar{x}}{\partial v}, \tag{6}$$

is parallel to $\bar{N}$ and is applicable to N'. Furthermore, the net $\bar{N}_1$ whose

* E. p. 428. A reference of this sort is to the author's *Differential Geometry*.

† E. p. 444, Ex. 14.

‡ These Transactions, vol. 17 (1916), pp. 437–458.

coördinates $\bar{x}_1, \bar{y}_1, \bar{z}_1$ are defined by

$$\bar{x}_1 = \bar{x} - \frac{\theta}{\theta'}\bar{x}' \tag{7}$$

is a T transform of $\bar{N}$.*

The common point equation of N' and $\bar{N}'$ admits the solution

$$\theta' = k(\sum \bar{x}'^2 - \sum x'^2), \tag{8}$$

where k is a constant, the symbol $\sum$ indicating the sum of three terms obtained from the three corresponding coördinates. We have shown† that for this value of θ' and the corresponding function θ, given by (4), the nets N_1 and $\bar{N}_1$ are applicable.

2. Transformations T of nets on a quadric.

Consider a net N on the general quadric, Q, whose equation is

$$ax^2 + by^2 + cz^2 + 2dxy + 2eyz + 2fzx + 2rx + 2sy + 2tz + w = 0. \tag{9}$$

Since the coördinates are solutions of an equation of the form (1), we have on differentiating (9) with respect to u and v

$$\begin{aligned} a\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + b\frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + c\frac{\partial z}{\partial u}\frac{\partial z}{\partial v} + d\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)& \\ + e\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} + \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right) + f\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial v}\frac{\partial x}{\partial u}\right) &= 0. \end{aligned} \tag{10}$$

Any net $N'(x')$ parallel to N is given by equations of the form (2). Consequently we have an equation of the form (10) in which x, y, z are replaced by x', y', z'. From this it follows that the function

$$\theta' = ax'^2 + by'^2 + cz'^2 + 2dx'y' + 2ey'z' + 2fz'x' \tag{11}$$

is a solution of the point equation of N'.‡ It is readily found that θ' and θ, given by

$$\begin{aligned} \theta = 2[axx' + byy' + czz' + d(xy' + yx') + e(yz' + zy')& \\ + f(zx' + xz') + rx' + sy' + tz'&], \end{aligned} \tag{12}$$

satisfy equations (4).

When these values are substituted in (5), it is found that the T transform $N_1(x_1)$ lies on Q. It can be shown that any congruence conjugate to a net N can be obtained by drawing through points of N lines whose direction-

* M_2. p. 170.

† M_2. p. 170.

‡ The function $\theta' \neq 0$, since N' cannot lie on a cone.

parameters are the coördinates of some net parallel to N. Hence we have the theorem of Ribaucour:*

Any congruence conjugate to a net on a quadric meets the quadric again in a net to which it is conjugate.

Let $N'(x')$ and $N''(x'')$ be two nets parallel to a net N, and $N_1(x_1)$ and $N_2(x_2)$ the T transforms of N determined by the pairs of corresponding functions θ_1, θ_1' and θ_2, θ_2'', where θ_1 and θ_2 are two solutions of (1). In place of (4) we have

$$(13)\quad \frac{\partial \theta_1'}{\partial u} = h_1 \frac{\partial \theta_1}{\partial u}, \qquad \frac{\partial \theta_1'}{\partial v} = l_1 \frac{\partial \theta_1}{\partial v}, \qquad \frac{\partial \theta_2''}{\partial u} = h_2 \frac{\partial \theta_2}{\partial u}, \qquad \frac{\partial \theta_2''}{\partial v} = l_2 \frac{\partial \theta_2}{\partial v},$$

where h_1, l_1 and h_2, l_2 are pairs of solutions of (3). In addition there are functions θ_1'' and θ_2', determined to within additive constants by the quadratures

