CONCERNING APPROACHABILITY OF SIMPLE CLOSED AND OPEN CURVES*

BY

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Schoenflies† was the first to formulate the converse of the fundamental theorem of C. Jordan‡ that a simple closed curve§ lying wholly within a plane decomposes the plane into an inside and an outside region. The statement of this converse theorem is as follows: Suppose $K$ is a closed, bounded set of points lying in a plane $S$ and that $S - K = S_1 + S_2$, where $S_1$ and $S_2$ are point-sets such that (1) every two points of $S_i$ ($i = 1, 2$) can be joined by an arc lying entirely in $S_i$, (2) every arc joining a point of $S_1$ to a point of $S_2$ contains at least one point of $K$ (3) if $O$ is a point of $K$ and $P$ is a point not belonging to $K$, then $P$ can be joined to $O$ by an arc that has no point except $O$ in common with $K$. Every point-set that satisfies these conditions is a simple closed curve. Schoenflies used metrical hypotheses in his proof. Lennes gave a proof of this converse theorem using straight lines.|| R. L. Moore pointed out that a proof similar in large part to that of Lennes can be carried through with the use of arcs and closed curves on the basis of his system of postulates $\Sigma_3$, thus furnishing a non-metrical proof of the converse theorem.¶

In all these proofs the condition numbered three, the condition of approachability (erreichbarkeit) plays a fundamental rôle. It is the purpose of the present paper to study the effect of substituting for the condition of approach-

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§ If $A$ and $B$ are distinct points, then a simple continuous arc from $A$ to $B$ is defined by Lennes as a bounded, closed, connected set of points containing $A$ and $B$, but containing no proper connected subset containing both $A$ and $B$. Cf. N. J. Lennes Curves in non-metrical analysis situs with an application in the calculus of variations, American Journal of Mathematics, vol. 33 (1911), p. 308. A simple closed curve is a set of points composed of two arcs $AXB$ and $AYB$ having no point in common other than $A$ and $B$. Hereafter in this paper “arc” and “closed curve” will be considered synonymous with “simple continuous arc” and “simple closed curve,” respectively.
ability, the condition that the set is "connected in kleinem."* The results obtained are embodied in the following theorem:

**Theorem A.** Suppose $K$ is a closed plane point-set, $S$ is the set of all points of the plane, while $S - K = S_1 + S_2$, where $S_1$ and $S_2$ are two mutually exclusive domains† such that every point of $K$ is a common boundary point of $S_1$ and $S_2$. Then a necessary and sufficient condition that $K$ be either a simple closed curve or an open curve‡ is that $K$ be connected in kleinem.

That the condition stated in Theorem A is necessary is evident. I will proceed to show that it is sufficient. Suppose $K$ is a connected in kleinem set satisfying the conditions stipulated in Theorem A. Then the following lemmas hold true:

**Lemma A.** Every arc joining a point of $S_1$ to a point of $S_2$ contains a point of $K$.

**Proof.** Suppose it were possible to draw an arc from a point $P_1$ of $S_1$ to a point $P_2$ of $S_2$ that contains no point of $K$. Then let us divide the arc $P_1 P_2$ into two sets, $M_1$ and $M_2$, where $M_1$ is the set of all points of $P_1 P_2$ that belong to $S_1$, while $M_2$ is the set of all points of $P_1 P_2$ which belong to $S_2$. As $P_1 P_2$ is a connected point-set either $M_1$ contains a limit point of $M_2$ or $M_2$ contains a limit point of $M_1$.

**Case I.** A point $F$ of $M_1$ is a limit point of $M_2$. As $F$ is a point of the domain $S_1$, there exists a region containing $F$ and lying entirely in $S_1$. As $S_1$ and $S_2$ are mutually exclusive domains, this region contains no point of $M_2$. Hence $F$ cannot be a limit point of $M_2$.

**Case II.** A point $G$ of $M_2$ is a limit point of $M_1$. This is impossible as in Case I.

Hence we are led to a contradiction if we suppose our lemma false.

