ON THE LOCATION OF THE ROOTS OF THE JACOBIAN OF TWO BINARY FORMS, AND OF THE DERIVATIVE OF A RATIONAL FUNCTION*

BY

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1. **Introduction.** There have recently been published the following results:†

THEOREM I. *If the points z_1, z_2, z_3 vary independently and have circular regions as their respective loci, then the locus of the point z_4 defined by the real constant cross ratio*

$$\lambda = (z_1, z_2, z_3, z_4)$$

is also a circular region.

THEOREM II. *Let f_1 and f_2 be binary forms of degrees p_1 and p_2 respectively, and let the circular regions C_1, C_2, C_3 be the respective loci of m roots of f_1, the remaining $p_1 - m$ roots of f_1, and all the roots of f_2. Denote by C_4 the circular region which is the locus of points z_4 such that*

$$(z_1, z_2, z_3, z_4) = \frac{p_1}{m},$$

when z_1, z_2, z_3 have the respective loci C_1, C_2, C_3. Then the locus of the roots of the jacobian of f_1 and f_2 is composed of the region C_4 together with the regions C_1, C_2, C_3, except that among the latter the corresponding region is to be omitted‡ if any of the numbers m, $p_1 - m$, p_2 is unity. If a region C_i $(i = 1, 2, 3, 4)$ has no point in common with any other of those regions which is a part of the locus of the roots of the jacobian, it contains precisely $m - 1$, $p_1 - m - 1$, $p_2 - 1$, or 1 of those roots according as $i = 1, 2, 3$, or 4.

It is the primary object of the present paper to consider extensions of Theorem II in various directions. Chapter I studies the possibility of extend-

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† These Transactions, vol. 22 (1921), pp. 101–116; this paper will be referred to as II. It was preceded by a paper, these Transactions, vol. 19 (1918), pp. 291–298, which will be referred to as I, and was followed by a third paper, these Transactions, vol. 23 (1922), pp. 67–88, which will be referred to as III. We shall also have occasion to refer to two other of our papers, using the letters A and S respectively: Annals of Mathematics (2), vol. 22 (1920), pp. 128–144; *Comptes Rendus du Congrès International des Mathématiciens*, Strasbourg, 1920, pp. 349–352. The term *locus* in Theorem I of the present paper replaces the term *envelope* used in II.

‡ The corresponding region is to be omitted in this *enumeration* of the points of the locus; the corresponding region may nevertheless be in whole or in part, a portion of the locus of the roots of the jacobian.

ing Theorem II to include regions which are not circular. It is found (Theorem III) that the reasoning formerly used cannot be directly extended, and specific examples bring out the nature of the difficulties in supplying other modes of reasoning. Chapter II treats the extension of Theorem II by increasing the number of circular regions which are allowed to be loci of roots of the ground forms of the jacobian. Theorems VI and XI are fairly general results obtained by this extension, the principal results of the paper. Chapter III is a short chapter which deals with centers of gravity; the results are mainly generalizations of well known results for polynomials and their derivatives. Finally, Chapter IV deals with the case of the roots of the jacobian of real forms. Theorem XV is a rather general result which applies to the roots of the derivative of a polynomial which has only real roots.

Chapter I: On the extension of Theorem II to other than circular regions

2. **A distinctive property of circular regions.** We shall now undertake to consider the extension of Theorem II to regions which are not circular. Inasmuch as rather large extensions of Theorem I in this direction can be obtained without difficulty, and in fact have been obtained in III, our first tendency is to attempt to extend Theorem II by repeating our previous reasoning of II, p. 113. This turns out to be impossible, due to the failure of Lemma I (II, p. 102) to admit of large extension to other than circular regions:

Theorem III. *If a closed region C has the property that the force at any external point P due to every set of k unit particles in C is equivalent to the force at P due to k unit particles coinciding at some point of C, then C is a circular region.*

In proving this theorem we do not need to assume the property stated for every k, but merely for any one particular k (except of course $k = 1$, for which the result is absurd). We shall suppose $k = 2$ and leave to the reader the modifications for the other values of k.

If C is the whole plane there is no external point P and we may consider the theorem true, since C is a circular region. Similarly, if C is a single point the theorem is true. In the sequel we assume C to be neither the entire plane nor a single point.

It will be noted that the force at P due to equal particles at two points M and N is equivalent to the force at P due to two coincident particles situated at Q, the harmonic conjugate of P with respect to M and N,* which situation is invariant under linear transformation. We shall proceed to prove the

Lemma. *If P is exterior to C, and M and N are two points of C, then C contains every point on that arc of the circle MNP bounded by M and N which does not contain P.*

* This fact is quite easy to prove; see for example A, pp. 128–129.

Choose P at infinity and equal particles at M and N; the point Q which is the mid point of the line segment MN is seen to be in C. Then the mid points of the segments MQ, QN are also in C, and in fact we have a set of points in C everywhere dense on the segment MN. Hence this entire segment belongs to C.

The region C contains at least two points. It follows from the lemma that C contains an infinity of points. Transform one point R of the boundary of C to infinity, and consider two other distinct points V and W of that boundary. We proceed to prove that every point of the segment VW is a point of C. We can find a sequence of points not belonging to C but approaching R. Hence there is a sequence of points (the harmonic conjugates of the former sequence with respect to V and W) belonging to C and approaching the mid point U of VW, so U belongs to C. Then the mid points of UV and VW also belong to C, and in this way we prove that every point of VW belongs to C. But R also belongs to C, and hence we can prove that either every point of that infinite segment of RV which does not contain W belongs to C or every point of that infinite segment RW which does not contain V belongs to C; for definiteness suppose the latter.

There exists an infinite sequence of points not belonging to C but approaching W, and we assume as of course we may do that we have oriented the line VW horizontally and that we have an infinite sequence of such points $\{Z_k\}$ all lying in the lower half plane. If a line through Z_k cuts the segment VWR, then every point of that line which lies in the upper half plane belongs to C, by the lemma. For any preassigned point Y in the upper half plane we can choose a point Z_k such that YZ_k cuts the segment VWR, so every point of the upper half plane belongs to C as does also every point of the line VW.

If the region C does not consist of precisely the upper half plane including its boundary, there is a point X of C in the lower half plane. We can make use of the fact that R is a point of the boundary of C as before, and prove that the entire finite segment joining X to an arbitrary point of the line VW belongs to C. Then V and W are not boundary points of C, contrary to our assumption. The demonstration of the theorem is now complete.

Theorem III is quite easily proved, although by essentially the same methods, if we assume C to be bounded by a regular curve. Thus the property considered is invariant under linear transformation, for the position of the k coincident particles is uniquely determined by P and the original particles, and by Theorem I (I, p. 291 = II, p. 101) P is a root of the jacobian of the two binary forms each of degree k and whose roots are respectively the k original particles in C and the k coincident particles. If C is not bounded by a circle, its boundary can be transformed into a contour which is not convex.* Then

* See Theorem III of a note by the present writer, Annals of Mathematics (2), vol. 22 (1921), pp. 262–266.

there are two points A and B on the boundary of C such that the segment AB is exterior to C and such that there is a point P exterior to C and on the line AB but not on the segment AB. The force at P due to one particle at A and $k-1$ particles at B is equal to the force at P due to k coincident particles at a point which is on the segment AB but which coincides with neither A nor B and hence which is exterior to C.

The essence of Lemma I (II, p. 102) was proved and applied by Laguerre;* his formulation of the result was quite different from the present formulation, although the application was to the location of the roots of algebraic equations. Thus Theorem III appears as a sort of converse of the theorem of Laguerre, as well as of Lemma I (II, p. 102).

3. **Successive application of Theorem II to the determination of loci.** The property of circular regions stated in Theorem III seems to be conclusive in showing that the reasoning of II, p. 113, cannot be reproduced to give large extensions of Theorem II. Theorem III justifies, moreover, the somewhat artificial use of circular regions in III, Theorem XIII.

We now point out by two simple examples how results can be found from successive applications of Theorem II. These examples are not given as large extensions of Theorem II, but rather to show how difficult is that extension to regions which are not circular. The proofs of these theorems are left to the reader; the proof of the latter depends on III, Theorem IX.

THEOREM IV. *Suppose we have a finite or infinite number of sets of regions $C_1^{(n)}$, $C_2^{(n)}$, $C_3^{(n)}$, $C_4^{(n)}$, of Theorem* I *corresponding to the value* $\lambda = p_1/m$, *and suppose that no $C_i^{(n)}$ has a point in common with $C_j^{(k)}$ unless $i = j$ $(i, j = 1, 2, 3, 4)$. Denote by T_1, T_2, T_3, T_4, the regions common to all the $C_1^{(n)}$, $C_2^{(n)}$, $C_3^{(n)}$, $C_4^{(n)}$, respectively. Then if T_1 contains m roots of a bilinear form f_1, if T_2 contains all the remaining $p_1 - m$ roots of f_1, and if T_3 contains all the p_2 roots of a second form f_2, then the regions T_1, T_2, T_3, T_4 contain all the roots of the jacobian of f_1 and f_2. No two of the regions T_1, T_2, T_3, T_4 have a point in common, and they contain respectively $m-1$, $p_1 - m - 1$, $p_2 - 1$, 1 of the roots of the jacobian.*

There is no reason to suppose that the actual locus of the roots of the jacobian is composed of T_1, T_2, T_3, T_4, when T_1, T_2, T_3 are the loci of roots of the ground forms. But if the regions $C_1^{(n)}$, $C_2^{(n)}$, $C_3^{(n)}$, $C_4^{(n)}$ have the disposition suggested in the first part of § 11 (III) or more generally if T_4 is the locus of the point z_4 determined by its cross ratio p_1/m with the points z_1, z_2, z_3 whose loci are T_1, T_2, T_3, these four regions form that locus, except of course that among these latter three the corresponding region is to be omitted if any of the numbers m, $p_1 - m$, p_2 is unity.

Œuvres, pp. 56–63; p. 59.