$$(14)\quad \frac{\partial \theta_1''}{\partial u} = h_2 \frac{\partial \theta_1}{\partial u}, \qquad \frac{\partial \theta_1''}{\partial v} = l_2 \frac{\partial \theta_1}{\partial v}, \qquad \frac{\partial \theta_2'}{\partial u} = h_1 \frac{\partial \theta_2}{\partial u}, \qquad \frac{\partial \theta_2'}{\partial v} = l_1 \frac{\partial \theta_2}{\partial v},$$

since θ_1 and θ_2 are solutions of (1). We have shown† that a net N_1''' of coordinates x_1''', y_1''', z_1''' is given by equations of the form

$$(15)\qquad x_1''' = x'' - \frac{\theta_1''}{\theta_1'} x',$$

and that N_1''' is parallel to N_1; also that the functions

$$(16)\qquad \theta_{12} = \theta_2 - \frac{\theta_1}{\theta_1'} \theta_2', \qquad \theta_{12}''' = \theta_2'' - \frac{\theta_1''}{\theta_1'} \theta_2'$$

are corresponding solutions of the point equations of N_1 and N_1'''. Moreover, we have shown also that the functions x_{12}, y_{12}, z_{12}, of the form

$$(17)\qquad x_{12} = x_1 - \frac{\theta_{12}}{\theta_{12}'''} x_1''',$$

are the coördinates of a net N_{12} which is a T transform of N_1 and also of N_2. Since θ_1'' and θ_2' are determined only to within arbitrary additive constants, there are accordingly ∞^2 such nets N_{12}.

We apply these results true for any net to the particular case when N is on the quadric Q, and also N_1 and N_2, that is when θ_1 and θ_2 are of the form (12). In order that θ_{12}''' and θ_{12} be of the form (11) and (12) with x', y', z'; x, y, z replaced by x_1''', y_1''', z_1'''; x_1, y_1, z_1 respectively, we must have

$$(18)\qquad \begin{aligned} \theta_1'' + \theta_2' - 2\,[\,ax'x'' + by'y'' + cz'z'' + d\,(x'y'' + x''y') \\ + e\,(y'z'' + y''z') + f\,(z'x'' + z''x')\,] = 0. \end{aligned}$$

*Comptes Rendus, vol. 74 (1872), p. 1491.

† M_1, p. 111.

By differentiation it is found that the left-hand member of this equation is constant, and consequently the additive constants in θ_1'' and θ_2' can be chosen in ∞^1 ways so that (18) shall hold. Hence

If N_1 and N_2 are T transforms of N and all three nets lie on Q (9), there are ∞^1 other nets N_{12} on Q which are T transforms of N_1 and N_2; they can be found by a quadrature.

3. Transformations T of nets C on the deforms of a central quadric

We say that a net N is C when it admits an applicable net $\bar{N}(\bar{x})$. We consider in particular the case when $\bar{N}$ lies on the central quadric Q,

$$a\bar{x}^2 + b\bar{y}^2 + c\bar{z}^2 = 1, \tag{19}$$

where no two of the coefficients a, b, c are equal.

Let $N'(x')$ and $\bar{N}'(\bar{x}')$ be a pair of applicable nets parallel to N and $\bar{N}$ respectively, the coördinates being given by (2) and (6). We have seen that N_1 and $\bar{N}_1$ defined by (5) and (7) are applicable, if θ' is given by (8). Moreover, from (11) it follows that $\bar{N}_1$ will lie on Q, if

$$a\bar{x}'^2 + b\bar{y}'^2 + c\bar{z}'^2 = k(\textstyle\sum\bar{x}'^2 - \sum x'^2). \tag{20}$$

There are two cases to consider, according as k is equal to a, b, or c or when k is not equal to one of these constants.

