OSCILLATION THEOREMS FOR THE REAL, SELF-ADJOINT
LINEAR SYSTEM OF THE SECOND ORDER*

BY

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INTRODUCTION

It is the object of this paper to determine the number of oscillations of a linear combination of the form (2) for the systems (3) and (4). From these results, an oscillation theorem for the solution $u_p(x)$, corresponding to the $p$th characteristic number of (4), is obtained.

Given the second order self-adjoint linear differential equation

$$\frac{d}{dx} \left[ K(x, \lambda) \frac{du}{dx} \right] - G(x, \lambda) u = 0 \quad (1)$$

and two linear combinations of a solution which does not vanish identically and its first derivative,

$$L_i[u(x, \lambda)] = \alpha_i(x, \lambda) u(x, \lambda) - \beta_i(x, \lambda) K(x, \lambda) u_x(x, \lambda) \quad (i = 1, 2), \quad (2)$$

we shall impose the following conditions and shall assume that they are satisfied throughout this paper:

I. $K(x, \lambda), G(x, \lambda), \alpha_i(x, \lambda), \beta_i(x, \lambda)$ are continuous, real functions of $x$ in the interval 

$$(X) \quad (a \leq x \leq b)$$

and for all real values of $\lambda$ in the interval

$$(\Lambda) \quad (\lambda_1 < \lambda < \lambda_2).$$

II. $K(x, \lambda)$ is positive everywhere in $(X, \Lambda)$ and

$$|\alpha_i| + |\beta_i| > 0$$

in $(X, \Lambda)$.

III. For each value of $x$ in $(X)$, $K$ and $G$ decrease (or do not increase)

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* Presented to the Society, Sept. 6, 1917.

† $f_{ix}(x, \lambda) = \frac{\partial}{\partial x} f_i(x, \lambda)$.
as λ increases. In no sub-interval of (X) are K and G simultaneously independent of λ and in no sub-interval of (X) is G identically zero.

IV. Either β_i = 0 for all values of x and λ in (X, A); or else β_i ≠ 0 in a < x < b for all λ's in (A) and one of the following is true,

either (a) β_i(a) ≠ 0 for all λ's in (A), β_i(b) ≠ 0, - α_i(b)/β_i(b) decreases (or does not increase) as λ increases;

or (b) β_i(b) = 0, β_i(a) ≠ 0 for all λ's in (A), α_i(a)/β_i(a) decreases (or does not increase) as λ increases;

or (c) β_i(a) ≠ 0, β_i(b) = 0, α_i(a)/β_i(a) and - α_i(b)/β_i(b) decrease (or do not increase) as λ increases.

V.*

\[
\lim_{\lambda \to \ell_1} - \frac{\min G}{\min K} = -\infty, \\
\lim_{\lambda \to \ell_2} - \frac{\max G}{\max K} = +\infty.
\]

I. The Sturmian system

Concerning the system

\[
\frac{d}{dx} \left[ K(x, \lambda) \frac{du}{dx} \right] - G(x, \lambda) u = 0, \\
L_1[u(a, \lambda)] = 0, \quad L_1[u(b, \lambda)] = 0,
\]

Sturm's oscillation theorem† may be stated with Bôcher‡ substantially as follows:

The system (3) satisfying conditions I–V has an infinite set of characteristic numbers such that

\[ \lambda_1 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_2 \]

and \( U(x, \lambda_p) \), the pth characteristic function, vanishes exactly p times on a < x < b.

We seek to determine the number of oscillations of \( L_1[U(x, \lambda_p)] \). We notice first that if \( \beta_1 = 0 \) for all values of x and λ in (X, A), then all the zeros of \( U(x, \lambda_p) \) and \( L_1[U(x, \lambda_p)] \) coincide, since \( \alpha_1(x, \lambda) = 0 \). Hence we have for \( \beta_1 = 0 \) precisely p zeros of \( L_1[U(x, \lambda_p)] \) on a < x < b.