**Lemma B.** The set $K$ is connected.

**Proof.** Suppose $K$ were not connected. Then it could be divided into two mutually exclusive sets $K_1$ and $K_2$, neither of which contains a limit point of the other one. Let $P_i (i = 1, 2)$ denote a point of $K_i$. Put about $P_i$ a circle $R_i$ having $P_i$ as center and such that $R_i$ and its interior lie entirely

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* Cf. Hans Hahn, *Ueber die allgemeinste ebene Punktmenge die stetiges Bild einer Strecke ist*, Jahresbericht der Deutschen Mathematiker Vereinigung, vol. 23 (1914), pp. 318–22. According to Hahn, a set of points $C$ is said to be connected in kleinem if, whenever $P$ is a point of $C$, $ε$ a positive number and $K$ a circle of radius $1/ε$ with center at $P$, then there exists within $K$ and with center at $P$, another circle $K_ε, P$ such that if $X$ is a point of $C$ within $K_ε, P$ then $X$ and $P$ lie together in some connected subset of $C$ that lies entirely within $K$.

† A domain is a connected set of points $M$ such that if $P$ is a point of $M$, then there is a region that contains $P$ and lies in $M$.

‡ An open curve is defined by R. L. Moore as a closed, connected, set of points $M$ such that if $P$ is a point of $M$, then $M - P$ is the sum of two mutually exclusive connected point-sets, neither of which contains a limit point of the other.
without $R_{i+1}$. As $K$ is connected in kleinem, there exists a circle $\tilde{R}$, lying within $R_i$ and with center at $P_i$ such that if $X_i$ is a point of $K$ within $\tilde{R}$, then $X_i$ and $P_i$ lie on some connected subset of $K$ lying within $R_i$. It may easily be shown that $X_i$ can be joined to $P_i$ by a simple continuous arc of $K$ lying entirely within $R_i$. As every point of $K$ is a common boundary point of $S_1$ and $S_2$, then there exists within $\tilde{R}_i$ a point $M_{ij}$ ($j = 1, 2$) belonging to $S_j$. As $S_j$ is a domain, then there exists a simple continuous arc $M_{ij}, K_j M_{ij}$ lying entirely in $S_j$. Join $M_{ij}$ to $P_i$ by a simple continuous arc $M_{ij} L_{ij} P_i$ lying entirely within $\tilde{R}_i$ and let $G_{ij}$ denote the first point of $K$ on the arc $M_{ij} L_{ij} P_i$ following $M_{ij}$. Then we may join $G_{i1}$ to $G_{i2}$ by an arc $G_{i1} F_i G_{i2}$ belonging to $K$ and lying entirely within $R_i$. The point-set $G_{i1} N_{i1}$ (on $M_{i1} L_{i1} P_1$) + $M_{i1} K_{i1} M_{i1} + M_{i1} G_{i1}$ (on $M_{i1} L_{i1} P_2$) contains a subset a simple continuous arc $G_{i1} H_{i1} G_{i1}$ lying except for its endpoints entirely in $S_1$, while the set $G_{i2} N_{i2}$ (on $M_{i2} L_{i2} P_1$) + $M_{i2} K_{i2} M_{i2} + M_{i2} G_{i2}$ (on $M_{i2} L_{i2} P_2$) contains a subset a simple continuous arc $G_{i2} H_{i2} G_{i2}$ lying except for its endpoints entirely in $S_2$. We then have a closed curve $G_{i1} F_i G_{i2} H_{i2} G_{i2} - F_i G_{i1} H_{i1} G_{i1}$ such that the arcs $G_{i1} F_i G_{i2}$ and $G_{i1} F_i G_{i2}$ lie entirely on $K$ and within $R_1$ and $R_2$, respectively, while $G_{i1} H_{i1} G_{i1} + G_{i2} H_{i2} G_{i2}$ belong to $S_1$ and $S_2$, respectively.

All points of $G_{i1} F_i G_{i2}$ belong to $K_1$. For suppose a point $H$ of $G_{i1} F_i G_{i2}$ belonged to $K_2$. As $H$ is joined to $G_{i1}$, which in turn can be joined to $P_1$ by an arc of $K$ lying entirely within $R_1$, it follows that $H$ can be joined to $P_1$ by an arc $HFP_1$ of $K$ lying entirely within $R_1$. Let $[\tilde{H}]$ denote the set of all points of $HFP_1$ belonging to $K_1$ while $[\tilde{H}_2]$ denotes the set of all points of $HFP_1$ belonging to $K_2$. Clearly neither of these sets contains a limit point of the other. Hence the arc $HFP_1$ is not a connected point-set. Hence the supposition that $H$ belongs to $K_2$ has led to a contradiction. In like manner, all points of $G_{i2} F_i G_{i2}$ belong to $K_2$.