If $k = a$, equation (20) may be written

$$x'^2 + y'^2 + z'^2 + t'^2 + w'^2 = 0, \tag{21}$$

where

$$t' = \sqrt{\frac{b}{a} - 1}\,\bar{y}', \qquad w' = \sqrt{\frac{c}{a} - 1}\,\bar{z}'. \tag{22}$$

As thus defined t' and w' are solutions of the common point equation of N' and $\bar{N}'$. The functions

$$t = \sqrt{\frac{b}{a} - 1}\,\bar{y}, \qquad w = \sqrt{\frac{c}{a} - 1}\,\bar{z} \tag{23}$$

are the corresponding solutions of the common point equation (1) of N and $\bar{N}$. Moreover, since $\sum\bar{x}^2 - \sum x^2$ is a solution of (1), so also is $x^2 + y^2 + z^2 + t^2 + w^2$, in consequence of (19). Consequently we have

$$\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v} + \frac{\partial t}{\partial u}\frac{\partial t}{\partial v} + \frac{\partial w}{\partial u}\frac{\partial w}{\partial v} = 0. \tag{24}$$

Hence x, y, z, t, w may be looked upon as the coördinates of an *orthogonal net* in 5-space, that is a generalization of a net composed of lines of curvature in ordinary space. Since an equation of the form (24) in $x', \cdots, w'$ is a consequence of (21), it follows that our problem consists in finding parallel nets in 5-space satisfying (21). We consider this problem in the next section and show that there are ∞^1 such nets parallel to an orthogonal net in 5-space.

When N and $\bar{N}$ are known, we have by (23) an orthogonal net in 5-space. Suppose we have a parallel net satisfying (21). Then $\bar{y}'$ and $\bar{z}'$ follow directly from (22), and $\bar{x}'$ is obtained by the quadrature (6). Thus $\bar{x}'$ involves an arbitrary additive constant e. Consequently there result ∞^1 T transforms N_1 of N. The coördinates of N_1 are of the form

$$x_1 = x - \frac{2\left[a\bar{x}(\bar{x}'+e)+b\bar{y}\bar{y}'+c\bar{z}\bar{z}'\right]}{a(\bar{x}'+e)^2+b\bar{y}'^2+c\bar{z}'^2}\,x'. \tag{25}$$

Corresponding points of these nets N_1 lie on the same line of the congruence G_1 conjugate to N of direction parameters x', y', z'. The function θ of this transformation is

$$\theta = 2(a\bar{x}\bar{x}'+b\bar{y}\bar{y}'+c\bar{z}\bar{z}')+2ea\bar{x},$$

and hence from Section 1 it follows that the tangent planes at corresponding points of the nets $\bar{N}_1$ envelop a cone. If ξ_0, η_0, ζ_0 are the coördinates of the vertex, the equation of the tangent plane is of the form

$$(\xi-\xi_0)X_1+(\eta-\eta_0)Y_1+(\zeta-\zeta_0)Z_1 = 0,$$

where ξ, η, ζ are current coördinates, and X_1, Y_1, Z_1 are direction-parameters of the normal to N_1. When the expressions for the latter are calculated, it is found that they involve e to the second degree. Consequently the tangent planes envelop a quadric cone.

Since the direction-parameters of the congruences $\bar{G}$ of the transformations from $\bar{N}$ into $\bar{N}_1$ are $\bar{x}'+e$, $\bar{y}'$, $\bar{z}'$, it follows that corresponding points of the ∞^1 nets $\bar{N}_1$ lie on a conic, the section of Q by a plane parallel to the plane of lines from O to the points $(\bar{x}'+e, \bar{y}', \bar{z}')$ as e varies.

When $k = b$ or c, we get ∞^1 other transformations of the same kind, and Theorem A has been established, except for the statement concerning the congruences G, which will be proved in the next section.

If k in (20) is any constant other than a, b, c, or 0, this equation may be written

$$x'^2+y'^2+z'^2+s'^2+t'^2+w'^2 = 0, \tag{26}$$

where

$$s' = \sqrt{\frac{a}{k}-1}\,\bar{x}', \qquad t' = \sqrt{\frac{b}{k}-1}\,\bar{y}', \qquad w' = \sqrt{\frac{c}{k}-1}\,\bar{z}'. \tag{27}$$

Evidently s', t', w' are solutions of the common point equation of N' and $\bar{N}'$, and s, t, w, defined by

$$s = \sqrt{\frac{a}{k}-1}\,\bar{x}, \qquad t = \sqrt{\frac{b}{k}-1}\,\bar{y}, \qquad w = \sqrt{\frac{c}{k}-1}\,\bar{z},$$

are solutions of the common point equation of N and N, as is also the function $x^2 + y^2 + z^2 + s^2 + t^2 + w^2$. Hence x, y, z, s, t, and w may be looked upon as the cartesian coördinates of an orthogonal net in 6-space, and our problem reduces to the determination of parallel nets in 6-space for which (26) holds. In the next section we show that there are ∞^2 such parallel nets. When a set of functions satisfying (26) is known, the functions $\bar{x}', \bar{y}', \bar{z}'$ follow directly from (27). Hence there is only one transform N_1 of N for each set of solutions, and Theorem B has been proved.