THEOREM V. *In the situation of* III, *Theorem* VIII, *suppose the point* P *which is a center of external similitude for every pair of the circles* C_1, C_2, C *to be actually external to all those circles. Denote by* T_1, T_2, T *the portions of the interiors of those circles which lie between two half lines through* P *cutting those circles. Then if* T_1 *and* T_2 *are the respective loci of* m_1 *and* m_2 *roots of a polynomial* $f(z)$, *the regions* T_1, T_2, T *are the loci of the roots of its derivative* $f'(z)$, *except that* T_1 *or* T_2 *is to be omitted if* m_1 *or* m_2 *is unity. If* T_1, T_2, T *are mutually external, they contain respectively* $m_1 - 1$, $m_2 - 1$, 1 *of the roots of* $f'(z)$.

A theorem similar to Theorem V will be obtained by cutting the circles C_1, C_2, C by any convex contour, but no result can generally be stated in this case concerning the actual locus of the roots of $f'(z)$.

In discussing the possibility of the extension of Theorem II by reproducing the reasoning of II, p. 113, we reached the impossibility of extending Lemma I (II, p. 102), by which we may replace the force at an arbitrary point P exterior to a region C due to k particles in C by the force at P due to k coincident particles in C. To obtain certain facts, however, concerning the location of the roots of the jacobian, it may not be necessary to replace the k particles in C by k coincident particles in C *for an arbitrary point* P exterior to C but merely *for certain points* P exterior to C. This fact is notably true if the ground forms are real, as we shall show in Chapter IV; it is also true for certain other cases, as we shall now indicate by an extremely simple example.

We consider the polynomial

$$f(z) = z(z - \alpha_1)(z - \alpha_2),$$

where α_1 and α_2 have as their common locus the interior and boundary of the circle C_1 whose center is $z = 6$ and radius unity. Then $f'(z)$ has a root z_1 which has as its locus the interior and boundary of the circle C_1 and a root z_2 which has as its locus the interior and boundary of the circle C whose center is $z = 2$ and radius $1/3$. Under the given conditions, moreover, z_1 and z_2 remain separate and distinct.

Let us now consider the same polynomial but assign to the roots α_1 and α_2 as their common locus the right-hand semicircular region S_1' of C_1. Any point of the right-hand semicircular region S' of C is by III, Theorem IX, a point of the locus of z_2, and we shall prove that no other point is a point of this locus. Suppose a point $\bar{z}$ to be a point of the locus of z_2; $\bar{z}$ is evidently in or on C. The force at $\bar{z}$ due to particles at α_1 and α_2 is equivalent to the force at $\bar{z}$ due to two particles coinciding at some point α. We know that $\bar{z}$ is in or on C, hence exterior to C_1, so α is in or on C_1. Moreover, $\bar{z}$ is not in the half plane bounded by and lying to the right of the line $x = 6$, and hence α is in that half plane. Then α is in S_1', so $\bar{z}$ is in S'.

The locus of z_1 under the conditions stated includes of course S_1', but other points as well. In fact, if we choose $\alpha_1 = 6 + i$, $\alpha_2 = 6 - i$, the two roots of the real polynomial $f'(z)$ cannot be conjugate imaginary, so z_1 is real but to the left of the point $z = 6$ and therefore not a point of S_1'.

We consider anew the same polynomial and assign the left-hand semicircle S_1'' of C_1 as the locus of α_1 and α_2. Any point of S_1'' is evidently a point of the locus of z_1, and no other point belongs to this locus. For any such point would lie in the semicircle S', which is impossible, according to the theorem of Lucas. Under our conditions the locus of z_2 includes the left-hand semicircle S'' of C, but also other points. We choose as before $\alpha_1 = 6 + i$, $\alpha_2 = 6 - i$, and find z_2 to be real and on or within S' as previously noted. But $z_2 \neq 2$, and hence is not a point of S''.

Results which are large extensions of Theorem II to other than circular regions seem difficult to prove, as is shown by Theorems IV and V, even when the regions involved are common to two or more circular regions. But theorems of a certain type are easily established; we give simply one example:

Let the intersecting circles C_1 and C_2 with centers at the points α_1 and α_2 and radii r_1 and r_2 be the respective loci of m_1 and m_2 roots of a polynomial $f(z)$. Let the region common to C_1 and C_2 be the locus of $f(z)$, which is supposed to have no roots other than those mentioned. Then the roots of $f'(z)$ lie in C_1, C_2, and the region common to the two circles whose centers are the points

$$\frac{(m_2 + m_3)\alpha_1 + m_1\alpha_2}{m_1 + m_2 + m_3}, \qquad \frac{m_2\alpha_1 + (m_1 + m_3)\alpha_2}{m_1 + m_2 + m_3},$$

and whose radii are respectively

$$\frac{(m_2 + m_3)r_1 + m_1 r_2}{m_1 + m_2 + m_3}, \qquad \frac{m_2 r_1 + (m_1 + m_3)r_2}{m_1 + m_2 + m_3}.$$

The region mentioned in this theorem is not the *locus* of the roots of $f'(z)$, for the intersection of the last two circles mentioned in the theorem can never be a root of $f'(z)$. The proof of these facts is left to the reader.

4. **The number of roots of the jacobian in a circular region.** A question concerning the distribution of the roots of the jacobian which is very closely connected with the question of extending Theorem II to regions other than circular is that of the number of roots of the jacobian in the regions C_i when two or more of those regions have common points. Thus it might be supposed that C_1, C_2, C_3, C_4 contain *always* m, $p_1 - m$, p_2, 1 of the roots of the jacobian except for the possibility that this number be exceeded when C_i has one or more points in common with another of the regions. This supposition is false, however, as we proceed to show by an example. Thus consider the circles C_1 and C_2 in the form of Theorem II corresponding to III, Theorem VIII

(i.e., S, Theorem I), such that C intersects C_2 but is exterior to C_1. Let the line of centers of the circles intersect C_1 in A_1 (the intersection nearest C and C_2) and intersect C_2 in A_2 and C in A (the intersections farthest from C_1). When m_1 roots of $f(z)$ coincide at A_1 and m_2 roots at A_2, we have $m_1 - 1$ roots of $f'(z)$ at A_1, $m_2 - 1$ roots at A_2, and one root at A. When the m_2 roots at A_2 move slightly so that they do not all coincide but all remain on C_2 and symmetric with respect to the line $A_1 A_2$, the force corresponding at A and in the neighborhood of A becomes equivalent to the force due to m_2 particles coinciding *exterior* to C_2, since these particles can be considered to lie in the circular region consisting of the circle C_2 and all exterior points. Hence the root of $f'(z)$ at A moves and becomes exterior to C; the m_2 roots at A_2 remain in the vicinity of A_2 and there is no root of $f'(z)$ on or interior to C.

The question which we raised has thus been answered so far as concerns a region C which contains no roots of the ground forms. The result is essentially the same for a region which does contain a number of roots of the ground forms. Consider the case of the derivative of a polynomial, the second theorem on page 115 of II, locate m_1 roots of $f(z)$ at the null circle C_1, and locate the remaining m_2 roots at two points A and B different from C_1. There are but two roots z_1 and z_2 of $f'(z)$ distinct from A, B, and C_1, and these are interior to the triangle ABC_1, so a circle C_2 can be drawn which includes A and B (that is, m_2 roots of $f(z)$) but includes neither z_1 nor z_2 and hence contains only $m_2 - 2$ roots of $f'(z)$.

The question we have been considering is closely connected with the following:* Suppose a circle C contains at least r roots of a polynomial $f(z)$ of degree n. What can be said of the number of roots of $f'(z)$ in C?

On the one hand, C may contain all the roots of $f(z)$ and hence all the $n - 1$ roots of $f'(z)$. On the other hand, C may contain r roots of $f(z)$ and yet no root of $f'(z)$ if merely $r < n$. In fact, we prove that C may contain precisely $n - 1$ roots of $f(z)$ and contain a preassigned number p of the roots of $f'(z)$, if merely $p < n - 1$. Locate one root of $f(z)$ at a point P and the other $n - 1$ roots at $n - p - 1$ distinct points which lie on a line L not passing through P. Then we can describe a circle C which includes the $n - p - 1$ distinct points on L and hence p roots of $f'(z)$ but which contains no other roots of $f'(z)$.

* Still another allied question is: Suppose a circle C is known to contain at least r roots of a polynomial of degree n; determine the smallest (concentric) circle C' which always contains at least m roots of the derived polynomial.

The circle C' exists only if $m < r$. For $n = r = m + 1$, the answer is given by Lucas's Theorem. For the case $r = 2$, the circle C' is readily determined by means of a theorem due to Grace, to which reference is made in A, § 4. For the case $n - 1 = r = m + 1$, the circle C' is easily found by the second theorem of II, p. 115. For other cases the problem seems considerably more complicated.

Chapter II: On the extension of Theorem II to a larger number of circular regions

5. **Problem of the locus corresponding to any number of circular regions.** Our attempt in Chapter I to extend Theorem II in a form to apply to the jacobian of two particular binary forms by considering regions other than circular as loci of the roots of the ground forms and finding the corresponding locus of the roots of the jacobian was not particularly fruitful. This seems to result rather from the nature of the problem itself than from the precise methods employed. We now take up the possibility of extending Theorem II so as to consider not merely three circular regions but any number of circular regions. Let us suppose explicitly that we have the binary forms f_1 and f_2 of respective degrees p_1 and p_2, and that the circular regions C_1', C_2', $\cdots$, C_m' are the loci respectively of p_1', p_2', $\cdots$, p_m' roots of f_1 and the circular regions C_1'', C_2'', $\cdots$, C_n'' are the loci respectively of p_1'', p_2'', $\cdots$, p_n'' roots of f_2, where we have

$$p_1' + p_2' + \cdots + p_m' = p_1,$$
$$p_1'' + p_2'' + \cdots + p_n'' = p_2.$$

For convenience in phraseology, we shall suppose that none of these regions is either a point or the entire plane unless otherwise stated. We wish then to find the location of the roots of the jacobian of f_1 and f_2; not merely to determine certain regions in which lie or do not lie the roots of the jacobian, but to determine the actual *locus* of those roots under the assigned conditions, as in Theorem II.