With Bôcher§ we define

\[ \{ \alpha_1 \beta_1 \} = \beta_1 \alpha_{1x} - \alpha_1 \beta_{1x} + \frac{\alpha_{1}^2}{K} - \beta_{1}^2 G. \]

* This condition may be replaced by other sets of conditions. See Bôcher, Leçons sur les Méthodes de Sturm (hereafter referred to as Leçons) (1917), chap. III, paragraphs 13–15.
‡ Leçons, p. 63 ff.
§ Leçons, p. 45.
Let $\beta_1 \neq 0$ in $(X, \Lambda)$, and consider the zeros of $U(x, \lambda)$ and $L_1[U(x, \lambda)]$. For $\lambda = \lambda_0$, $L_1[U(a, \lambda_0)] = 0$ and $L_1[U(b, \lambda_0)] = 0$, but $U(x, \lambda_0)$ does not vanish on $(X)$. Hence $\{\alpha_1 \beta_1\}_{\lambda=\lambda_0} < 0$ must vanish for some value of $x$ in $(X)$, since if $\{\alpha_1 \beta_1\}_{\lambda=\lambda_0} < 0$ throughout $(X)$, $L_1[U(x, \lambda_0)]$ could vanish but once in $(X)$;* and if $\{\alpha_1 \beta_1\}_{\lambda=\lambda_0} > 0$ for every $x$ in $(X)$, $U(x, \lambda_0)$ would vanish once in $(X)$.† If $\{\alpha_1 \beta_1\} > 0$ for $\lambda \equiv \lambda_1$ for all values of $x$ in $(X)$, then the zeros of $L_1[U(x, \lambda)]$ and $U(x, \lambda)$ separate.† But $U(x, \lambda_p)$ vanishes exactly $p$ times on $a < x < b$. Hence $L_1[U(x, \lambda_p)]$ vanishes $p - 1$ times on $a < x < b$ for $p \geq 1$.

If $\beta_1(a, \lambda) = 0$ in $(\Lambda)$, $\beta_1(x, \lambda) \neq 0$ for $a < x \leq b$, then $\alpha(a) \neq 0$ in $(\Lambda)$. Then the zeros of $L_1[U(x, \lambda_p)]$ and $U(x, \lambda_p)$ separate† each other on $a < x \leq b$, provided $\{\alpha_1 \beta_1\}_{\lambda=\lambda_p} > 0$. Let $x_1$ be the first zero of $U(x, \lambda_p)$ on $a < x \leq b$. Then

$$L_1[U(x_1, \lambda_p)] = -\beta_1(x_1, \lambda_p) K(x_1, \lambda_p) U_x(x_1, \lambda_p).$$

Now direct computation shows that

$$L_{1x}[U(a)] = \frac{K(a) U_x(a)}{\alpha_1(a)}. $$

But $K(a) U_x(a)$ and $K(x_1, \lambda_p) U_x(x_1, \lambda_p)$ have opposite signs, and $\text{sgn} \beta_1(x_1, \lambda_p) = \text{sgn} \beta_1(b)$.‡ Hence

$$\text{sgn} L_1[U(x_1, \lambda_p)] = \text{sgn} L_{1x}[U(a)] \cdot \text{sgn} \beta_1(b) \text{ sgn} \alpha_1(a).$$

Thus $L_1[U(x, \lambda_p)]$ vanishes or does not vanish in $a < x < x_1$ according as $\alpha_1(a) \beta_1(b)$ is negative or is positive respectively. Hence if $\alpha_1(a) \beta_1(b) > 0$, $L_1[U(x, \lambda_p)]$ has $p - 1$ zeros on $a < x < b$, and if $\alpha_1(a) \beta_1(b) < 0$, $L_1[U(x, \lambda_p)]$ has $p$ zeros on $a < x < b$, for $p \geq 1$.

If $\beta_1(b, \lambda) = 0$ in $(\Lambda)$, $\beta_1(x, \lambda) \neq 0$ for $a \leq x < b$, then a similar argument can be made with the following result: if $\alpha_1(b) \beta_1(a) > 0$, $L_1[U(x, \lambda_p)]$ has $p - 1$ zeros on $a < x < b$, and if $\alpha_1(b) \beta_1(a) < 0$, $L_1[U(x, \lambda_p)]$ has $p$ zeros on $a < x < b$, for $p \geq 1$. Therefore

**Oscillation Theorem I.** If $U(x, \lambda_p)$ is the $p$th characteristic function of the system (3) satisfying conditions I-V and if $\{\alpha_1 \beta_1\} > 0$ for $\lambda \equiv \lambda_1$ for every $x$ in $(X)$, then $L_1[U(x, \lambda_p)]$ will vanish on $a < x < b$ for $p \geq 1$,

