THE GEOMETRY OF HERMITIAN FORMS*

BY

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1. The general case

We mean by a *hermitian form* an expression which is linear and homogeneous in each of two sets of conjugate imaginary variables and which is a constant multiple of its own conjugate imaginary. Such a form may be written

$$\sum a_{ij} x_i \bar{x}_j \qquad (i, j, = 0, 1, \cdots n), \tag{1}$$

where

$$a_{ij} = \rho \bar{a}_{ji}, \qquad \rho^2 = 1, \qquad |a_{ij}| \neq 0.$$

The last inequality is added for convenience, and is not needed in all parts of the work. Strictly speaking, we are dealing with hermitian forms of non-vanishing discriminant.

It is easy to write down a hermitian form in the Clebsch-Aronhold symbolism. We here replace (1) by

$$a_x \bar{a}_{\bar{x}}. \tag{2}$$

For the discriminant we have the expression

$$\frac{1}{(n+1)!} |a^{(1)} a^{(2)} \cdots a^{(n+1)}| \cdot |\bar{a}^{(1)} \bar{a}^{(2)} \cdots \bar{a}^{(n+1)}|,$$

each of the last two factors being a symbolic $(n+1)$-rowed determinant.

In order to live up to the geometric title of our paper, we shall speak of a set of $(n+1)$ homogeneous values (x), not all simultaneously zero, as the homogeneous coördinates of a *point* in a projective space of n-dimensions. Following the conventions of this sort of geometry, we shall define the system of all points whose coördinates satisfy a linear homogeneous equation

$$\sum_i u_i x_i \equiv (ux) = 0 \qquad (i = 0, 1, \cdots n) \tag{3}$$

as forming a *hyperplane*, or S_{n-1}. If there be k such equations, linearly independent, the corresponding points shall be said to generate an S_{n-k}. The coördinates of all points of the system are linearly dependent on those of any

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$n - k + 1$ linearly independent individuals of their number. Conversely, such dependence will always lead back to k independent equations, and an S_{n-k}.

Two points (x) and (y) whose coördinates are connected by one and, hence, all of the equations

$$\sum_{i,j} a_{ij}\, x_i\, \bar{y}_j = \sum_{i,j} a_{ji}\, y_j\, \bar{x}_i = a_x\, \bar{a}_{\bar{y}} = a_y\, \bar{a}_x = 0 \tag{4}$$

are said to be *conjugate* with regard to the form (1). The totality of points conjugate to a given point will generate a hyperplane which shall be defined as *polar hyperplane* of the given point. More generally, we may state

THEOREM I. *All points of an S_{n-k} are conjugate to all of an S_{k-1} with regard to a hermitian form.*

Let us take any point whose coördinates do not reduce the form (1) to zero. Such a point will not lie in its polar hyperplane, will not be self-conjugate, and may be chosen in ∞^n ways. A second point may then be chosen conjugate to the first but not to itself, then a third conjugate to the first two, etc. We arrive finally at a system of points, each two of which are conjugate with regard to the form. Remembering the identity

$$n + (n - 1) + \cdots - + 2 + 1 = \tfrac{1}{2}\, n\,(n + 1),$$

we have

THEOREM II. *There are $\infty^{\frac{1}{2}(n(n+1))}$ systems of $(n + 1)$ points, each two of which are conjugate with regard to a given hermitian form of a non-vanishing discriminant.*

Let us take such a system of points as the basis of our coördinate apparatus; we find immediately that our form takes the canonical shape

$$\sum_i a_{ii}\, x_i\, \bar{x}_i, \tag{5}$$

where all of the coefficients are real. Evidently, when the discriminant is zero, there is a similar canonical expression, some of the coefficients vanishing. A further reduction is effected by the transformation

$$x_i = \sqrt{|a_{ii}|}\, x_i.$$

THEOREM III. *Every hermitian form of non-vanishing discriminant may be reduced in $\infty^{\frac{1}{2}(n(n+1))}$ ways to an expression of the type*

$$\sum_i \pm\, x_i\, \bar{x}_i. \tag{6}$$

It remains to be seen what is the significance of the distribution of positive and negative signs. Suppose that things are so arranged that the first k

terms in (6) have one sign, while the remaining terms have the other sign, and that $k \leqq n - k + 1$.