Let us consider, for any particular values of the roots of f_1 and f_2 which satisfy our hypothesis, a root ζ of the jacobian exterior to all the circular regions C_1', $\cdots$, C_m', C_1'', $\cdots$, C_n''. This root ζ is an analytic function of α, any root of f_1 or f_2, and hence when α varies over a certain two-dimensional continuum, ζ also varies over a certain two-dimensional continuum. We thus have a certain number of regions which may or may not be distinct and may or may not have common points which are the loci of the points ζ. We see by the analyticity of the transformation that all the points α must be on the boundaries of their proper regions whenever a point ζ corresponding is on the boundary of its locus.* Moreover, if a point ζ is on the boundary of its locus, and exterior to all the regions C_1', $\cdots$, C_m', C_1'', $\cdots$, C_n'', we know by Lemma I (II, p. 102) that all the points α pertaining to any one circular region can be considered to coincide on the boundary of that region. But the precise manner of simultaneous variation of these coincident roots on the

* There is an exception to this reasoning if the algebraic equation defining ζ degenerates and if ζ is independent of a particular α, but in that case α can be moved at will without changing ζ and so α can be considered as on the boundary of its locus. A similar remark applies also below.

boundaries of their loci in such a manner that a point ζ or several points ζ remain on the boundaries of their loci and trace out those boundaries is as yet unknown.

Let us restrict ourselves for the moment to the situation where the circular regions $C_1', \cdots, C_n''$ are relatively small, or to be more precise, such that for no choice of the roots of the ground forms in their proper regions can two roots of the jacobian coalesce exterior to those circular regions; we suppose further that no two of the regions $C_1', \cdots, C_n''$ and the regions R which are the loci each of one of the roots of the jacobian exterior to those circular regions when the roots of the ground forms have their proper regions as loci—no two of all these regions have a point in common. We may allow the roots of the ground forms to coalesce in their proper regions; we notice that the circular regions $C_1', C_2', \cdots, C_m', C_1'', C_2'', \cdots, C_n''$ contain and are therefore the loci of respectively $p_1' - 1, p_2' - 1, \cdots, p_m' - 1, p_1'' - 1, p_2'' - 1, \cdots, p_n'' - 1$ roots of the jacobian. There are then $m + n - 2$ regions R each of which is the locus of one root of the jacobian. When we allow the circular regions to become larger and larger, of course the regions R expand also, need not preserve their identity (for example, two of them may coincide), and finally these regions cover the entire plane.

Very little is known of the precise nature of the boundaries of these regions R.* Their boundaries are not, except in very special instances, circular regions, but are curves which presumably have interesting properties with reference to the boundaries of the regions $C_1', \cdots, C_n''$; which properties can be expressed in a manner so as to be invariant under linear transformation. It is evidently true that if we start with any situation $C_1', \cdots, C_n''$ and if we allow two of the regions $C_1', \cdots, C_m'$ or two of the regions $C_1'', \cdots, C_n''$ to coalesce, one of the regions R will coalesce with them, and we shall have precisely the situation of $m - 1$ regions C_i' or $n - 1$ regions C_i''.

The exact determination of the regions R in any very general case seems difficult. If all the original circular regions reduce to points except one of them, say C_1', we can determine the path of the roots ζ of the jacobian as α, a p_1'-fold root of f_1, traces the circle C_1'. These roots ζ, in their totality, trace closed curves, for the situation when α returns to its initial position is exactly the same as the initial situation. The boundaries of the regions R must be composed of these closed curves, or at least of portions of them. If now we allow a second one of our circular regions, say C_1'', to be a non-degenerate region, the new locus of the roots of the jacobian will be a number of regions

* The writer conjectures that when there are q roots of the jacobian in these regions R, these regions are in their totality bounded by a degenerate or non-degenerate q-circular $2q$-ic; only the degenerate cases of this curve have ever been treated in detail, except for $q = 1$. Compare Walsh, Proceedings of the National Academy of Sciences, vol. 8 (1922), pp. 139–141.

R'. The boundaries of the regions R' will be curves which are envelopes of the curves R corresponding to the region C_1' and the null regions $C_2', \cdots, C_m', C_1'' = \beta, C_2'', \cdots, C_n''$, while the point β traces the circle C_1''. By continuing in this way, we have a process for the generation of the regions R in any case desired. But the actual determination of the boundaries in a very general case would presumably be too laborious by this process; more powerful methods will have to be devised.

The statement has been made that the regions R are not in general circular regions; it is perhaps worth while to present a specific instance to illustrate this fact. We consider the polynomial

$$f(z) = (z - \alpha_1)(z - \alpha_2)(z - \alpha_3),$$

where $\alpha_1 = i$, $\alpha_2 = -i$, and the locus of α_3 is chosen to be a circle C_3 and its exterior, whose center is the origin and radius so large that the loci of the two roots of $f'(z)$ are entirely distinct. In the field of force to determine the roots of $f'(z)$, the force at a point of either coördinate axis due to the two particles at i and $-i$ is in direction along that axis. Hence, whenever $f'(z)$ has a root on an axis, α_3 must also be on that axis. The root ζ of $f'(z)$ larger in absolute value is determined on the positive half of the axis of reals by the particle α_3 at the right-hand intersection of C_3 and that axis, and a second particle of twice the mass at the harmonic conjugate* of ζ with respect to the points i and $-i$; this harmonic conjugate lies *to the left of* the origin. This root ζ is determined on the positive half of the axis of imaginaries by α at the upper intersection of C_3 and that axis, and a second particle of twice the mass at the harmonic conjugate of ζ with respect to i and $-i$; this harmonic conjugate lies *above* the origin. The curve bounding the locus of ζ is symmetric with respect to the coördinate axes and hence is not a circle.

The characteristic of Theorem II (and indeed also of Theorems VI and XI) in comparison with the more general results indicated in this present section seems to be a certain *linearity*. This fact is brought out very clearly in S, but also in II, since by Lemma II (II, p. 102) the position of equilibrium is determined by its cross ratio with three points, a relation which is essentially linear. It is as a result of that linearity that for our particular situations the loci of the roots of the jacobian are all bounded by circles.

6. **A condition that a root of the jacobian be on the boundary of its locus.** We return now to the general case of the preceding section, and shall obtain a geometric relation between the roots of the ground forms and a root of the jacobian, when all of those roots are on the boundaries of their loci.

Consider the points α_1 and α_2 at which coincide all the p_1' roots in C_1' and all the p_2' roots in C_2' respectively; we consider also a root ζ of the jacobian which

* Compare §§ 2, 9; also A, pp. 128–129.

is supposed not to lie at a common root of the two ground forms or at a multiple root of either form, so that ζ is given by a certain algebraic equation which actually contains α_1 and α_2. We may write this equation in the form

$$k + \frac{p_1'}{\zeta - \alpha_1} + \frac{p_2'}{\zeta - \alpha_2} = 0. \tag{1}$$

We can hold fast ζ and all roots of the ground forms other than α_1 and α_2, and move α_1 and α_2 depending on each other so as to satisfy (1). Since (1) is linear in α_1 and α_2, when one of these points is made to trace a circle the other also traces a circle. When α_1 moves so as to trace C_1', α_2 moves so as to trace a circle tangent to C_2'. In fact, if α_2 were to trace a circle intersecting C_2', α_2 would at some time move *interior* to the region C_2', and still α_1 and α_2 would be in their proper loci. Then motion of α_2 holding α_1 fast would cause ζ to move over a two-dimensional continuum, so ζ would not be on the boundary of its locus.

It will be useful to study in some detail the relation between α_1 and α_2 defined by (1). The two double points of the transformation (α_1, α_2) are

$$\alpha_1 = \alpha_2 = \zeta; \qquad \alpha_1 = \alpha_2 = \zeta + \frac{p_1' + p_2'}{k};$$

denote this latter point by α. We have, of course, $p_1' + p_2' \neq 0$, so that these two points are distinct. The cross ratio of these fixed points with α_1 and α_2 in any position is readily computed:

$$\frac{[\alpha_1 - \alpha_2]\left[\zeta - \left(\zeta + \dfrac{p_1' + p_2'}{k}\right)\right]}{[\alpha_2 - \zeta]\left[\left(\zeta + \dfrac{p_1' + p_2'}{k}\right) - \alpha_2\right]} = \frac{p_1' + p_2'}{p_2'}.$$

Inasmuch as this cross ratio is real, the four points $\alpha_1, \alpha_2, \zeta, \alpha$ lie on a circle C.

The circle C is self-corresponding under the transformation (α_1, α_2). In fact, two of its points ζ and α are unchanged while a third point α_1 is transformed into a point α_2 of the circle; this is sufficient.

If p_1' and p_2' are both negative instead of both positive, we have the case of the roots of f_2 located in C_1'' and C_2''. In either of these situations, the first case we consider, the points α_1 and α_2 are separated by ζ and α. For a transformation can be made so that $k = 0$. The value of the cross ratio gives us

$$\frac{\alpha_1 - \alpha_2}{\zeta - \alpha_2} = \frac{p_1' + p_2'}{p_2'}, \qquad \frac{\alpha_1 - \zeta}{\zeta - \alpha_2} = \frac{p_1'}{p_2'};$$

so α_1 and α_2 are indeed separated by ζ and α.

We thus choose ζ as fixed on the boundary of its locus; the fixed points

α_1' and α_2' (particular values of α_1 and α_2 respectively) corresponding are also on the boundaries of their respective loci. We have already remarked that when α_1 moves from α_1' along C_1', α_2 moves from α_2' along a circle tangent to C_2'. When α_1 moves from α_1' interior to the region C_1', α_2 moves from α_2' exterior to the region C_2'. When α_1 moves on C, α_2 moves in the opposite direction but also on C. It follows that *C cuts C_1' and C_2' at angles of the same magnitude, and when C is transformed into a straight line the tangents to C_1' at α_1' and C_2' at α_2' are parallel.** There are different possibilities here according to whether C cuts C_1' and C_2' at equal angles or at supplementary angles. We leave it for the reader to notice that if the loci C_1' and C_2' are both interior or both exterior to their bounding circles, these two angles are equal; if one locus is interior and the other exterior to its bounding circle, these two angles are supplementary.