$p$ times if

- either $\beta_1 = 0$ in $(X, \Lambda)$;
- or $\beta_1(a) = 0$ in $(\Lambda)$, $\beta_1(x, \lambda) \neq 0$ in $a < x \leq b$ and $\alpha_1(a) \beta_1(b) < 0$;
- or $\beta_1(b) = 0$ in $(\Lambda)$, $\beta_1(x, \lambda) \neq 0$ in $a \leq x < b$ and $\alpha_1(b) \beta_1(a) < 0$;

$p - 1$ times if

* Böcher, Leçons, p. 51.
† Ibid., p. 50.
‡ sgn $f = \text{sign} f$. 
either $\beta_1 \neq 0$ in $(X, \Lambda)$;  
or $\beta_1(a) = 0$ in $(\Lambda), \beta_1(x, \lambda) \neq 0$ in $a < x \leq b$ and $\alpha_1(a) \beta_1(b) > 0$;  
or $\beta_1(b) = 0$ in $(\Lambda), \beta_1(x, \lambda) \neq 0$ in $a \leq x < b$ and $\alpha_1(b) \beta_1(a) > 0$.

Let $L[u(x, \lambda)] = \alpha(x, \lambda) u(x, \lambda) - \beta(x, \lambda) K(x, \lambda) u_x(x, \lambda)$, where $\alpha, \beta, \alpha_x, \beta_x$ are continuous in $(X, \Lambda)$. Then we may state

Oscillation theorem II. If $U(x, \lambda_p)$ is the $p$th characteristic function of $(3)$ satisfying conditions I-V, where $\{\alpha_1 \beta_1\} > 0, \{\alpha \beta\} > 0$ for $\lambda \equiv \lambda_1$ and $(\alpha_1 \beta - \alpha \beta_1) \neq 0$ in $(X, \Lambda)$, then $L[U(x, \lambda_p)]$ will vanish on $a < x < b$, for $p \geq 1$,
p times if

either $\beta_1 \neq 0$ in $(X, \Lambda)$;  
or $\beta_1(a) = 0$ in $(\Lambda), \beta_1(x, \lambda) \neq 0$ in $a < x \leq b$ and $\alpha_1(a) \beta_1(b) > 0$;  
or $\beta_1(b) = 0$ in $(\Lambda), \beta_1(x, \lambda) \neq 0$ in $a \leq x < b$ and $\alpha_1(b) \beta_1(a) > 0$;  
p + 1 times if

either $\beta_1 = 0$ in $(X, \Lambda)$;  
or $\beta_1(a) = 0$ in $(\Lambda), \beta_1(x, \lambda) \neq 0$ in $a < x \leq b$ and $\alpha_1(a) \beta_1(b) < 0$;  
or $\beta_1(b) = 0$ in $(\Lambda), \beta_1(x, \lambda) \neq 0$ in $a \leq x < b$ and $\alpha_1(b) \beta_1(a) < 0$.

Proof: The zeros of $L_1[U(x, \lambda_p)]$ and $L_2[U(x, \lambda_p)]$ separate one another on $(X)$. But $L_1[U(a, \lambda_p)] = 0$ and $L_1[U(b, \lambda_p)] = 0$. Hence $L[U(x, \lambda_p)]$ has one more zero on $a < x < b$, for $p \geq 1$, than $L_1[U(x, \lambda_p)]$. This proves Theorem II.

II. THE GENERAL SELF-ADJOINT SYSTEM

Consider the system

\[
\frac{d}{dx} \left[ K(x, \lambda) \frac{du}{dx} \right] - G(x, \lambda) u = 0,
\]

$L_1[u(a, \lambda)] = L_1[u(b, \lambda)], \quad L_2[u(a, \lambda)] = L_2[u(b, \lambda)],$

where $L_1$ and $L_2$ are defined as in (2) and $\beta_2 \neq 0$† in $(X, \Lambda)$. We impose further conditions:

VI. $\{\alpha_i \beta_i\} \neq 0$ for $\lambda \equiv \lambda_1$,

VII. $\alpha_1 \beta_2 - \alpha_2 \beta_1 = -1$ in $(X, \Lambda)$,

VIII.‡

\[
\begin{bmatrix}
\alpha_1(a) & \beta_1(a) & \alpha_1(b) & \beta_1(b) \\
\alpha_2(a) & \beta_2(a) & \alpha_2(b) & \beta_2(b) \\
\Delta \alpha_1(a) & \Delta \beta_1(a) & \Delta \alpha_1(b) & \Delta \beta_1(b) \\
\Delta \alpha_2(a) & \Delta \beta_2(a) & \Delta \alpha_2(b) & \Delta \beta_2(b)
\end{bmatrix} \equiv 0.
\]

*Bocher, Leçons, p. 50.