4. Orthogonal nets in n-space

Let $x_1, \cdots, x_n$ be the cartesian coördinates of an orthogonal net in n-space. These coördinates are solutions of an equation of the form (1) such that

$$\sum_{i=1}^{n} \frac{\partial x_i}{\partial u} \frac{\partial x_i}{\partial v} = 0. \tag{28}$$

We put

$$\frac{\partial x_i}{\partial u} = \sqrt{E}\, \xi_i, \qquad \frac{\partial x_i}{\partial v} = \sqrt{G}\, \eta_i \qquad (i = 1, \cdots n), \tag{29}$$

the functions ξ_i and η_i being chosen so that

$$\sum \xi^2 = \sum \eta^2 = 1, \qquad \sum \xi\eta = 0.$$

In this case equation (1) can be given the form

$$\frac{\partial^2 \theta}{\partial u \partial v} = \frac{\partial \log \sqrt{E}}{\partial v} \frac{\partial \theta}{\partial u} + \frac{\partial \log \sqrt{G}}{\partial u} \frac{\partial \theta}{\partial v}, \tag{30}$$

and we have

$$\frac{\partial \xi}{\partial v} = \frac{1}{\sqrt{E}} \frac{\partial \sqrt{G}}{\partial u} \eta, \qquad \frac{\partial \eta}{\partial u} = \frac{1}{\sqrt{G}} \frac{\partial \sqrt{E}}{\partial v} \xi.$$

By algebraic processes we can find $n(n-2)$ functions $Y_j^{(i)}$ $(i = 1, \cdots n;$ $j = 1, \cdots n-2)$ such that

$$\Delta = \begin{vmatrix} Y_1^{(1)} & \cdots & Y_1^{(n)} \\ \cdot & \cdot & \cdot \\ Y_{n-1}^{(1)} & \cdots & Y_{n-1}^{(n)} \\ \xi_1 & \cdots & \xi_n \\ \eta_1 & \cdots & \eta_n \end{vmatrix}$$

is an orthogonal determinant. Since

$$\sum_{i=1}^{n} Y_j^{(i)} \xi_i = 0, \qquad \sum_{i=1}^{n} Y_j^{(i)} \eta_. = 0,$$

we have equations of the form

$$\frac{\partial Y_j^{(i)}}{\partial u} = \sum_k P_{jk} Y_k^{(i)} + A_j \xi_i,$$

$$\frac{\partial Y_j^{(i)}}{\partial v} = \sum_k Q_{jk} Y_k^i + B_j \eta_i,$$

$$\frac{\partial \xi_i}{\partial u} = -\sum_k A_k Y_k^{(i)} - m\eta_i, \qquad \frac{\partial \xi_i}{\partial v} = n\eta_i,$$

$$\frac{\partial \eta_i}{\partial u} = m\xi_i, \qquad \frac{\partial \eta_i}{\partial v} = -\sum_k B_k Y_k^{(i)} - n\xi_i,$$

where

$$m = \frac{1}{\sqrt{G}} \frac{\partial \sqrt{E}}{\partial v}, \qquad n = \frac{1}{\sqrt{E}} \frac{\partial \sqrt{G}}{\partial u}.$$