The second case we shall consider is that of two roots α_1 and α_2, of the forms f_1 and f_2, of multiplicities p_1' and p_1'', and loci C_1' and C_1'', respectively. Essentially the same formulas apply, except that in (1) and the succeeding formulas the numbers p_1' and p_2' are replaced by $p_2 p_1'$ and $-p_1 p_1''$ respectively; we suppose $p_2 p_1' - p_1 p_1'' \neq 0$. The cross ratio of the four points $\alpha_1, \alpha_2, \zeta, \alpha$ shows that α_1 and α_2 are not separated by ζ or α. Hence if α_1 and α_2 trace the self-corresponding circle C, they trace it in the same sense. The angles which C cuts on C_1' and C_1'' are then equal or supplementary according as the regions C_1' and C_1'' lie one inside and the other outside their bounding circles, or as these regions lie both inside or both outside their bounding circles. When C is transformed into a straight line, the lines tangent to C_1' at α_1 and to C_1'' at α_2 are parallel.

The third case we have to treat is the remaining situation under the second case, where $p_2 p_1' - p_1 p_1'' = 0$; here the transformation (α_1, α_2) has but the one double point ζ. It is still true that the circle C through $\alpha_1, \alpha_2, \zeta$ is self-corresponding under this transformation. For if we denote by γ the point α_2 corresponding to $\alpha_1 = \alpha_2'$, where α_1' and α_2' are fixed values of α_1 and α_2, we shall have

$$-k = \frac{p_2 p_1'}{\zeta - \alpha_1'} - \frac{p_1 p_1''}{\zeta - \alpha_2'} = \frac{p_2 p_1'}{\zeta - \alpha_2'} - \frac{p_1 p_1''}{\zeta - \gamma},$$

$$\frac{(\alpha_1' - \alpha_2')(\gamma - \zeta)}{(\alpha_2' - \gamma)(\zeta - \alpha_1')} = -1,$$

so $\alpha_1', \alpha_2', \zeta, \gamma$ are concyclic; the three points $\alpha_1', \alpha_2', \zeta$ of C are transformed

* We cannot prove here, as in III, Theorem II, that this property holds also for the tangent to the boundary of the locus of ζ at the point ζ. In fact, if we choose another pair of points α_1', α_2', leading to the circle C', it is in general impossible for C' to cut at equal angles C_1' at α_1' and the boundary of the locus of ζ at ζ. For a specific example, see the illustration used at the close of § 5.

into three points α_2', γ, ζ of C which, therefore, is self-corresponding. If ζ is transformed to infinity, we have

$$\frac{\alpha_1' - \alpha_2'}{\alpha_2' - \gamma} = 1,$$

so γ is obtained from α_2' by translation by an amount equal to $\alpha_1' - \alpha_2'$. Then α_1 and α_2 trace C in the same sense. As in our second case, the angles which C cuts on C_1' and C_1'' are equal or supplementary according as the regions C_1' and C_1'' lie one inside and the other outside their bounding circles, or as these regions lie both inside or both outside their bounding circles. When C is transformed into a straight line, the lines tangent to C_1' at α_1 and to C_1'' at α_2 are parallel.*

We have now considered all typical cases of two roots α_1 and α_2 of the ground forms. In particular if we choose two roots of a single form which have the same locus, the reasoning we have used shows that when ζ is on the boundary of its locus α_1 and α_2 must be on the boundary of their common locus and *must coincide*. This may be used to replace Lemma I (II, p. 102). The reader may be interested in applying the remarks of the present section to the situations of Theorems II and VI.

7. **A special case of coaxial circles.** We have pointed out in § 5 that for the situation there described of $m + n$ circular regions $C_1', \cdots, C_m', C_1'', \cdots, C_n''$ as the loci of the roots of the ground forms, the locus of the roots of the jacobian is not in general a number of circular regions or of regions bounded by several circles. But of course there are special situations for which the locus of the roots of the jacobian is bounded by circles. This is evidently true, for example, if $C_1', \cdots, C_m', C_1'', \cdots, C_n''$ are bounded by $m + n$ concentric circles or more generally by $m + n$ coaxial circles having no common point.† For such a situation, moreover, the methods used in I can be applied; we shall give an illustration of that fact.

We use the notation of I, p. 293 ff., and suppose the situation simplified

* This remark enables us immediately to state something about the locus to be determined in certain cases. Thus consider the situation and notation of Theorem X (we might indeed choose that of Theorem VI). Choose the line through $\alpha_1, \cdots, \alpha_n$ horizontal and draw a parallel L' through any point z on the boundary of its locus. If one point ζ_1 (notation of § 9) is not on L', say below L', and if z is exterior to all the circles $C_1, \cdots, C_n$, all other points ζ_i are also below L'. Then by Lucas's Theorem, z cannot be a root of $f'(z)$. Therefore all points ζ_i lie on L' and z lies on a circle C_i'.

If $m_1, \cdots, m_n$ are all greater than unity, no point z interior to a circle C_i need be considered. If any m_i is unity, however, the points z interior to C_i must be considered. The writer has been unable to treat similarly (by the method just indicated) this last case, and thus completely to prove Theorem X.

† If the coaxial system is composed of all circles through two points or all circles tangent at a single point, we may consider all the roots of both forms to coincide at a single point, the jacobian vanishes identically, and the locus of the roots of the jacobian is the entire plane.

by transformation as in I. Suppose C_1 to contain k roots of f_1 ($p_2\,k$ positive particles) and l roots of f_2 ($p_1\,l$ negative particles). If l is sufficiently small in comparison with k, and if C_2 and C_3 are sufficiently remote, it seems reasonable to suppose that we can obtain a region near but exterior to C_1, which region contains no root of the jacobian. The circle C_1 contains, then, k particles each of mass p_2 and l particles each of mass $-p_1$. Outside of C_2 there are $p_1 - k$ particles each of mass p_2 and outside of C_3 there are $p_2 - l$ particles each of mass $-p_1$. If Q is a position of equilibrium, we must have, in the notation of I,

$$\frac{p_2\,k}{a+r} \leqq \frac{p_1\,l}{r-a} + \frac{p_2\,(p_1-k)}{b-r} + \frac{p_1\,(p_2-l)}{c+r},$$

which can be put into the equivalent form

$$\begin{aligned} 0 \leqq r^2[&-p_2\,k(a+b) - p_1\,l(a+c) + p_1\,p_2(b+c)] \\ &+ r[p_2\,k(a+b)(a-c) + p_1\,l(a+c)(b-a)] \\ &+ [-p_1\,p_2\,a^2(b+c) + p_2\,kac(a+b) + p_1\,lab(a+c)]. \end{aligned} \tag{2}$$

This form does not simplify materially. Denote by C_4 and C_5 the circles whose centers are O and radii the roots of the right-hand member. The cross ratio of the points C_4', C_4'', C_5', C_5'' (notation as in I, p. 294) with the collinear points C_1'', C_2', C_3'' can easily be calculated, but this cross ratio contains a, b, c explicitly and is not independent of their ratios; we therefore use a different method to describe C_4 and C_5. We are supposing implicitly that the roots of the right-hand member of (2) are positive or that at least one of these roots is positive.

If C_4 and C_5 lie between C_1 and C_2 and between C_1 and C_3, they bound an annulus which contains no root of the jacobian. For if $r = a$, the right-hand member of (2) reduces to

$$2p_1\,l(a+c)(b-a),$$

so that inequality is satisfied for $r = a$ and therefore is not satisfied when r lies between the two roots.

Under this hypothesis we can determine the precise number of roots of the jacobian in the smaller of the new circles by allowing the roots of f_1 and f_2 in C_1 to move continuously and to coincide at O. When the $p_2\,k$ and $p_1\,l$ particles are all in coincidence at O, the circle C_1 contains precisely $k + l - 1$ roots of the jacobian, so this is the original number of roots interior to or on the inner boundary of the annulus.

Hence we have, under the assumptions already made:

1. *If the circles C_4 and C_5 lie between C_1 and C_3, then the annular region between C_4 and C_5 contains no root of the jacobian of f_1 and f_2. The region which*

is bounded by C_4 or C_5 and contains the region C_1 contains precisely $k + l - 1$ roots of the jacobian.

2. *If C_4 and C_5 are separated by C_3, there are no roots of the jacobian in the annular region which is part of the annular region bounded by C_4 and C_5 and which contains no point of the region C_3. The circular region bounded by C_4 or C_5 which contains the region C_1 but no point of the region C_3 contains precisely $k + l - 1$ roots of the jacobian.*

This theorem can readily be expressed in general form so as to include the situation after linear transformation; compare the corresponding statement in I.