Consider the S_{k-1} given by the equations

$$x_0 - x_k = x_1 - x_{k+1} = \cdots x_{k-1} - x_{2k-1} = 0,$$

$$x_{2k} = x_{2k+1} \cdots = x_n = 0.$$

Every point of this S_{k-1} lies in its polar hyperplane with regard to the form. On the other hand there could not be an S_k each of whose points was in its polar hyperplane. For if there were such a variety, that would have at least one point in common with the S_{n-k} given by the equations

$$x_0 = x_1 = \cdots x_{k-1} = 0,$$

and for such a point (x) we should have the absurd equation

$$x_k \bar{x}_k + x_{k+1} \bar{x}_{k+1} + \cdots + x_n \bar{x}_n = 0.$$

Our reasoning here is reversible throughout. We thus reach

THEOREM IV. *There will exist an S_{k-1} all of whose points are self-conjugate with regard to a given hermitian form of non-vanishing discriminant, but no S_k possessing this property when $k \leqq \frac{1}{2} n$, and when the form can be reduced to a sum of products of conjugate imaginary coördinate values, whereof just k products have one algebraic sign, and the remainder the other sign.*

THEOREM V. *Sylvester's law of inertia holds for hermitian forms.**

2. PSEUDO-ORTHOGONAL RELATIONS

We shall, from now on, confine ourselves to the special case of the hermitian form

$$\sum_i x_i \bar{x}_i \equiv (x\bar{x}). \tag{7}$$

Since this expression can never vanish, there are no points which are self-conjugate. An S_k and an S_{n-k+1}, so related that each point of one is conjugate to each of the other shall be said to be *pseudo-orthogonal.* The same unwieldy adjective shall be applied to every transformation that leaves the form unaltered except for a constant non-vanishing factor. If such a transformation be a collineation of the type

$$x_i = \sum_j a_{ij} x_j \qquad (|a_{ij}| \neq 0), \tag{8}$$

the fundamental equations of condition are

* Given without proof by Segre in the paper, *Un nuovo campo di ricerche geometriche*, Atti della R. Accademia delle Scienze di Torino, vol. 25 (1890), p. 605.

$$\sum_k a_{kp}\,\bar{a}_{kq} = 0 \qquad (p \neq q), \tag{9a}$$

$$\sum_k a_{kp}\,\bar{a}_{kp} = \sum_k a_{kq}\,\bar{a}_{kq}. \tag{9b}$$

Let us see what equations (9a) mean geometrically, as that will lead us to discover an explicit form for our pseudo-orthogonal collineation. They tell us that the members of two different columns of the matrix of the collineation are proportional to the coördinates of two points which are conjugate with regard to the form. Equations (9b) are satisfied by affecting such coördinates with proper factors of proportionality.

The first point whose coördinates shall go into this matrix may be selected at random. The second may be chosen linearly dependent on the first and on an arbitrary point, yet conjugate to the first. The third may be taken linearly dependent on the first two and another independent arbitrary point but conjugate to the first two, and so on. The ith row of the matrix of the transformation would thus take the general form

$$||\,a_i \qquad pa_i + qb_i \qquad ra_i + sb_i + tc_i \qquad \cdots\,||.$$

It is our business to choose the multipliers in such a way that the point whose coördinates go to make up any column is conjugate to those which appear in the columns to the left. Since the condition that two points (x) and (y) should be conjugate takes the form

$$(x\bar{y}) = (y\bar{x}) = 0, \tag{10}$$

the coördinates of the point which is linearly dependent on $(a)(b)\cdots(k)(l)$ may be expressed in the form

$$\begin{vmatrix} a_i & b_i & \cdots & k_i & l_i \\ (a\bar{a}) & (b\bar{a}) & \cdots & (k\bar{a}) & (l\bar{a}) \\ (a\bar{b}) & (b\bar{b}) & \cdots & (k\bar{b}) & (l\bar{b}) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ (a\bar{k}) & (b\bar{k}) & \cdots & (k\bar{k}) & (l\bar{k}) \end{vmatrix}.$$

Our equations (9a) are thus completely satisfied. If we indicate by (y) the coördinates of a point appearing in any column, we have merely to multiply them all through by a factor η where

$$|\eta| = \frac{r}{\sqrt{(y\bar{y})}}.$$

If thus

$$\Delta_l = \begin{vmatrix} a & b & \cdots & k & l \\ (a\bar{a}) & (b\bar{a}) & \cdots & (k\bar{a}) & (l\bar{a}) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ (a\bar{k}) & (b\bar{k}) & \cdots & (k\bar{k}) & (l\bar{k}) \end{vmatrix} \tag{11}$$

we reach our final statement:

THEOREM VI. *The general term of the matrix of a pseudo-orthogonal collineation can be expressed in the form*

$$(12)\qquad re^{i\theta_i}\frac{\begin{vmatrix} a_i & b_i & \cdots & k_i & l_i \\ (a\bar{a}) & (b\bar{a}) & \cdots & (k\bar{a}) & (l\bar{a}) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ (a\bar{k}) & (b\bar{k}) & \cdots & (k\bar{k}) & (l\bar{k}) \end{vmatrix}}{\sqrt{\sum_{p,q}(p\bar{q})\frac{\partial\Delta_l}{\partial p}\frac{\partial\bar{X}_l}{\partial\bar{q}}}}.$$