The interior of $G_{i1} F_i G_{i2} H_{i2} G_{i2} F_i G_{i2} H_{i1} G_{i1}$ must contain at least one point of $K$. For suppose it does not contain a point of $K$. Then the interior of $G_{i1} F_i G_{i2} H_{i2} G_{i2} F_i G_{i2} H_{i1} G_{i1}$ is a subset of $S_1 + S_2$. Suppose it contains a point $H$ of $S_1$. Then $H$ can be joined to $H_2$ by an arc $HXH_2$ lying except for $H_2$ entirely within $G_{i1} G_{i2} H_{i2} G_{i2} F_i G_{i2} H_{i1} G_{i1}$. Let $[W]$ denote the set of all points of $HXH_2$ belonging to $S_1$ while $[W_2]$ denotes the set of all points of $HXH_2$ which are points of $S_2$. Clearly neither of these sets contains a limit point of the other. Hence the arc $HXH_2$ is not a connected point-set. Hence the supposition that $H$ belongs to $K_2$ has led to a contradiction. In like manner, all points of $G_{i2} F_i G_{i2}$ belong to $K_2$.

* It is understood that subscripts are reduced modulo 2.

† Cf. R. L. Moore, A theorem concerning continuous curves, Bulletin of the American Mathematical Society, vol. 23 (1917). While Professor Moore's theorem states that every two points of a continuous curve can be joined by a simple continuous arc lying entirely on the given continuous curve, it is clear that his methods suffice to prove the above stronger statement.

‡ If $AXB$ is an arc, then the symbol $AXB$ will denote $AXB - A - B$.


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a limit point of the other. Hence the arc $HXH_2$ is not a connected point-set. In like manner the supposition that there is within $G_{11} F_1 G_{12} H_2 G_{22} F_2 - G_{21} H_1 G_{11}$, a point of $S_2$ leads to a contradiction. Hence $G_{11} F_1 G_{12} H_2 G_{22} F_2 - G_{21} H_1 G_{11}$ must enclose a point of $K$.

Let $[V_2]$ denote the set of all points $V_2$ such that either (1) $V_2$ is a point of $G_{21} F_2 G_{22}$, or (2) $V_2$ is a point such that there exists a closed connected set $V_2 X F_2$ belonging to $K$ and lying within or on $G_{11} F_1 G_{12} H_2 G_{22} F_2 G_{21} H_1 G_{11}$ and such that $F_2$ is a point of $G_{21} F_2 G_{22}$. As $K$ is connected in kleinem it may easily be proved that $[V_2]$ is a closed set. It is also true that all points of $[V_2]$ belong to $K_2$. Hence no point of $G_{11} F_1 G_{12}$ either belongs to or is a limit point of $[V_2]$. It may also be proved with the use of the in kleinem property that no point of $[V_2]$ is a limit point of a set of points of $K$ lying within $G_{11} F_1 G_{12} H_2 G_{22} F_2 G_{21} H_1 G_{11}$ and containing no point of $V_2$. There exists an arc $H_1 Y H_2$ such that (1) $H_1 Y H_2$ is a subset of the interior of $G_{11} F_1 G_{12} H_2 G_{22} F_2 G_{21} H_1 G_{11}$ and (2) $H_1 Y H_2$ contains no points of $[V_2]$.*