8. **Theorem VI, a general theorem for circles having an external center of similitude.** There is another fairly general class of loci other than the very simple class just considered for which the locus of the roots of the jacobian as treated in § 5 is bounded by circles. We shall now use a method which is novel in some respects but which makes use of our former results to establish

THEOREM VI. *Let the interiors and boundaries of the circles $C_1', C_2', \cdots, C_{n'}'$, whose centers are $\alpha_1', \alpha_2', \cdots, \alpha_{n'}'$, respectively, be the loci of $m_1', m_2', \cdots, m_{n'}'$ roots of the form f_1 which has no other roots. Let the interiors and boundaries of the circles $C_1'', C_2'', \cdots, C_{n''}''$, whose centers are $\alpha_1'', \alpha_2'', \cdots, \alpha_{n''}''$, respectively, be the loci of $m_1'', m_2'', \cdots, m_{n''}''$ roots of the form f_2 which has no other roots. Suppose further that a point P is interior to no circle C_i' or C_i'' and is an external center of similitude for every pair of the circles C_i', C_j' and for every pair of the circles C_i'', C_j'' and an internal center of similitude for every pair C_i', C_j''. The distinct roots of $\bar{J}$, the jacobian of $\bar{f}_1$ and $\bar{f}_2$, when all the roots of the ground forms are concentrated at the centers of their proper circles, are denoted by $\alpha_1, \alpha_2, \cdots, \alpha_n$ (all of which points are collinear with $P, \alpha_1', \cdots, \alpha_{n'}', \alpha_1'', \cdots, \alpha_{n''}''$) of multiplicities $m_1, m_2, \cdots, m_n$, and by $C_1, C_2, \cdots, C_n$ are denoted the circles which have these points as centers and radii such that P is an external or internal center of similitude for every pair of the circles $C_1', \cdots, C_{n'}', C_1'', \cdots, C_{n''}'', C_1, \cdots, C_n$. Then the locus of the roots of J, the jacobian of f_1 and f_2, is composed of the interiors and boundaries of the circles $C_1, C_2, \cdots, C_n$. A circle C_i exterior to all the other circles C_j contains precisely m_i roots of J.*

Limiting cases of the circles $C_1', \cdots, C_{n'}', C_1'', \cdots, C_{n''}''$ are the points P and P', the point at infinity. We shall admit these circles as possibilities in the demonstration of the theorem, providing, however, that there is at least one proper circle, which we shall suppose to be C_1'. If there is no proper circle C_1' or C_1'', either the only roots of J are P and P', in which case the theorem remains true, or every point of the plane is a root of J, in which case the theorem is not true.

The configuration of the three sets of circles has some obvious but interesting properties relative to $\bar{J}$. Let us choose as horizontal the line L on which lie

P, α_1', $\cdots$, $\alpha_{n'}'$, α_1'', $\cdots$, $\alpha_{n''}''$, with the C_i' to the right of P, let us number in order the two sets of circles commencing with C_1' and C_1'', the nearest circles to P, and let us denote by μ_i' the left-hand intersection of C_i' with L and by ν_i' the right-hand intersection, with the opposite conventions for μ_i'' and ν_i'', the intersections of C_i'' with L. The points μ_1', $\cdots$, $\mu_{n'}'$, μ_1'', $\cdots$, $\mu_{n''}''$ may be obtained from the points α_1', $\cdots$, $\alpha_{n'}'$, α_1'', $\cdots$, $\alpha_{n''}''$' by a similarity transformation with P as center, and as a line L' is allowed to rotate about P its intersections with C_i' and C_i'' have always this same property. In fact, we may consider properly chosen intersections of L with C_i' and C_i'' to be the roots of $\bar{f}_1$ and $\bar{f}_2$; then the roots of $\bar{J}$ trace the circles C_i. In particular, when L' is tangent to the circles C_1', $\cdots$, $C_{n'}'$, C_1'', $\cdots$, $C_{n''}''$, the points of tangency have this relation to the points of tangency with C_1, $\cdots$, C_n.

To prove Theorem VI, we consider as usual the field of force given by Theorem I (I, p. 291 = I, p. 101). We can obtain immediately a qualitative idea of the locus of the roots of J. No point Q above both of the tangents T and T' common to the circles C_1, $\cdots$, C_n, C_1', $\cdots$, $C_{n'}'$, C_1'', $\cdots$, $C_{n''}''$ can be a root of J. For the force at Q due to the particles situated at the roots of f_1 has a component normal to PQ and that due to the particles situated at the roots of f_2 has a component normal to PQ and in the same sense; at least one of these components is different from zero. Thus also no point below both T and T' can be a root of J and no point above T (or T') but not above T' (or T) yet lying to the right of C_1'' and to the left of C_1' can be a root of J. No point Q of T (or T') can be a root of J unless all the roots of f_1 and of f_2 are on T (or T') which can only occur if these roots lie at the points of tangency of T (or T') with the circles bounding their proper loci; that is, if Q lies at a point of tangency of T (or T') and a C_i. Inasmuch as the locus of the roots of J is a closed point set there must be some sort of a boundary of that locus between any two of the circles C_i.

By means of the similarity transformation with P as center, we see that every point of the locus as stated in Theorem VI is really a point of the locus. To complete the determination of the locus we have merely to prove that if a point Q is a point of the boundary of the locus, that point is on one of the circles C_i.

The interior and boundary of the circle C_i' or C_i'' which is the locus of more than one root of f_1 or of f_2 is also the interior and boundary of one of the circles C_i; every point interior to or on such a circle is a point of the locus of the roots of J. A point on or interior to two circles C_i' and C_{i+1}' (and so of course C_i'' and C_{i+1}'') is also on or within a circle C_j; in fact there is a root α_j of $\bar{J}$ between α_i' and α_{i+1}', and the circle C_j whose center is α_j contains in its interior the region common to C_i' and C_{i+1}', for this is true of any circle whose center lies between α_i' and α_{i+1}' if the circle has the two common tangents T and T'.

It merely remains to consider points Q exterior to all the circles C_1', $\cdots$, $C_{n'}'$, C_1'', $\cdots$, $C_{n''}''$ or interior to or on at most one circle C_i' or C_i'' which is the locus of but one root of f_1 or f_2. In any case the particles at the roots of f_1 and f_2 may by Lemma I (II, p. 102) be considered to coincide in their respective loci so far as the force at Q is concerned.

9. **Proof of Theorem VI; replacing of two particles by a single particle.** The point Q is then to be considered as fixed, and for definiteness to lie to the right of P and of course above one of the lines T and T' but below the other, so that the line PQ actually cuts all the circles C_1', $\cdots$, $C_{n'}'$, C_1'', $\cdots$, $C_{n''}''$. We know the loci of certain particles each representing several of the roots of f_1; we shall replace two of these particles by a single equivalent particle, then the new particle and a third of the original particles by a single equivalent particle, and so on until we have replaced all the particles representing the roots of f_1 by a single equivalent particle, and similarly for the particles representing the roots of f_2. A study of the properties of the loci of these various particles will enable us to prove that Q is on one of the circles C_i. We suppose for the present that Q is exterior to all the circles C_j'.

To replace two particles at ζ_1 and ζ_2 of masses m_1 and m_2 by a single particle ζ of mass $m_1 + m_2$ so that the force at Q shall be unchanged, we have the equation for ζ

$$\frac{m_1}{\zeta_1 - q} + \frac{m_2}{\zeta_2 - q} = \frac{m_1 + m_2}{\zeta - q},$$

where q is the complex number representing the point Q. This equation is equivalent to

$$\frac{(\zeta_1 - q)(\zeta_2 - \zeta)}{(\zeta_2 - q)(\zeta - \zeta_1)} = \frac{m_1}{m_2}.$$

We wish to replace the particles ζ_1 and ζ_2, whose loci are the interiors and boundaries of C_1' and C_2' and which represent all the roots of f_1 having these regions as loci, by a single equivalent particle. If we transform Q to infinity, we shall have precisely the conditions of III, Theorem VIII; m_1 and m_2 are both positive and the points ζ_1 and ζ_2 are always separated by ζ and Q. Then the variable circle C of Lemma IV (II, p. 105) moves so as always to cut C_1' and C_2' at the same angle, and cuts also S_1', the circle bounding the locus of ζ, also at this same angle. When Q is transformed back to the finite part of the plane, it remains true that C cuts C_1' and C_2' at the same angle; C cuts S_1' at this same angle or the supplementary angle according as the locus S_1' is interior or exterior to its bounding circle.* We leave this fact to be verified by the reader; this can be done by considering any one circle C under the

* This difference in behavior, which we shall constantly meet, disappears entirely if we project stereographically on to the sphere.

transformation of Q from the point at infinity to its original finite position. In particular it will be noticed that the line PQ is one of the circles C cutting C_1', C_2', S_1' at the proper angles. When ζ_1 and ζ_2 are on PQ and are chosen as the right-hand (left-hand) intersections of PQ with C_1' and C_2', ζ is on PQ and at the right-hand or left-hand (left-hand or right-hand) intersection of PQ with S_1' according as the locus S_1' is interior or exterior to its bounding circle. The converse of this statement is also true; such a choice of ζ leads to a unique determination of ζ_1 and ζ_2 as described.

We have supposed Q to be exterior to all the circles C_i'; suppose now Q exterior to C_1' but interior to C_2'; we need not consider Q interior to the two circles. When Q is transformed to infinity we have a special case of Theorem I, but no longer a special case of III, Theorem VIII. The circle C which generates as in Lemma IV (II, p. 105) the boundary of S_1' cuts C_1' and C_2' at supplementary angles. In fact, if we assume C to cut C_1' and C_2' at equal angles, but not at supplementary angles, when ζ_1, ζ_2, ζ are on the boundaries of their loci, the line $\zeta_1\zeta_2\zeta$ can be rotated about ζ so that ζ_1 and ζ_2 move into the interiors of their loci, so ζ cannot be on the boundary of its locus. The circle S_1' is found to be cut by C at an angle equal to that cut on C_2' and supplementary to that cut on C_1'.

We shall not consider in detail the case that Q is on C_1' or C_2'; we need not consider Q on both circles. Whether Q is on or within one circle or exterior to all the circles C_i' it is always true that when Q is in its original finite position C cuts C_1' and C_2' at the same angle, and cuts S_1' at this same angle or the supplementary angle according as the locus S_1' is interior or exterior to its bounding circle. When ζ_1 and ζ_2 are chosen properly as the intersections of C_1' and C_2' with PQ, one of the circles C, ζ is on PQ and on S_1', and conversely. The tangents to these three circles at those three points are parallel.