Expressing the condition of integrability of the first two of these equations, we find that the functions P, Q, A, and B must satisfy

$$\begin{gathered} \frac{\partial}{\partial v} P_{jl} - \frac{\partial}{\partial u} Q_{jl} = \sum_k Q_{jk} \cdot \sum_l P_{kl} - \sum_l Q_{kl} \cdot \sum_k P_{jk}, \\ \frac{\partial B_j}{\partial u} = \sum_k P_{jk} B_k + A_j m, \qquad \frac{\partial A_j}{\partial v} = \sum_k Q_{jk} A_k + B_j n. \end{gathered} \tag{31}$$

We wish to show that it is posisble to define a set of functions $y_l^{(k)}$, which are the elements of an orthgonal substitution of order $n - 2$, so that the functions $X_k^{(i)}$, defined by

$$X_k^{(i)} = y_1^{(k)} Y_1^{(i)} + \cdots + y_{n-2}^{(k)} Y_{n-2}^{(i)} \qquad \begin{pmatrix} i = 1, & \cdots & n, \\ k = 1, & \cdots & n-2 \end{pmatrix},$$

satisfy the equations of the form

$$\frac{\partial X_k^{(i)}}{\partial u} = a_k \xi_i, \qquad \frac{\partial X_k^{(i)}}{\partial v} = b_k \eta_i. \tag{32}$$

In fact, substituting in these equations and making use of the above results, we find that the y's must satisfy the equations

$$\frac{\partial y_l^{(k)}}{\partial u} + \sum_{j=1}^{n-2} y_j^{(k)} P_{jl} = 0, \qquad \frac{\partial y_l^{(k)}}{\partial v} + \sum_{j=1}^{n-2} y_j^{(k)} Q_{jl} = 0. \tag{33}$$

It is readily found that the conditions of integrability of this system are satisfied, in consequence of (31), which establishes the existence of the functions $X_k^{(i)}$ forming with the ξ's and η's the elements of an orthogonal determinant, such that (32) holds.* In addition to these equations we have also

$$\begin{gathered} \frac{\partial \xi_i}{\partial u} = -\sum_k a_k X_k^{(i)} - m\eta_i, \qquad \frac{\partial \xi_i}{\partial v} = n\eta_i, \\ \frac{\partial n_i}{\partial u} = m\xi_i, \qquad \frac{\partial \eta_i}{\partial v} = -\sum_k b_k X_k^{(i)} - n\xi_i. \end{gathered} \tag{34}$$

* Guichard, Annales de l'École Normale, ser. 3, vol. 14 (1897), pp. 498–516.

The conditions of integrability of (32) and (34) are readily found to be

$$\frac{\partial a_k}{\partial v} = mb_k, \qquad \frac{\partial b_k}{\partial u} = na_k,$$

(35)

$$\frac{\partial m}{\partial v} + \frac{\partial n}{\partial u} + \sum_k a_k b_k = 0.$$

In terms of these functions the coördinates of the given net may be given the form

$$x_i = \sum_{k=1}^{n-2} p_k X_k^{(i)} + q\xi_i + v\eta_i, \tag{36}$$

where p_k, q, and r are determinate functions.

When we express the condition that the functions x_i, given by (36), satisfy (29), we find that

$$\begin{aligned} &\frac{\partial p_k}{\partial u} = a_k q, && \frac{\partial p_k}{\partial v} = b_k r, \\ &\frac{\partial q}{\partial u} = -\sum_k a_k p_k - mr + \sqrt{E}, && \frac{\partial q}{\partial v} = nr, \\ &\frac{\partial r}{\partial u} = mq, && \frac{\partial r}{dv} = -\sum_k b_k p_k - nq + \sqrt{G}. \end{aligned} \tag{37}$$

If we put

$$2\omega = \sum_{i=1}^{n} x_i^2, \tag{38}$$

we have

$$\frac{\partial \omega}{\partial u} = \sqrt{E}\, q, \qquad \frac{\partial \omega}{\partial v} = \sqrt{G}\, r. \tag{39}$$