Let $[V_1]$ denote the set of all points of $K$ within or on the closed curve, $H_1 Y H_2 G_{22} F_2 G_{21} H_1$, not belonging to $[V_2]$. The set $[V_1]$ is closed. Put about each point of $[V_1]$ a circle lying entirely within $G_{11} F_1 G_{12} H_2 G_{22} F_2 - G_{21} H_1 G_{11}$ and containing within it or on its boundary no point of $[V_2]$. By the Heine-Borel Property, there exists a finite number of circles of the above set, $C_1, C_2, \cdots, C_n$, covering $[V_1]$. With the use of Theorems 41, 42, 43, and 44 of Professor Moore's Foundations we may easily obtain from the set $C_1, C_2, \cdots, C_n$ and the closed curve $G_{11} F_1 G_{12} H_2 Y H_1 G_{11}$, a new closed curve $G_{11} F_1 G_{12} H_2 Z H_1 G_{11}$, where the arc $H_1 G_{11} F_1 G_{12} H_2$ of the new closed curve $G_{11} F_1 G_{12} H_2 Z H_1 G_{11}$ is the arc $H_1 G_{11} F_1 G_{12} H_2$ of $G_{11} F_1 G_{12} H_2 Y H_1 G_{11}$ and where $H_2 Z H_1$ is free from points of $K$ and lies within $G_{11} F_1 G_{12} H_2 G_{22} F_2 - G_{21} H_1 G_{11}$. But then we have a point of $S_1$ joined to a point of $S_2$ by an arc containing no point of $K$. Thus the supposition that $K$ is not connected, leads to a contradiction.

**LEMMA C.** If $K$ contains one simple closed curve $J$, then all points of $K$ belong to $J$.

**Proof.** Suppose Lemma C is not true. Then $K$ contains a closed curve $J$ and at least one point $P$ not on $J$. Two cases may arise:

**Case I.** $P$ is within $J$. As every point of $K$ is a common boundary point of $S_1$ and $S_2$, the interior of $J$ contains a point $P_1$ of $S_1$ and a point $P_2$ of $S_2$. The exterior of $J$ cannot contain a point $\overline{P}_1$ of $S_1$. For suppose it did. Then any arc from $P_1$ to $\overline{P}_1$ would contain a point of $J$ and hence a point of $K$, contrary to the fact that $S_1$ is a domain. In like manner no point of $S_2$ can be in the exterior of $J$. Hence the exterior of $J$ must be a subset of $K$, while

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$S_1$ and $S_2$ are subsets of the interior of $J$. But this is impossible because no point without $J$ is a limit point of a set of points lying entirely within $J$ thus making it impossible that every point of $K$ be a common boundary point of $S_1$ and $S_2$. Hence the supposition that $P$ is within $J$ has led to a contradiction.

Case II. $P$ is without $J$. Case II may be proved impossible by an argument similar to that used in Case I.

An immediate consequence of Lemma C is that if $K$ is not a simple closed curve, then there is but one $K$-arc from a point $A$ of $K$ to a distinct point $B$ of $K$.

**Lemma D.** The set $K$ does not contain three arcs $OP_1$, $OP_2$, and $OP_3$, no two of which have a common point other than $O$.

**Proof.** Suppose Lemma D were false. Then there would exist three arcs $OP_1$, $OP_2$, and $OP_3$, no two of which have a point in common other than $O$. Put about $P_i$ ($i = 1, 2, 3$) a circle $C_i$ such that the point-set $OP_{i+1} + OP_{i+2}$ is a subset of the exterior of $C_i$ and such that $C_i$ has no point in common with $C_{i+1} + C_{i+2}$. As $K$ is connected in kleinem, there exists within $C_i$ and with center at $P_i$, another circle $C_{P_i, C_i}$ such that if $X_i$ is a point of $K$ within $C_{P_i, C_i}$, then there is an arc from $X_i$ to $P_i$ every point of which is a point of $K$ and which lies entirely within $C_i$.† As all points of $K$ are limit points of both $S_1$ and $S_2$, $C_{P_i, C_i}$ must contain at least one point $P_{i, 1}$ of $S_1$. As $S_1$ is a domain, there is an arc $P_{i, 1}P_{i, 2}$ from $P_{i, 1}$ to $P_{i, 2}$ all points of which belong to $S_1$. Join $P_{i, 1}$ to $P_i$ by an arc $P_{i, 1}P_i$ lying entirely within $C_{P_i, C_i}$ and let $X_i$ denote the first point of the arc $P_{i, 1}P_i$ after $P_{i, 1}$, which belongs to $K$. There exists an arc $X_iP_i$ from $X_i$ to $P_i$ belonging to $K$ and lying entirely within $C_i$. Let $P_i$ denote the first point of the arc $X_iP_i$ which is on $OP_i$. The point-set $P_1X_1 + X_1P_{11} + P_{11}P_{21} + P_{21}X_2 + X_2P_2$ contains as a subset an arc $P_1F_1P_2$ such that (1) $P_1F_1P_2$ has no point in common with $OP_1 + OP_2 + OP_3$, (2) all points of $P_1F_1P_2$ belong to either $K$ or $S_1$, (3) at least one point, $F_1$, of $S_1$ is a point of $P_1F_1P_2$. By methods similar to those just employed, we may construct an arc $Q'_1H_2Q'_2$ from a point $Q'_1$ of $OP_1$ to a point $Q'_2$ of $OP_2$ such that (1) $Q'_1$ is on $OP_1$ between $P_1$ and $O$, (2) all points of $Q'_1H_2Q'_2$ belong to either $S_2$ or $K$, (3) except for $Q'_1$ and $Q'_2$, $Q'_1H_2Q'_2$ has no point in common with $P'_1FP'_2 + OP_1 + OP_2 + OP_3$, (4) at least one point $H_2$ of $Q'_1H_2Q'_2$ belongs to $S_2$. Two cases may arise:

Case I. $Q'_1H_2Q'_2$ is entirely within $OP_1F_1P_2O$. Then the interior of $OP_1F_1P_2O = Q'_1H_2Q'_2 +$ the interior of $Q'_1H_2Q'_2 +$ the interior of $P'_1F_1P_2Q'_2H_2Q'_1P'$. The point-set $OP_3 + P_3$ is either entirely within or entirely without $Q'_1H_2Q'_2O$.

(a) Suppose $OP_3 + P_3$ is entirely within $Q'_1H_2Q'_2O$. Then $Q'_1H_2Q'_2O$

* It is understood throughout this argument that subscripts are reduced modulo 3.
† See an earlier footnote.

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must enclose at least one point $L$ of $S_1$. But then an arc from $L$ to $F_1$ must contain at least one point of $OQ_1' H_2 Q_2' O$. Hence, as $OQ_1' H_2 Q_2' O$ is a subset of $K + S_2$, no such arc $LF_1$ can lie entirely in $S_1$, contrary to the fact that $S_1$ is a domain.

(b) Suppose $OP_3 + P_3$ is entirely without $OQ_1' H_2 Q_2' O$. It follows that $OP_3 + P_3$ is entirely without $OP_1' F_1 P_2' O$. Then the exterior of $OP_1' F_1 P_2' O$ contains at least one point $M$ of $S_2$. Then any arc from $M$ to $H_2$ must contain at least one point of $OP_1' F_1 P_2' O$ and hence at least one point not in $S_2$. But this is contrary to the fact that $S_2$ is a domain.

Thus in Case I we are led to a contradiction.

Case II. $Q_1' H_2 Q_2'$ is without $OP_1' F_1 P_2' O$. We may show that Case II is impossible by methods similar to those used in Case I.

Lemma E. If $O$ is a point of $K$ and $P$ is a point of $S_i$ ($i = 1, 2$) then there exists at least one arc $OP$ such that $OP + P$ is a subset of $S_i$.

Proof. Two conceivable cases may arise.