We have thus replaced the particles at ζ_1 and ζ_2 by a single equivalent particle. So far as the force at Q is concerned, we can replace ζ_1 and ζ_2 at any positions in their loci by ζ in its locus S_1', and for any position of ζ in S_1' we can determine ζ_1 and ζ_2 in their loci so that the force at Q is the same. If Q is in C_1' or C_2', the force at Q can be made as large as desired, so Q must be in S_1', and conversely. If Q is on C_1' or C_2', the force at Q can be made as large as desired but only in certain special directions, so Q is on S_1', and conversely. If the region S_1' is external to its bounding circle, the force at Q is zero for proper choice of ζ and hence of ζ_1 and ζ_2, and conversely.

Next we replace by a single equivalent particle ζ' the particle ζ as just determined and ζ_3, the particle which represents the roots of f_1 whose common locus is C_3' (assuming the existence of this set of roots). No further detailed discussion is required of this new situation; as before, a system of circles C' cuts S_1', C_3', and the boundary S_2' of the locus of ζ' at equal angles or at angles supplementary to the angle cut on C_3' according as the loci S_1' and S_2' do not or do include the point at infinity. The line PQ is as before one of the system of circles C'. When ζ' is on PQ and on S_2', ζ_3 is on PQ and on C_3' and ζ is on PQ and S_1', so ζ_1 and ζ_2 are on PQ and C_1' and C_2' respectively; moreover, the tangents to C_1', C_2', C_3' at ζ_1, ζ_2, ζ_3, respectively, are parallel. The converse of this statement is also true.

We continue in this same manner to replace pairs of particles by equivalent particles, and finally replace all the particles $\zeta_1, \zeta_2, \cdots, \zeta_{n'}$ representing the roots of f_1 by a single particle η_1 whose locus is a circular region S_1 whose boundary is cut by PQ at an angle supplementary or equal to the angle cut on $C_1', C_2', \cdots, C_{n'}'$ according as the locus contains or does not contain the point at infinity. When η_1 is on PQ and on S_1, we know that $\zeta_1, \cdots, \zeta_{n'}$ are on $C_1', \cdots, C_{n'}'$ respectively and that the tangents to these circles at these points are parallel. It follows from reasoning to be given later that at no intermediate stage is the locus of one of our auxiliary particles the entire plane.

Similarly the particles $\xi_1, \xi_2, \cdots, \xi_{n''}$ representing the roots of f_2 are replaced by a single particle η_2 whose locus is a circular region S_2 which is either one of the points P or P' or is bounded by a circle S_2 which is cut by PQ at an angle equal to the angles cut on the circles $C_1'', C_2'', \cdots, C_{n''}''$. In fact, if all the roots of f_2 are not concentrated at P' the particles corresponding to the roots of f_2 always exert at Q a force not zero, so the locus of η_2 does not include the point at infinity. When η_2 is on PQ and S_2, we know that $\xi_1, \xi_2, \cdots, \xi_{n''}$ are on $C_1'', C_2'', \cdots, C_{n''}''$ respectively, and that the tangents to these circles at these points are parallel.

10. **Theorem VI: proof completed.** For Q to be a root of J, the loci S_1 and S_2 must have at least one point in common, at which point are to coincide η_1 and η_2 so that their resultant force at Q shall be zero. Such a common point cannot be Q, for Q is not a point of S_2. The loci S_1 and S_2 cannot overlap if Q is on the boundary of its locus, for when Q varies slightly in any direction, S_1 and S_2 vary but slightly. If S_1 and S_2 overlap, we may vary Q slightly in any direction but so little that S_1 and S_2 still have common points, so that Q remains a root of J for some choice of η_1 and η_2 and hence is not on the boundary of the locus of the roots of J. We defer until later the possibilities that the two circles S_1 and S_2 coincide or that S_1 or S_2 may be the entire plane or a point.

The regions S_1 and S_2 have but a single point in common, and since PQ cuts the circles S_1 and S_2 at equal or supplementary angles according as S_1 does or does not contain the point at infinity, a single point common to these two loci must lie on PQ. That is, η_1 and η_2 lie on S_1 and S_2 respectively, and on PQ, so $\zeta_1, \zeta_2, \cdots, \zeta_{n'}, \xi_1, \xi_2, \cdots, \xi_{n''}$ all lie on PQ and the lines tangent to the circles $C_1', C_2', \cdots, C_{n'}', C_1'', C_2'', \cdots, C_{n''}''$ at these points are parallel. Then Q lies on one of the circles C_i.*

The possibility that the circles S_1 and S_2 coincide is readily treated. We may choose η_1 and η_2 to coincide on S_1 and on S_2, and on PQ. These points are still on the boundaries of their respective loci, and hence the previous reasoning is valid.

According to the assumptions already made, the locus S_1 cannot be a point. The locus S_2 will be a point when and only when the roots of f_2 are concentrated at P or P' or both. But in such a case the single point S_2 is on the line PQ and the preceding reasoning holds.

The possibility that S_1 or S_2 may be the entire plane remains to be considered. If one of these loci is the entire plane, the other must be a point; otherwise we have essentially the case of overlapping already disposed of. The locus S_2 is either the point P' or does not contain P', so is never the entire plane. If S_1 is the entire plane, we may suppose S_2 to be a point which of course lies on PQ. We prove our former result by a limiting process. When a point Q' is very near Q but external to the locus of the roots of J, the circles $S_1', S_2', \cdots, S_1$ are very near the corresponding circles for Q; for Q' the locus S_1 is certainly not the entire plane. Denote by Σ_2 the point at which is located the single particle representing all the roots of f_2, so far as concerns the force at Q'; Σ_2 is not in the locus S_1. When Q' approaches Q always remaining exterior to the locus of the roots of J, Σ_2 approaches the point S_2. The circle S_1 corresponding to Q' becomes smaller and smaller, the locus S_1 never contains Σ_2, so the circle S_1 approaches the point S_2. We may choose η_1 an intersection with PQ' of the circle S_1 corresponding to Q', and we shall have the points $\zeta_1, \cdots, \zeta_{n'}$ on PQ' and on the circles C_1', $\cdots, C_{n'}'$. When Q' approaches Q, PQ' approaches PQ, the point η_1 approaches S_2 and the points $\zeta_1, \cdots, \zeta_{n'}$ approach points on PQ and on C_1', $\cdots, C_{n'}'$. These limiting points can be taken as corresponding to η_1 coinciding with S_2 and thus give our result that Q lies on one of the circles $C_1, \cdots, C_n$.

Theorem VI is now completely proved except for its last sentence. When we notice the number of roots of $\bar{J}$ in a region C_i and remark that if the roots of f_1 and f_2 are varied continuously then the roots of J vary continuously, and that if C_i is exterior to every other circle C_j no root can enter or leave C_i, this last sentence is seen to be true. It hardly need be added that a number of circles C_i which may have common points but which have no point in common with any other circle C_j contain a number of roots of J equal to the sum of the multiplicities m_i corresponding to their centers as roots of $\bar{J}$.

* The mere fact that the tangents at these points are parallel does not rule out certain isolated points Q on the line $P\alpha_1' \cdots \alpha_{n'}' \alpha_1'' \cdots \alpha_{n''}''$ but a more detailed consideration of the loci $S_1', S_2', \cdots$ does rule them out without difficulty.

11. **Generalization of Theorem VI by transformation.** In Theorem VI we have assumed that P is interior to none of the circles $C_1', \cdots, C_{n'}', C_1'', \cdots, C_{n''}''$. The theorem is still true if this assumption is omitted, even if we permit roots of one or both forms to lie at infinity, except that the locus of the roots of J may be the entire plane, and will surely be the entire plane if P is interior to or on the circles $C_1', \cdots, C_{n'}', C_1'', \cdots, C_{n''}''$, and if no roots of either ground form are constrained to lie at P'. The proof as given requires only a few minor modifications to apply to this new configuration. If the circles $C_1', \cdots, C_{n'}', C_1'', \cdots, C_{n''}''$ all have the common center P, the circles bounding the locus of the roots of J are not determined by the position of their centers as described in the statement of Theorem VI, but are to be determined for example by their points of intersection with an arbitrary line through P, precisely as we considered the points μ_i' and μ_i'' in § 8.

Theorem VI has the advantage over Theorem II of being entirely symmetrical with respect to the two forms f_1 and f_2. The special case of Theorem VI where there are two circles C_1', C_2' and merely one circle C_1'' leads to merely one circle C_1 distinct from the three original circles. For this case, Theorems II and VI give the same result. But of course Theorem II is more general than this particular situation. Thus, the result for the jacobian problem of the theorem stated in II, pp. 114–115, or indeed of the problem of § 9 where Q is interior to C_2' is included in Theorem II but not in Theorem VI. There is, however, a general theorem concerned with an indefinite number of circular regions $C_1', \cdots, C_{n'}', C_1'', \cdots, C_{n''}''$ which generalizes all possible situations of Theorem II and which is to be proved in §§ 13–15 by the methods we have been using.

Theorem VI as stated is not invariant under linear transformation. If we perform such a transformation we obtain the following new result:

THEOREM VII. *If the loci of the roots of f_1 are circular regions bounded by circles each of which is tangent internally to a circle L_1 and externally to a circle L_2 (tangent to both L_1 and L_2 internally) and if the loci of the roots of f_2 are circular regions bounded by circles each of which is tangent externally to L_1 and internally to L_2 (tangent to both L_1 and L_2 externally), and if these loci are so related to their bounding circles that they contain neither the entire circle L_1 nor the entire circle L_2, then the locus of the roots of the jacobian J of f_1 and f_2 is a number of circular regions bounded by circles each tangent internally to L_1 and externally to L_2 or tangent externally to L_1 and internally to L_2 (tangent to L_1 and L_2 internally or to L_1 and L_2 externally) and such that each region is so related to its bounding circle that it contains neither the entire circle L_1 nor the entire circle L_2. The exact location of the circles bounding the locus of the roots of J can be determined by allowing the roots of the ground forms to coincide on L_1 or on L_2, always remaining in their proper loci; the roots of J are the points of tangency with L_1*

and L_2 of the circles desired. The number of roots of J in any of these circular regions having no point in common with any other of these regions is the multiplicity as a root of J of the points of tangency of its boundary with L_1 and L_2 under these conditions.