†This restriction does not involve a loss of generality, since if $\beta_1 \neq 0, \beta_2 = 0$ then we may interchange $L_1$ and $L_2$.

‡ $\Delta f = f(\lambda + \Delta \lambda) - f(\lambda)$ for $\Delta \lambda > 0$.  

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In a recent paper* the writer proved the following theorem concerning the system (4), satisfying conditions I-VIII:

There exists one and only one characteristic number of (4) between every pair of characteristic numbers of the Sturmian system (3). If \( \lambda_p \) represents the ordered characteristic numbers of (3) and \( l_p \) those of (4) (account being taken of their multiplicity) then

Case I. \( l_p \) is in the interval \((\lambda_p, \lambda_{p+1})\) if \( L_2[U(b, \lambda_0)] \cdot \phi(L_1 + \epsilon) > 0 \).

Case II. \( l_p \) is in the interval \((\lambda_{p-1}, \lambda_p)\), \( p \geq 1 \), if \( L_2[U(b, \lambda_0)] \cdot \phi(L_1 + \epsilon) < 0 \).

Let \( u_p(x) = u(x, l_p) \) be the \( p \)th characteristic function of (4). We proceed to consider the number of oscillations of \( L_1[u_p(x)] \). We notice first that if \( \lambda = l_p \) is a double characteristic number, then \( l_p \) coincides with \( \lambda_{p-1} \), \( \lambda_p \), or \( \lambda_{p+1} \), and \( L_1[u_p(x)] \) will have the number of oscillations designated by Theorem I.

If \( \lambda = l_p \) is a simple value, we may discriminate between Case I and Case II, following a method due to Birkhoff.† If \( \beta_1 = 0 \) in \((X, \Lambda)\), the sign of \( L_2[U(b, \lambda_0)] \) is the same as \(-\alpha_1(a) \alpha_1(b)\), which is negative, since \( \alpha_1 \neq 0 \) in \((X, \Lambda)\). Hence \( L_2[U(b, \lambda_0)] \) is negative. Also \( \phi(L_1 + \epsilon) \) has the sign of \(-\beta_2(a) \beta_2(b)\), but \( \beta_2 \neq 0 \) in \((X, \Lambda)\). Hence \( \phi(L_1 + \epsilon) \) is negative, and we have Case I where \( l_p \) is on the interval \( \lambda_p < \lambda < \lambda_{p+1} \). By Theorem I, \( L_1[U(x, \lambda_p)] \) vanishes exactly \( p \) times on \( a < x < b \) for \( p \geq 1 \), and \( L_1[U(x, \lambda_{p+1})] \) vanishes \( p + 1 \) times on \( a < x < b \) for \( p \geq 1 \). But \( L_1[U(b, \lambda)] \neq 0 \) for \( \lambda_p < \lambda < \lambda_{p+1} \). Hence \( L_1[U(x, \lambda)] \) vanishes \( p + 1 \) times on \( a < x < b \) for \( \lambda_p < \lambda < \lambda_{p+1} \). But the roots of \( L_1[U(x, \lambda)] \) and \( L_1[u(x, \lambda)] \)§ separate one another, and \( L_1[U(a, \lambda)] = 0 \). Hence \( L_1[u(x, \lambda)] \) vanishes \( p + 1 \) or \( p + 2 \) times on \( a < x < b \) for \( \lambda_p < \lambda < \lambda_{p+1} \), \( p \geq 1 \). Hence \( L_1[u_p(x)] \) vanishes either \( p + 1 \) or \( p + 2 \) times on \( a < x < b \), \( p \geq 1 \). But from (4) the number of zeros of \( L_1[u_p(x)] \) is always even. Therefore we have \( p + 2 \) roots if \( p \) is even and \( p + 1 \) roots for \( p \) odd, \( p \geq 1 \).