From (28) it follows that ω is a solution of (30). Conversely, if ω' is any other solution of (30), the functions q' and r' given by

$$\frac{\partial \omega'}{\partial u} = \sqrt{E}\, q', \qquad \frac{\partial \omega'}{\partial v} = \sqrt{G}\, r', \tag{40}$$

satisfy the fourth and fifth of equations (37). These functions and the functions p'_k obtained by the quadratures

$$\frac{\partial p'_k}{\partial u} = a_k q', \qquad \frac{\partial p'_k}{\partial v} = b_k r',$$

determine an orthogonal net N', parallel to N, whose coördinates are of the form

$$x'_i = \sum_k p'_k X_k^{(i)} + q' \xi_i + r' \eta_i.$$

Moreover,

$$2\omega' = \sum x_i'^2 = \sum_k p_k'^2 + q'^2 + r'^2.$$

From these results it follows that for the case

$$\sum x_i'^2 = 0,$$

it is necessary and sufficient that q' and r' be zero, and that

$$(41) \qquad x_i' = \sum_k e_k X_k^{(i)}, \qquad \sum e_k^2 = 0,$$

where the e's are constants.

In the particular case treated in the preceding section, where (21) was to be satisfied, we have that $k = 3$. Since the above equations are homogeneous, there are consequently only ∞^1 sets of functions x', y', z', t', w' satisfying (21). Moreover, when we eliminate the e's from $\sum e_k^2 = 0$ and the three expressions for x', y', and z', we get a homogeneous quadratic in x', y', and z', and consequently the lines of the transformations through a point of N form a quadric cone. Furthermore, the determination of the functions $y_l^{(k)}$ from (33) is the same problem as finding the direction-cosines of a trihedral in 3-space whose rotations are given. This requires the solution of a Riccati equation.*

5. Theorems of permutability of transformations G_k

Let $N_1(x_1)$ and $N_2(x_2)$ be transforms of a net N applicable to a net $\bar{N}$ on Q, and let $\bar{N}_1(\bar{x}_1)$ and $\bar{N}_2(\bar{x}_2)$ be the nets on Q to which N_1 and N_2 are applicable. Let θ_1 and θ_2 be the functions of these transformations, where

$$(42) \qquad \theta_1' = k_1(\sum \bar{x}'^2 - \sum x'^2), \qquad \theta_2'' = k_2(\sum \bar{x}''^2 - \sum x''^2),$$

$\bar{x}', \bar{x}''$; x', x'' being obtained from $\bar{x}$ and x by equations of the form (13). We seek under what conditions the net N_{12} whose coördinates are of the form (17) is applicable to a net $\bar{N}_{12}$ on Q.

If we put

$$(43) \qquad \theta_{12}''' = k_2(\sum \bar{x}_1'''^2 - \sum x_1'''^2),$$

where $\bar{x}_1'''$ is given by an equation similar to (15), we find that this expression for θ_{12}''' is equal to that given by (16), if

$$(44) \qquad k_2\,\theta_1'' + k_1\,\theta_2' - 2k_1 k_2(\sum \bar{x}''\,\bar{x}' - \sum x''\,x') = 0.$$

If the left-hand member of this equation is differentiated, it is found that it is constant, in consequence of (14) and (42). Hence the additive constants entering in θ_1'' and θ_2' can be chosen so that (44) holds. In order that $\bar{N}_{12}$

* E., p. 159.

given by an equation similar to (17) be on Q, we must have

$$\theta_1'' + \theta_2' = 2(a\bar{x}'\bar{x}'' + b\bar{y}'\bar{y}'' + c\bar{z}'\bar{z}''), \tag{45}$$

as follows from (18).

Consider first the case where $k_2 \neq k_1$. Solving (44) and (45) for θ_1'' and θ_2', we find expressions which satisfy (14) and (42). Hence Theorem C has been proved. This theorem is equally true when k_1 and k_2, or either is equal to a, b, or c.