There is a limiting case of this theorem yet different from Theorem VI where L_1 is a straight line and L_2 a proper circle, but we shall not state the result in detail.

12. **Application of Theorem VI to the roots of the derivatives of polynomials.** Theorem VI and in fact Theorem VII can be used to obtain results concerning the roots of the derivative of a rational function by means of the remark made in I, p. 297 (or II, p. 114). We thus consider $C'_1, \cdots, C'_{n'}$ as the loci of the roots and $C''_1, \cdots, C''_{n''}$ as the loci of the poles of the given function. The regions $C_1, \cdots, C_n$ together with the possibility of the point at infinity appear as the loci of the roots of the derivative. There is a peculiar difference, however, between this locus and the corresponding locus of the roots of the jacobian. A region C''_i which is the locus of more than one pole of the original function is the locus of at least one root of the derivative, with the exception that no point of the bounding circle C''_i can be a root of the derivative unless it is interior to another region C''_j; we leave to the reader the verification of this statement.

We shall dwell at some length on an application of the above remark applied to Theorem VI, which is indeed a special case of that theorem, concerning the derivative of a polynomial:

THEOREM VIII. *Let the interiors and boundaries of circles $C_1, \cdots, C_n$ whose centers are $\alpha_1, \cdots, \alpha_n$ be the loci of $m_1, \cdots, m_n$ roots respectively of a polynomial $f(z)$ which has no other roots; suppose these circles to have a common external center of similitude P actually exterior to all these circles. Denote by $g(z)$ the polynomial $f(z)$ when all its roots are concentrated at the centers of their proper circles, and denote by $\alpha'_1, \cdots, \alpha'_{n'}$ the distinct roots of its derivative $g'(z)$, of respective multiplicities $m'_1, \cdots, m'_{n'}$. Then the locus of the roots of $f'(z)$ is composed of the interiors and boundaries of the circles $C'_1, \cdots, C'_{n'}$ whose centers are $\alpha'_1, \cdots, \alpha'_{n'}$ and whose radii are such that P is a common external center of similitude for the circles $C_1, \cdots, C_n, C'_1, \cdots, C'_{n'}$. A circle C'_i which has no point in common with another circle C'_j contains m'_i roots of $f'(z)$.*

An extreme degenerate case of this theorem is when all the C_i are null circles, and $f(z)$ is identical with $g(z)$. The case of merely two circles brings us back to a theorem given in II, p. 115.

Inasmuch as the circles C'_i have P as a common external center of similitude, Theorem VIII can be applied again to the polynomial $f'(z)$ and shows that the roots of $f''(z)$ lie on or within certain circles $C''_1, \cdots, C''_{n''}$. The most obvious consideration of the geometric situation shows that any point on or within one of these circles actually is a point of the locus.

It is to be noticed, however, that this reasoning can be used only if we know that the circles C'_i contain respectively m'_i roots of $f'(z)$, and it is pointed out in § 4 that one of these circles does not necessarily contain precisely that number of roots of $f'(z)$ and in fact may contain no such root. Thus the reasoning can be used only if we know that the circles C'_i are mutually external. This is always the case for a given set of values of $\alpha_1, \cdots, \alpha_n$ if the circles C_i are sufficiently small, so in the following theorem we require that the circles C_i be sufficiently small. This means, more explicitly, that the theorem is true for a definite derivative $g^{(k)}_{(z)}$ if the circles C_i are so small that no circle $C^{(m)}_i$ has a point in common with a circle $C^{(m)}_j$ $(i \neq j)$, for $m = 1, 2, \cdots, k-1$.

Thus we have

THEOREM IX. *Let the interiors and boundaries of circles $C_1, \cdots, C_n$ whose centers are $\alpha_1, \cdots, \alpha_n$ be the loci of $m_1, \cdots, m_n$ roots respectively of a polynomial $f(z)$ which has no other roots; suppose these circles to have a common external center of similitude P actually exterior to all these circles. Denote by $g(z)$ the polynomial $f(z)$ when all its roots are concentrated at the centers of their proper circles, and denote by $\alpha^{(k)}_1, \cdots, \alpha^{(k)}_{n(k)}$ the distinct roots of its kth derivative $g^{(k)}(z)$, of respective multiplicities $m^{(k)}_1, \cdots, m^{(k)}_{n(k)}$. Then if the circles C_i are sufficiently small the locus of the roots of $f^{(k)}(z)$ is composed of the interiors and boundaries of the circles $C^{(k)}_1, \cdots, C^{(k)}_{n(k)}$ whose centers are $\alpha^{(k)}_1, \cdots, \alpha^{(k)}_{n(k)}$, and whose radii are such that P is a common external center of similitude for the circles $C_1, \cdots, C_n, C^{(k)}_1, \cdots, C^{(k)}_{n(k)}$. A circle $C^{(k)}_i$ which has a point in common with no other circle $C^{(k)}_j$ contains precisely $m^{(k)}_i$ roots of $f^{(k)}(z)$.*

The special case of this theorem where there are but two of the original circles C_1 and C_2 has already been proved by another method.* For this special case we make no restriction on the size of the circles C_i.

A limiting case of Theorem IX is that P is infinite but the points $\alpha_1, \cdots, \alpha_n$ finite, and the radii of $C_1, \cdots, C_n$ finite. The circles $C_1, \cdots, C_n$ are then all equal. The theorem is true for this limiting case. In fact, suppose a root R of $f^{(k)}(z)$ to be exterior to all the circles $C^{(k)}_1, \cdots, C^{(k)}_{n(k)}$. We can choose circles $S_1, \cdots, S_n$ having a finite point P as common external center of similitude and such that R is also exterior to all the circles $S^{(k)}_1, \cdots, S^{(k)}_{n(k)}$ corresponding. This shows that every point of the locus is on or within $C^{(k)}_1, \cdots, C^{(k)}_{n(k)}$; the converse is easily seen from translation of the situation for $g(z)$ and $g^{(k)}(z)$. This result may be expressed somewhat loosely as follows:

THEOREM X. *If the loci of the roots of a polynomial are the interiors and boundaries of sufficiently small equal circles whose centers lie on a line L, the locus of the roots of the kth derivative $f^{(k)}(z)$ consists of the interiors and bound-*

* Walsh, Paris Comptes Rendus, vol. 172 (1921), pp. 662–664.

aries of circles equal to these whose centers also lie on L and depend only on the centers of the original circles.

This new theorem for $k = 1$ is not a special case of Theorem VI and can easily be expressed in a form invariant under linear transformation, thus giving a new result for the jacobian of two binary forms (compare § 14) and for the derivative of a rational function.*

The approximate determination of the roots of the jacobian of two binary forms, of the derivative of a rational function, or of any derivative of a polynomial is thus made, by Theorems VI–X, to depend essentially on the determination of the roots of the jacobian, of the derivative of a rational function, or of any derivative of a polynomial which has all its roots real.

The extreme simplicity of Theorem X immediately raises the question of the truth of that theorem if the supposition of the collinearity of $\alpha_1, \cdots, \alpha_n$ is omitted. We can easily prove that the theorem is not true under this changed hypothesis by means of the remark of § 6. It is surely true under the changed hypothesis that every point on or within a circle C_i' that is equal to the C_i and whose center is α_i' is a point of the locus, but it is not true without further restrictions that the locus consists precisely of the points on and interior to the circles C_i'.

Consider a polynomial $g(z)$ with three simple roots $\alpha_1, \alpha_2, \alpha_3$, which are not collinear, so that neither root α_1', α_2' of $g'(z)$ is collinear with a pair of the points $\alpha_1, \alpha_2, \alpha_3$. Choose the equal circles C_1, C_2, C_3, with centers $\alpha_1, \alpha_2, \alpha_3$, of such small radius that for no possible choice of the points in their proper loci can we have two roots β_i and β_j of $f(z)$ collinear with a root β_k' of $f'(z)$. Suppose β_1' to be on C_1'; we choose β_i of such a nature that $\beta_i - \alpha_i = \beta_1' - \alpha_1'$. The circle C through $\beta_1, \beta_2, \beta_1'$ is not a straight line, the points β_1 and β_2 cannot satisfy the requirements of § 6 with regard to the circles C, C_1, and C_2, from which it follows that β_1' cannot be on the boundary of its locus.

13. **Theorem XI: an extension of Theorems II and VI.** We now come to the proof of the general theorem mentioned in § 11, which includes Theorem II as well as Theorem VI:

THEOREM XI. *Let f_1 and f_2 be two binary forms, and let circular regions $C_1', C_2', \cdots, C_{n'}'$; $C_1'', C_2'', \cdots, C_{n''}''$ be the respective loci of $m_1', m_2', \cdots, m_{n'}'$ roots of f_1 (which has no other roots) and of $m_1'', m_2'', \cdots, m_{n''}''$ roots of f_2 (which has no other roots). Suppose there is a family of coaxial circles S each of which cuts at the same angle all the circles C_i' which bound loci interior to them, and at*

* (Added in proof): It seems to the writer probable that Theorem IX is true with no restriction on the size of the circles C_i. The special case of this more general theorem where the circular regions C_i are half planes is for the case $k = 1$ contained by a limiting process in Theorem IX, and for all values of k has been established by Mr. B. Z. Linfield, in a paper to be published in these T r a n s a c t i o n s.

the supplementary angle all the circles C'_i which bound loci exterior to them, and which cuts at this same angle all the circles C''_j which bound loci exterior to them and at the supplementary angle all circles C''_j which bound loci interior to them. Then the locus of the roots of the jacobian of f_1 and f_2 is a number of circular regions bounded by circles $C_1, C_2, \cdots, C_n$ each of which is cut by every circle S at an angle equal or supplementary to the angles cut by S on C'_i and C''_j; the regions which are the loci of the roots of the jacobian may be either internal or external to their bounding circles. The circles $C_1, C_2, \cdots, C_n$ are included among the circles traced by the roots of the jacobian when all the roots of f_1 and f_2 are concentrated on the circles bounding their proper loci and move so that one of the circles S constantly passes through them all, while the lines tangent to these bounding circles C'_i, C''_j at the points which are the roots of f_1 and f_2 (and the lines tangent to C_k at the points which are the roots of the jacobian) all become parallel when S is transformed into a straight line. Any region C_i having no point in common with any other region C_j contains a number of roots of the jacobian equal to the multiplicity of the root of the jacobian which traces that circle C_i under these conditions.