If \( \beta_1(a) = 0 \) in \((\Lambda)\) and \( \beta_1(b) \neq 0 \), the sign of \( L_2[U(b, \lambda_0)] \) is that of \( \alpha_1(a)/\beta_1(b) \), and \( \phi(L_1 + \epsilon) \) has the sign of \( \beta_1(b)/\alpha_1(a) \). Hence we have Case I where \( l_p \) is on the interval \( \lambda_p < \lambda < \lambda_{p+1} \). If \( \alpha_1(a) \beta_1(b) < 0 \), by Theorem I, \( L_1[U(x, \lambda_p)] \) vanishes \( p \) times on \( a < x < b \) and \( L_1[U(x, \lambda_{p+1})] \) vanishes \( p + 1 \) times on \( a < x < b \). Reasoning exactly as above, we find that \( L_1[u_p(x)] \) vanishes \( p + 2 \) times on \( a < x < b \) if \( p \) is even and \( p + 1 \)

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* Existence Theorems for the General, Real, Self-Adjoint Linear System of the Second Order, these Transactions, vol. 19 (1918), p. 94.

† \( \phi(\lambda) = 0 \) is the characteristic equation of (4) whose roots are the characteristic numbers, \( l_p \). \( L_1 + \epsilon \) is a value of \( \lambda \) in \((\Lambda)\) near \( L_1 \).

‡ Existence and Oscillation Theorem for a Certain Boundary Value Problem, these Transactions, vol. 10 (1909), pp. 259-270.

§ Bôcher, Leçons, p. 48.
times if $p$ is odd, $p \geq 1$. If $\alpha_1(a) \beta_1(b) > 0$, by Theorem I, $L_1[U(x, \lambda_p)]$ vanishes $p - 1$ times on $a < x < b$, and $L_1[U(x, \lambda_{p+1})]$ vanishes $p$ times on $a < x < b$. But $L_1[U(b, \lambda)] \neq 0$ for $\lambda_p < \lambda < \lambda_{p+1}$, and $L_1[U(a, \lambda)] = 0$. The roots of $L_1[U(x, \lambda)]$ and $L_1[u(x, \lambda)]$ separate one another. Hence $L_1[u_p(x)]$ vanishes either $p$ or $p + 1$ times on $a < x < b$, $p \geq 1$. But from (4) the number of zeros of $L_1[u_p(x)]$ is always even. Therefore we have $p$ roots if $p$ is even and $p + 1$ roots if $p$ is odd.

If $\beta_1(b) = 0$ in $(\Lambda)$ and $\beta_1(a) \neq 0$, the sign of $L_2[U(b, \lambda_0)]$ is that of $-\beta_1(a)/\alpha_1(b)$, and $\phi(\mathcal{L} + \epsilon)$ has the sign of $-\beta_1(a)/\alpha_1(b)$. Hence we have Case I again. If $\alpha_1(a) \beta_1(a) < 0$ we proceed as before and obtain $p + 2$ zeros of $L_1[u_p(x)]$ on $a < x < b$ if $p$ is even and $p + 1$ zeros if $p$ is odd, $p \geq 1$. If $\alpha_1(a) \beta_1(a) > 0$, we obtain $p$ zeros of $L_1[u_p(x)]$ on $a < x < b$ if $p$ is even and $p + 1$ zeros if $p$ is odd, $p \geq 1$.

If $\beta_1 \neq 0$ in $(X, \Lambda)$, the sign of $L_2[U(b, \lambda_0)]$ is that of $\beta_1(a)/\beta_1(b)$, which is positive. The sign of $\phi(\mathcal{L} + \epsilon)$ is that of $\beta_1(a) \beta_2(b) - \beta_2(a) \beta_1(b)$ if $\beta_1(a) \beta_2(b) - \beta_2(a) \beta_1(b)$ does not vanish. If

$$\beta_1(a) \beta_2(b) - \beta_2(a) \beta_1(b) = 0,$$

the sign of $\phi(\mathcal{L} + \epsilon)$ is that of $\beta_1(a) \beta_1(b)$, which is positive. Accordingly, if $\beta_1(a) \beta_2(b) - \beta_2(a) \beta_1(b) \geq 0$, we have Case I, and $l_p$ is on $\lambda_p < \lambda < \lambda_{p+1}$. By Theorem I, $L_1[U(x, \lambda_p)]$ vanishes $p - 1$ times, and, proceeding as above, we find that $L_1[u_p(x)]$ has $p$ zeros if $p$ is even and $p + 1$ zeros if $p$ is odd, $p \geq 1$. Likewise, if $\beta_1(a) \beta_2(b) - \beta_2(a) \beta_1(b) < 0$, $l_p$ is on $\lambda_{p-1} < \lambda < \lambda_p$, and $L[u_p(x)]$ will vanish $p$ times if $p$ is even and $p - 1$ times if $p$ is odd, $p \geq 1$.