Here the subscript i indicates the row in which this term lies, while the order of the determinant Δ_l as defined in (11) *gives the column. The summation in the denominator covers all pairs of letters, one from each of the series $a, b, \cdot k, l$; $\bar{a}, \bar{b}, \cdots \bar{k}, \bar{l}$. The non-vanishing factor r is the same in every term.**

It is pleasant to note that the most important properties of a pseudo-orthogonal collineation are found with much less labor than is needed to establish the fundamental theorem. Let us look at the fixed points. Every collineation of space surely leaves at least one point fixed. That point can not be self-conjugate with regard to the fundamental hermitian form, and its polar hyperplane, which does not include the given point, is also fixed, the transformation therein being of exactly the same type in one less dimension.

THEOREM VII. *In every pseudo-orthogonal collineation there are necessarily $n+1$ fixed points, each two of which are conjugate with regard to the fundamental form.*

It is clear that we can take these fixed points as the basis of our coördinate system. We have then the canonical form for a pseudo-orthogonal collineation

$$(13)\qquad x_k' = re^{i\theta_k}x_k.$$

If $\theta_0 = \theta_1 = \cdots = \theta_p$, all points of the S_p, $x_{p+1} = x_{p+2} = \cdots = x_n = 0$ are invariant.

If $(a^{(0)})(a^{(1)})(a^{(2)})\cdots(a^{(n)})$ be $n+1$ distinct points, each two of which are conjugate with regard to our form, the ith row of a pseudo-orthogonal

* The only attempt that has been made to find the form for a pseudo-orthogonal collineation, so far as the writer is aware, is to be found in an article by Loewy, "*Ueber bilineare Formen mit conjugirt imaginaeren Variabeln,*" Abhandlungen der Kaiserlichen Leopoldinischen-Carolinischen Akademie, Halle, vol. 71 (1898). Loewy does not, however, give the transformation in explicit shape, but in a form that involves the solution of a certain matrix equation. It is to be noted that the number of apparently arbitrary quantities is really greater than that of the free parameters in the group of these transformations. It would be highly desirable to find an expression for the transformation in a form that involves no redundant parameters, the present writer regrets that all attempts which he has made so far have met with meager success.

collineation which transfers them to the fundamental points of the coördinate system will be

$$\left\| \frac{a^{(0)}}{\sqrt{(a^{(0)}\bar{a}^{(0)})}} \; \frac{a_i^{(1)}}{\sqrt{(a^{(1)}\bar{a}^{(1)})}} \cdots \frac{a_i^{(n)}}{\sqrt{(a^{(n)}\bar{a}^{(n)})}} \right\|.$$

Taking the product of this, and the inverse of a second such transformation which carries a similar set of points (b) into the fundamental set, we have a pseudo-orthogonal transformation carrying the set (a) into the set (b).

THEOREM VIII. *A pseudo-orthogonal collineation may be found to carry any set of points, mutually conjugate with regard to the fundamental form, into any other such set.*

We are now in a position to determine the number of parameters of the pseudo-orthogonal group. To begin with, the determination of the $n+1$ points which shall be carried into the fundamental points depends upon

$$n + (n-1) + \cdots 2 + 1 + 0 = \tfrac{1}{2}n(n+1)$$

complex parameters. On the other hand, we see by (13) that the general collineation which leaves the fundamental points invariant depends on n real parameters, $\theta_k - \theta_0$, the common factor r being irrelevant.

THEOREM IX. *The totality of pseudo-orthogonal collineations is a group depending on $\frac{1}{2}n(n+1)$ complex, and n real parameters.**

Let us see if there be any one-parameter groups of pseudo-orthogonal transformations. Since all transformations of such a group have the same fixed points, we may take these as the basis of the coördinate system; it becomes a question of finding one-parameter groups under (13). Let us first look for groups depending on a single complex parameter. We may, without loss of generality, take $re^{i\theta_0}$ for the independent variable, and write

$$re^{i\theta_k} = f_k(re^{i\theta_0}).$$

But in this case the function

$$\frac{f_k(z)}{z}$$

has a constant modulus and so is a constant. Hence the common factor $re^{i\theta_0}$ could be divided out of (13) and the group would not depend on any parameter.

THEOREM X. *There are no infinite groups of pseudo-orthogonal collineations depending analytically on a single complex parameter.*

The case is different when we come to consider groups depending on a single real parameter. We may take θ_0 for this parameter, and write

$$\theta_k = f_k(\theta_0).$$

* Study, *Kürzeste Wege im komplexen Gebiete*, Mathematische Annalen, vol. 60 (1905), p. 323.