In order that $k_2 = k_1$, it follows from (44), (45), and (27) that we must have

$$x'x'' + y'y'' + z'z'' + s's'' + t't'' + w'w'' = 0,$$

where from (41)

$$x' = \sum_j e_j' X_j, \qquad x'' = \sum_j e_j'' X_j, \qquad \cdots.$$

Consequently the constants e' and e'' must satisfy

$$\sum e_j'^2 = 0, \qquad \sum e_j''^2 = 0, \qquad \sum e_j' e_j'' = 0.$$

This is possible when j can take on the values $1, 2, 3, 4$, that is when k is not equal to a, b, or c, and then it is possible in an infinity of ways. We say that two such transformations G_k are complementary. In this case θ_1'' or θ_2' must be found by a quadrature and the other follows from (45). Hence

If N_1 and N_2 are complementary G_k transforms of a net N, there are ∞^1 nets N_{12} which are G transforms of N_1 and N_2; they can be found by a quadrature.

6. When Q is a central quadric of revolution, or a sphere

When $N(x)$ is applicable to a net on the quadric of revolution Q,

$$ax^2 + b(y^2 + z^2) = 1, \tag{46}$$

the transformations G_k, as described in Theorem B, exist as in the case of the general quadric. However, there is only one set of transformations of the type described in Theorem A. They are G_a.

When in equation (20) we put $k = b$, the result may be written

$$x'^2 + y'^2 + z'^2 + s'^2 = 0, \tag{47}$$

where

$$s' = \sqrt{\frac{a}{b} - 1}\,\bar{x}'. \tag{48}$$

If we put

$$s = \sqrt{\frac{a}{b} - 1}\,\bar{x},$$

the functions x, y, z, s are the cartesian coördinates of an orthogonal net in

4-space, since $x^2 + y^2 + z^2 + s^2$ also is a solution of (1). In this case the functions $y_i^{(k)}$ of Section 4 may be taken in the form

$$y_1^{(1)} = \cos\phi, \quad y_1^{(2)} = \sin\phi, \quad y_2^{(1)} = -\sin\phi, \quad y_2^{(2)} = \cos\phi.$$

Then the determination of ϕ from (33) becomes an algebraic problem. There are two sets of solutions satisfying (47). They are of the forms

$$x' = X_1^{(1)} + iX_2^{(1)}, \quad x'' = X_1^{(1)} - iX_2^{(1)}.$$

In either case $\bar{x}'$ follows directly from (48), but $\bar{y}'$ and $\bar{z}'$ are given by quadratures of the form (6) in which h and l are known. Hence both $\bar{y}'$ and $\bar{z}'$ involve arbitrary additive constants. Therefore either congruence conjugate to N is conjugate to ∞^2 T transforms N_1, which are applicable to Q.

If we keep the additive constant of $\bar{y}'$ or $\bar{z}'$ fixed and allow the other to vary, we see as in Section 3 that the tangent planes to these ∞^1 nets N_1 at corresponding points envelop a quadric cone, and the corresponding points on Q lie on a conic.

Ribaucour has shown that if the point equation of a net N on a surface S of coördinates x, y, z admits a solution R such that $x^2 + y^2 + z^2 - R^2$ also is a solution, the spheres with centers on S and of radius R are enveloped by two surfaces Σ_1 and Σ_2 upon which the lines of curvature correspond to the net N on S, and consequently the lines joining points of S to corresponding points of Σ_1 and Σ_2 form two normal congruences conjugate to N, which are the only normal congruences conjugate to N other than the one normal to S. Moreover, it can be shown that each of these congruences has for direction-parameters the coördinates x', y', z' of a net N' parallel to N for which $x'^2 + y'^2 + z'^2 = R'^2$, where x', y', z', R' are the solutions of the point equation of N' corresponding to x, y, z, R respectively.* This is the situation in the case of the above transformations. In fact the congruences of these transformations are the congruences of normals to the sheets of the envelopes of the spheres with centers on N and of radius $R = \sqrt{1 - (a/b)\,\bar{x}}$. We remark also that the distances from a point $(\bar{x}, \bar{y}, \bar{z})$ of $\overline{N}$ on Q to the foci on the axis of revolution are $R \pm 1/\sqrt{a}$. Evidently, the congruences normal to the sheets of the envelope of the spheres with centers on N and of radius $R \pm 1/\sqrt{a}$ are the same as the preceding.