Summarizing the various kinds of coefficients of the boundary conditions of (4) as follows:

$A$: $\beta_1 = 0$ in $(X, \Lambda)$;

$B^-$: Either $\beta_1(a) = 0$ in $(\Lambda)$, $\beta_1(x, \lambda) \neq 0$ in $a < x \leq b$, and $\alpha_1(a) \beta_1(b) < 0$;

or $\beta_1(b) = 0$ in $(\Lambda)$, $\beta_1(x, \lambda) \neq 0$ in $a \leq x < b$, and $\alpha_1(b) \beta_1(a) < 0$;

$B^+$: Either $\beta_1(a) = 0$ in $(\Lambda)$, $\beta_1(x, \lambda) \neq 0$ in $a < x \leq b$, and $\alpha_1(a) \beta_1(b) > 0$;

or $\beta_1(b) = 0$ in $(\Lambda)$, $\beta_1(x, \lambda) \neq 0$ in $a \leq x < b$, and $\alpha_1(b) \beta_1(a) > 0$;

$C^+$: $\beta_1 \neq 0$ in $(X, \Lambda)$ and $\beta_1(a) \beta_2(b) - \beta_2(a) \beta_1(b) \geq 0$ in $(\Lambda)$;

$C^-$: $\beta_1 \neq 0$ in $(X, \Lambda)$ and $\beta_1(a) \beta_2(b) - \beta_2(a) \beta_1(b) < 0$ in $(\Lambda)$;

we state

Oscillation Theorem III. If $u_p(x)$ is the $p$th characteristic function corresponding to a simple value of the system (4) satisfying conditions I–VIII,
then the number of zeros of \( L_1[u_p(x)] \) on \( a < x < b \) for \( p \geq 1 \) is given by the following table:

<table>
<thead>
<tr>
<th>Case</th>
<th>Number of zeros</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p = 2m )</td>
<td>( p + 2 )</td>
</tr>
<tr>
<td>( p = 2m + 1)</td>
<td>( p + 1 )</td>
</tr>
</tbody>
</table>

Furthermore, we notice that, since \( \alpha_1 \beta_2 - \alpha_2 \beta_1 \neq 0 \) in \((X, \Lambda)\), the zeros of \( L_1[u_p(x)] \) and \( L_2[u_p(x)] \) separate one another on \( X^* \), and, if \( \lambda = l_p \) is a double value, the number of zeros of \( L_2[u_p(x)] \) is given by Theorem II. If \( \lambda = l_p \) is a simple value, both \( L_1[u_p(x)] \) and \( L_2[u_p(x)] \) by (4) have an even number of zeros on \((X)\). Hence they oscillate the same number of times. Therefore

**Corollary.** The number of zeros of \( L_2[u_p(x)] \) on \( a < x < b \), for \( p \geq 1 \), is precisely the number given in the table of Theorem III.

From the foregoing results we may deduce the number of oscillations of \( u_p(x) \). If \( \lambda = l_p \) is a double value, then \( u_p(x) \) will differ from \( U(x, \lambda_{p-1}) \), \( U(x, \lambda_p) \), or \( U(x, \lambda_{p+1}) \) by at most a non-vanishing factor, and the number of zeros of \( u_p(x) \) will be given by the Sturmian Oscillation Theorem.

For a simple value we consider the following cases:

If \( \beta_1 = 0 \) in \((X, \Lambda)\) the zeros of \( u_p(x) \) and \( L_1[u_p(x)] \) coincide. Hence \( u_p(x) \) has \( p + 2 \) zeros if \( p = 2m \) and \( p + 1 \) zeros if \( p = 2m + 1 \).