The fundamental property of groups gives us

$$f_k(r) + f_k(s) = f_k(r+s), \qquad f_k(t) = a_k t.$$

Dividing out the constant factor r, we get the general form for a transformation of this group

$$x_k' = e^{ia_k t} x_k.$$

An infinitesimal transformation of the group will give

$$dx_k = ia_k x_k \, dt.$$

Integrating, we get the form for the "threads," that is the path-curves of points under the group, namely

$$x_k = \rho_k e^{ia_k t}. \tag{14}$$

When will two pseudo-orthogonal collineations be commutative? Let us pick a transformation of (8) and one of (13). The conditions for commutativity are

$$(e^{i\theta_i} - e^{i\theta_j}) a_{ij} \equiv 0 \qquad (i, j, = 0, 1 \cdots n).$$

If one of our transformations, which we assume to be (13), has only $n+1$ fixed points, this equation can only be satisfied by taking

$$a_{ij} = 0, \qquad i \neq j,$$

and the points fixed for the first transformation are also fixed for the second. Suppose, however, that our transformation (13) has an S_p fixed, so that

$$\theta_0 = \theta_1 \cdots = \theta_p.$$

Then

$$a_{i(p+r)} = a_{(p+s)j} = 0.$$

The first $p+1$ equations of (8) can, then, be written

$$\begin{aligned} x_0'' &= a_{00} x_0' + a_{01} x_1' + \cdots a_{0p} x_p', \\ x_1'' &= a_{10} x_1' + \cdots, \\ &\cdots\cdots\cdots \\ x_p'' &= a_{p0} x_0' + \cdots a_{pp} x_p'. \end{aligned}$$

These, in turn, can be reduced to

$$x_0'' = e^{i\phi_0} x_0', \qquad x_1'' = e^{i\phi_1} x_1, \qquad \cdots, \qquad x_p'' = e^{i\phi_p} x_p',$$

while the first $p+1$ transformations of (8) remain

$$x_0' = x_0, \qquad x_1' = x_1, \qquad \cdots, \qquad x_p' = x_p.$$

We are thus enabled to state the general theorem:

THEOREM XI. *A necessary and sufficient condition that two pseudo-orthogonal collineations should be commutative is that every S_p all of whose points are invariant in the first transformation, but which is not contained in an S_{p+1} of fixed points, should be invariant in the other transformation.*

It is easy enough to find relative invariants under pseudo-orthogonal collineations. For instance, the points $(a)(b)\cdots(k)$ will have the relative invariant

$$\begin{vmatrix} (a\bar{a}) & (a\bar{b}) & \cdots & (a\bar{k}) \\ (b\bar{a}) & \cdots & \cdots & (b\bar{k}) \\ \cdot & \cdot & \cdot & \cdot \\ (k\bar{a}) & \cdots & \cdots & (k\bar{k}) \end{vmatrix}.$$

This can vanish only when the points are linearly dependent, since it is the product of the two conjugate imaginary matrices

$$\begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ b_0 & b_1 & \cdots & b_n \\ \cdot & \cdot & \cdot & \cdot \\ k_0 & k_1 & \cdots & k_n \end{vmatrix} \times \begin{vmatrix} \bar{a}_0 & \bar{a}_1 & \cdots & \bar{a}_n \\ \bar{b}_0 & \bar{b}_1 & \cdots & \bar{b}_n \\ \cdot & \cdot & \cdot & \cdot \\ \bar{k}_0 & \bar{k}_1 & \cdots & \bar{k}_n \end{vmatrix}.$$

Let us take two points; remembering that the product of two conjugate imaginary expressions is necessarily positive, we see that

$$\begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ b_0 & b_1 & \cdots & b_n \end{vmatrix} \times \begin{vmatrix} \bar{a}_0 & \bar{a}_1 & \cdots & \bar{a}_n \\ \bar{b}_0 & \bar{b}_1 & \cdots & \bar{b}_n \end{vmatrix} > 0,$$

$$(a\bar{a})(b\bar{b}) > (a\bar{b})(b\bar{a}).$$

We are thus lead to a new system of non-euclidean geometry where the distance of two points (a) and (b) is defined by the equation

$$\cos\frac{d}{k} = \frac{\sqrt{(a\bar{b})}\,\sqrt{(b\bar{a})}}{\sqrt{(a\bar{a})}\,\sqrt{(b\bar{b})}}. \tag{15}$$

These considerations of hermitian metrics are, however, outside the domain of our present paper; they have, in fact, already been treated by others.*

* Study, loc. cit., p. 333; also Fubini, *Sulle metriche definite da una forma hermitiana*, Atti della R. Istituto Veneto, vol. 63 (1903).

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