Beltrami† has shown that if the lines of a normal congruence are invariably bound to a surface S, the congruence is changed into a new normal congruence when S is deformed. In consequence of this theorem and the facts set forth in the preceding paragraph, if lines are drawn from points of $\overline{N}$ to the foci of Q on the axis of revolution, these lines become the two congruences G_1 and G_2

* M_1, p. 110.

† E. p. 403.

of the two transformations, when $\bar{N}$ is applied to N. Hence we have the

THEOREM. *Let N be a net applicable to a net $\bar{N}$ on a central quadric of revolution Q; the lines joining points of $\bar{N}$ to the foci of Q on the axis of revolution become lines of two normal congruences, G_1 and G_2, conjugate to N when $\bar{N}$ is applied to N; there can be found by two quadratures ∞^2 nets N_1 conjugate to G_1 and ∞^2 nets N_2 conjugate to G_2 all of which are applicable to ∞^2 nets $\bar{N}_1$ and ∞^2 nets $\bar{N}_2$ on Q; the nets N_1 or N_2 can be grouped into ∞^1 families of ∞^1 nets such that their tangent planes at points on the same line of the congruence form a quadric cone and the corresponding points of the applicable nets on Q lie on a conic.*

The transformations G_k and G_a of nets applicable to a net $\bar{N}$ on Q (46) admit the theorem of permutability C. In addition there is an analogous theorem of permutability when N_1 is a transform of N in accordance with the above theorem and N_2 is a transform by a G_a or a G_k, when $k \neq b$.

When Q is a sphere, real or imaginary, with the equation

$$a(\bar{x}^2 + \bar{y}^2 + \bar{z}^2) = 1,$$

an applicable net N consists of the lines of curvature of a surface of constant gaussian curvature, since every net on Q is orthogonal. If we take $k = a$, we have from (20)

$$x'^2 + y'^2 + z'^2 = 0,$$

which is impossible for 3-space according to Section 4. Consequently only general transformations G_k exist, and the processes of finding them are the same as in Sections 3, 4 for the general quadric. From (20) and (11) we have $\theta' = \sum x'^2 \cdot ak/(k-a)$. Hence from Section 1 it follows that N and N_1 are in the relation of a transformation of Ribaucour, and consequently these are the transformations of surfaces of constant gaussian curvature discussed by Bianchi.*

7. Transformations T of nets applicable to a paraboloid

If the equation of the paraboloid P is taken in the form

$$a\bar{x}^2 + b\bar{y}^2 + 2\bar{z} = 0, \tag{49}$$

equations (11) and (12) become

$$\theta' = a\bar{x}'^2 + b\bar{y}'^2, \qquad \theta = 2(a\bar{x}\bar{x}' + b\bar{y}\bar{y}' + \bar{z}').$$

In place of (20) we have

$$\sum x'^2 + \left(\frac{a}{k} - 1\right)\bar{x}'^2 + \left(\frac{b}{k} - 1\right)\bar{y}'^2 - \bar{z}'^2 = 0. \tag{50}$$

When k is equal to a or b, we have transformations of the type described

* L. c.

in Theorem A, with the difference that there are only two such groups of transformations of nets applicable to P. In like manner there are ∞^2 transformations G_k, for each value of k other than a, b, or zero. Moreover, these transformations admit a theorem of permutability analogous to Theorem C.

When $b = a$ in (49) and P is a surface of revolution, Theorem B holds true. There are no transformations as described by Theorem A. When we put $k = a$ in (50), it becomes

$$\sum x'^2 = \bar{z}'^2.$$

Proceeding as in Section 6, we find that there are two sets of ∞^2 transforms of N, the ∞^2 transforms of either set being conjugate to a congruence which is normal to one of the sheets of the envelope of spheres with centers on N and of radius $\bar{z}$. From a well-known property of the parabola and the above theorem of Beltrami it follows that these normal congruences are obtained by drawing from points of $\bar{N}$ lines to the focus of P and lines perpendicular to the tangent plane to P at its vertex, and then applying $\bar{N}$ to N. Hence we have for nets applicable to a paraboloid of revolution a theorem the same as for a central quadric of revolution (section 6) except with reference to the method of generating the normal congruences of the transformations.

PRINCETON UNIVERSITY