If \( \beta_1 \neq 0 \) in \((X, \Lambda)\) or \( \beta_1(a) = 0, \beta_1(b) = 0 \) or \( \beta_1(b) = 0, \beta_1(a) = 0 \), we write the first boundary condition of (4)

\[
 u_p(a) \cdot P(a) = u_p(b) \cdot P(b),
\]

where

\[
P_1(x) = \alpha_1(x) - \beta_1(x) K(x) u_p'(x)/u_p(x).
\]

If \( P_1(a) P_1(b) > 0 \), \( u_p(a) u_p(b) \) will be positive and \( u_p(x) \) has an even number of roots. But the zeros of \( u_p(x) \) and \( L_1[u_p(x)] \) separate. Hence \( u_p(x) \) and \( L_1[u_p(x)] \) have the same number of zeros on \( a < x < b \). If \( P_1(a) P_1(b) < 0 \), \( u_p(a) u_p(b) \) will be negative and \( u_p(x) \) has an odd number of zeros. Hence \( u_p(x) \) will have one more or one less zero than \( L_1[u_p(x)] \) on \( a < x < b \).

It is also to be noticed that in Case I, since \( l_p \) is on \((\lambda_p, \lambda_{p+1})\), \( u_p(x) \) can vanish not more than \( p + 2 \) times nor less than \( p - 1 \) times. In Case II, since \( l_p \) is on \((\lambda_{p-1}, \lambda_p)\), \( u_p(x) \) can vanish not more than \( p + 1 \) times nor less than \( p - 2 \) times.

Designating the condition \( P_1(a) \cdot P_1(b) > 0 \) by \( P^+ \) and the condition \( P_1(a) \cdot P_1(b) < 0 \) by \( P^- \), we may state

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* Bocher, *Leçons*, p. 50.
† Cf. Cor., p. 96 of the paper of the author referred to above.
Oscillation theorem IV. If $u_p(x)$ is the $p$th characteristic function corresponding to a simple value of the system (4) satisfying conditions I-VIII, then the number of zeros of $u_p(x)$ on $a < x < b$, for $p \geq 1$, is given by the following table:

<table>
<thead>
<tr>
<th>Case</th>
<th>Number of zeros $p = 2m$</th>
<th>Number of zeros $p = 2m + 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$p + 2$</td>
<td>$p + 1$</td>
</tr>
<tr>
<td>$B^+ P^+$</td>
<td>$p + 2$</td>
<td>$p + 1$</td>
</tr>
<tr>
<td>$B^- P^+$</td>
<td>$p + 1$</td>
<td>$p + 2$ or $p$</td>
</tr>
<tr>
<td>$B^+ P^+$</td>
<td>$p$</td>
<td>$p + 1$</td>
</tr>
<tr>
<td>$B^- P^-$</td>
<td>$p + 1$ or $p - 1$</td>
<td>$p + 2$ or $p$</td>
</tr>
<tr>
<td>$C^+ P^+$</td>
<td>$p$</td>
<td>$p + 1$</td>
</tr>
<tr>
<td>$C^- P^+$</td>
<td>$p + 1$ or $p - 1$</td>
<td>$p + 2$ or $p$</td>
</tr>
<tr>
<td>$C^- P^-$</td>
<td>$p$</td>
<td>$p - 1$</td>
</tr>
</tbody>
</table>

Note: if in particular we choose

$$\alpha_i(x, \lambda) = \frac{(b - x) \alpha_i(\lambda) + (x - a) \gamma_i(\lambda)}{b - a}$$

$$\beta_i(x, \lambda) = \frac{(b - x) \beta_i(\lambda) + (a - x) \delta_i(\lambda)}{b - a}$$

the system (4) becomes identical with that of (4) of the paper by the writer to which reference has been made above. If $\beta_i \delta_i > 0$, it will be necessary to modify the boundary conditions of (4) by taking

$$L_i[u(a, \lambda)] = -L_i[u(b, \lambda)]$$

with condition VII replaced by

$$\alpha_1(a) \beta_2(a) - \alpha_2(a) \beta_1(a) = \alpha_1(b) \beta_1(b) - \alpha_2(b) \beta_2(b) = -1,$$

if only one of the boundary conditions is modified. If both boundary conditions are modified, condition VII remains unchanged. In either case conditions I-VI and VIII will be satisfied. The oscillation theorems will be true with the modification that $L_i[u_p(x)]$ will have an odd number of zeros.

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