Case I. There exist points $A_1$ and $A_2$ of $K$ [$A_1 \neq O \neq A_2$] such that $O$ is a point of the arc $A_1 O A_2$ belonging to $K$. By the same methods as were used in the preceding lemma we may construct an arc $A_1' F_1 A_2'$ such that (1) on $A_1 O A_2$ the order $A_1 A_1' O A_2'$ holds, (2) $A_1' F_1 A_2'$ is a subset of $S_1 + K$, (3) at least one point $F_1$ of $A_1' F_1 A_2'$ is a point of $S_1$, (4) no point of $A_1' F_1 A_2'$ belongs to $A_1 O A_2$. The point $O$ is not a limit point of $K - A_1' O A_2'$. For suppose it were. Then it would be a sequential limit point of a set of points $P_1, P_2, \ldots$, every one of which belongs to $K - A_1' O A_2'$. Put about $O$ as center a circle $M$ such that $A_1' O A_2'$ are both without $M$. As $K$ is connected in kleinem there exists another circle $\overline{M}$ lying within $M$ and having its center at $O$ such that if $X$ is a point of $K$ within $\overline{M}$, then $X$ and $O$ can be joined by an arc of $K$ lying entirely within $\overline{M}$. Let $\overline{P}$ denote that point of the set $P_1, P_2, \ldots$ of lowest subscript which lies within $\overline{M}$, while $PO$ denotes an arc of $K$ from $\overline{P}$ to $O$ lying entirely within $\overline{M}$. Let $O'$ denote the first point of $\overline{PO}$ which is on $A_1' O A_2'$. Then the set $K$ contains three arcs $A_1' O' A_1', A_1' O' A_2'$, and $A_1' O A_2'$, no two of which have a point in common other than $O'$. But this is contrary to Lemma D. Hence $O$ cannot be a limit point of $K - A_1' O A_2'$. There exists a closed curve $G$ enclosing $O$ but enclosing no points of $A_1' F_1 A_2' + [K - A_1' O A_2']$. Then there exist two closed curves $J_1' + J_2'$ such that (1) every point of $J_1'$ or $J_2'$ belongs either to $G$ or to $A_1' F_1 A_2' O A_1'$ (2) $O$ is on $J_1'$ and on $J_2'$ (3) every point within $J_1'$ is within $A_1' F_1 A_2' O A_1'$ while every point within $J_2'$ is without $A_1' F_1 A_2' O A_1'$ (4) every point within either $J_1'$ or $J_2'$ is within $G$.* It is clear that either the interior of $J_1'$ or the interior of $J_2'$ is a subset of $S_1$ while the interior of the other of these two closed curves is a subset of $S_2$. Let $J_1$ denote that one whose interior is a subset of $S_1$ while

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J₂ denotes the one whose interior is a subset of S₂. Let E denote a point within J₁, while P₁ is any other point of S₁. There exists an arc EO such that EO - 0 is a subset of the interior of J₁. As S₁ is a domain, there is an arc EP₁ lying entirely in S₁. The point-set EO + EP₁ contains as a subset an arc from P₁ to 0 lying except for 0 entirely in S₁. In like manner we may show that any point P₂ of S₂ can be joined to 0 by an arc lying except for 0 entirely in S₂.

Case II. There do not exist two distinct points A₁ and A₂ of K such that 0 is on an arc of K from A₁ to A₂. Let A denote a point of K different from 0 while AR₀ denotes an arc of K from A to 0. By an argument similar to that employed in Case I we may show that if 0 were a limit point of K - AR₀, then either there would exist three arcs AR', R'₀, and R'₁, no two of which have a point in common other than R' or there would exist a point A', (A ≠ A' ≠ 0) such that 0 is an arc of K from A' to A. But the first of these possibilities contradicts Lemma D while the second is contrary to the hypothesis of Case II. Hence 0 cannot be a limit point of K - AR₀. Put about 0 a circle C that neither contains or encloses any point of K - AR₀. Let P₁, P₂, ⋯ denote a set of points of S₁ approaching 0 as their sequential limit point. It is possible to pass at least one simple continuous arc through AR₀ + P₁ + P₂ + ⋯. Let P₁ OR₀ denote one such arc. If the interval OP₁ of the arc P₁ OR₀ does not lie entirely within C, let P' denote the first point which it has in common with C. Otherwise let P' denote P₁. Let P denote that point of the set P₁, P₂, ⋯ of lowest subscript lying on OP'. It is clear that the sub-arc OP of P₁ OR₀ lies, except for 0, entirely in S₁.

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The condition is sufficient. For suppose $S_1$ is bounded while $K$ is unbounded. Since $S_1$ is bounded, there exists a circle $C$ enclosing $S_1$. Since $K$ is unbounded, it contains a point $P$ without $C$. The point $P$ cannot be a limit point of $S_1$. But this is contrary to hypothesis.

Proof of Theorem A. Two cases may arise:

Case I. $K$ is bounded. Then, by Schoenflies’ Theorem and the preceding lemmas, it follows that $K$ is a simple closed curve.

Case II. $K$ is unbounded. It follows, by Lemma $F$ that neither $S_1$ nor $S_2$ is bounded. Then $K$ is an open curve. For a proof of this statement see my paper, “The converse of the theorem concerning the division of a plane by an open curve.”

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* Cf. these Transactions, vol. 18 (1917), pp. 177–184.