We shall first undertake to prove this theorem for the simplest case, namely, that the circles S form a coaxial family having no point common to all those circles. We transform so that the circles S have a common center P. If the given circles C'_i, C''_j are all straight lines, all the regions C'_i, C''_j have a common point, the locus of the roots of the jacobian is the entire plane, and the theorem is proved. In any other case, all the circles $C'_1, \cdots, C'_{n'}, C''_1, \cdots, C''_{n''}$ are equal; the loci corresponding to the former can be considered to lie inside the bounding circles and those corresponding to the latter to lie outside the bounding circles.

We shall use the same method of proof as was used in § 9, namely, the replacing of all the particles in the field of force corresponding to the roots of f_1 by a single particle η_1, the replacing of all the particles corresponding to the roots of f_2 by a single particle η_2, and the study of the loci of η_1 and η_2. We shall prove that no matter what may be the location of the circular regions which are the loci or the distribution of the roots of the ground forms among these loci, *the locus of the roots of the jacobian is always the entire plane.*

If a point z is exterior to all the circles C'_i, C''_j and exterior to all the circles S which actually cut those circles C'_i, C''_j, then z is a root of the jacobian. For if $n'' > 1$, z lies in C''_1 and C''_2, may lie at a multiple root of f_2, and hence is a point of the locus. If $n'' = 1$, replace the particles ζ_1 and ζ_2 of masses m'_1 and m'_2 whose loci are C'_1 and C'_2 by a single equivalent particle ζ of mass $m'_1 + m'_2$ whose locus is a circular region S'_1. Then the circle S'_1 is larger than C'_1; this follows from the fact that there are two circles through z tangent to S'_1, C'_1, C'_2, and having the same kind of contact with all three of these circles; neither intersection of those two tangent circles with each other separates any two of the points of tangency of the circles S_1', C_1', C_2'. The locus of ζ is the *interior* and boundary of S_1'. If $n' > 2$, we now replace ζ and the particle ζ_3 whose mass is m_3' and whose locus is C_3' by an equivalent particle ζ' of mass $m_1' + m_2' + m_3'$. The locus S_2' of ζ' is the interior of a circle which does not contain z and which is larger than C_3', by the reasoning just used. We continue in this way and finally replace all the particles representing the roots of f_1 by a single equivalent particle η_1 whose locus S_1 is larger than the circles C_i' (or if $n'' = 1$, equal to them). In any case, the region S_1 has at least one point in common with the region C_1'', so z is a point of the locus.

Denote by Σ_1 the larger and by Σ_2 the smaller of the two circles of the family S which are tangent to C_i' and C_j''. If z is interior to Σ_2, and if Σ_2 is exterior to C_i', C_j'', the reasoning just given applies with practically no change. If $n'' = 1$, the locus of η_1 which represents all the roots of f_1 is a region S_1 whose bounding circle is larger than the circles C_i', so the region S_1 must have at least one point in common with the region C_1'', and z is a point of the locus.

Let us now consider a point z between Σ_1 and Σ_2 under the assumption that Σ_2 is not interior to the circles C_i', C_j''. If $n'' = 1$, we find as before circles S_1', S_2', $\cdots$, S_1 all larger than C_1'. In fact, we need consider only points z interior to or on at most one circle C_i'. Describe a circle Σ through z and through the points of tangency of Σ_1 with C_1' and C_2'. When ζ_1 and ζ_2 lie at these two points, the point ζ corresponding lies on Σ, and is such that z and ζ separate ζ_1 and ζ_2. Hence ζ is exterior to Σ_1. Similarly there is a point ζ interior to Σ_2, so S_1' is indeed larger than C_1'. Thus we find z to be a point of the locus, for S_1 and C_1'' have a common point.

If $n'' > 1$, we need consider only points z interior to at most one circle C_i' and exterior to at most one circle C_j''. The circles S_1', S_2', $\cdots$, S_1 are all larger than C_1' (or equal to C_1' if $n' = 1$). The region S_1'' which is the locus of ξ, the particle equivalent to the particles ξ_1 and ξ_2 whose loci are C_1'' and C_2'', is a circular region which contains all the region common to C_1'' and C_2''. Hence the circle S_1'' is smaller than C_1''. The region S_1 which is the locus of the particle η_1 representing all the roots of f_1 is larger than each circular region C_i'. The region S_2 which is the locus of the particle η_2 representing all the roots of f_2 is bounded by a circle smaller than the bounding circle of each region C_i'', so S_1 and S_2 have at least one common point and z is a point of the locus.

It remains to consider points z interior to Σ_2, if Σ_2 is interior to C_i' and C_j'', but this treatment is so similar to the results already given that it is omitted. It remains also to consider points z on Σ_1 and on Σ_2, but since all other points of the plane are points of the locus and the locus of the roots of the jacobian is a closed point set, these points also belong to the locus. Theorem XI is now completely proved if the circles S have no point in common.

14. **Theorem XI, proof continued.** We next undertake to prove Theorem

XI for the case that the circles S form a coaxial family of circles all tangent at a single point P, which point we transform to infinity. If none of the original circular regions $C_1', \cdots, C_{n'}', C_1'', \cdots, C_{n''}''$ is the point at infinity, our results just proved for the case of circles S having no point in common hold without essential change; the entire plane is the locus of the roots of the jacobian. But P may be considered a null circle, the locus of a number of roots of f_1 and of f_2; in this case the entire plane need not be the locus of the roots of the jacobian.

If all the roots of f_2 are concentrated at P, then either all the roots of f_1 are also concentrated at P and the locus of the roots of the jacobian is the whole plane, or there are a number of fixed equal circles $C_1', \cdots, C_{n'}'$ bounding loci interior to them. In this latter case the field of force is precisely the field corresponding to Theorem X, so for this case Theorem XI is already proved. The case that there is at least one finite circle C_j'' requires some further consideration.

Denote by Σ_1 and Σ_2 the lines which belong to the coaxial family S and which are tangent to all the circles $C_1', \cdots, C_{n''}''$ and transform so that Σ_1 and Σ_2 are horizontal, with Σ_1 above Σ_2. Any point z on the boundary of the locus of the roots of the jacobian must lie on one of the circles traced by the roots of the jacobian when the roots of the ground forms trace the boundaries of their respective loci all constantly lying on one variable circle S and tracing the circles $C_1', \cdots, C_{n''}''$ in the same sense; this is the location of the roots of the ground forms described in the statement of Theorem XI. This fact is proved precisely as in §§ 9, 10, if z lies on a circle S which actually cuts the circles $C_1', \cdots, C_{n''}''$. We replace the particles at the roots of f_1 by a single equivalent particle η_1, the particles at the roots of f_2 by a single equivalent particle η_2, and notice that when z is on the boundary of its locus the loci of η_1 and η_2 cannot overlap. To complete the proof of Theorem XI in our special case it is sufficient to consider points z say above Σ_1 and to prove that all such points are points of the locus of the roots of the jacobian.

If z is a point above Σ_1 and if there are two or more finite circles C_j'', z is common to two or more of those circular regions and is therefore a point of the locus. If there is but one circle C_j'' other than at infinity, further consideration is required.

The locus of η_2 is the exterior of a circle S_2 obtained from C_1'' by similarity transformation with z as center, and S_2 is farther from z than is C_1''. Thus if there is but one finite circle C_1', the locus of η_1 is the interior of a circle S_1' obtained from C_1' by similarity transformation with z as center, and the loci S_1' and S_2 must have at least one common point, so z belongs to the locus.

If there are two finite circles C_i', we replace the particles whose loci are C_1' and C_2' by a single equivalent particle whose locus is S_1', and then replace that particle and the particles at infinity which represent the roots of f_1 by a single equivalent particle whose locus is S_1. There are two circles through z tangent to S_1', C_1', C_2'; one of these circles contains S_1', C_1', C_2' and the other contains none of these circles. The external tangents to S_1' and C_1' intersect on the horizontal line through z, the radical axis of the circles through z tangent to S_1', C_1', C_2'.* Hence there is a circle Σ tangent to Σ_1 and Σ_2 and such that Σ and S_1' have z as common external center of similitude. It follows that the regions S_1 and S_2 have at least one common point. In fact, if there is no line through z which cuts both C_1'' and S_1', S_1 is entirely interior to S_2. If there is a line through z which cuts both C_1'' and S_1', a line through z and tangent to C_1'' cuts S_1' and S_1 and lies wholly in S_2. Thus z is a point of the locus.

If there are three finite circles C_i', we find S_1' as before; the external center of similitude of S_1' and C_3' lies on the horizontal line through z, so that line is the radical axis of the two circles through z and tangent to S_1', S_2', C_3'; one of these tangent circles contains S_1', S_2', C_3', and the other contains none of these three circles. Then the external center of similitude of S_2' and C_3' lies on the horizontal line through z, and as before we find that S_1 and S_2 have at least one common point. This reasoning is general for any number of circles C_i'.

Every point z above Σ_1 and hence every point below Σ_2 is a point of the locus. We may show either by similar reasoning or as in § 13 that every point on either of these lines belongs to the locus.

Theorem XI is thus proved for circles S all tangent at a single point. It is worth while, perhaps, to point out explicitly that there actually exist situations with one or more circles C_j'', and where the locus of the roots of the jacobian is not the entire plane. Thus, let there be merely two finite circles C_j'' and let z be interior to both of them. Then S_1'' is a region exterior to a circle which surrounds z. If z is exterior to all the circles C_i' and if the locus S_1 is interior to its bounding circle, it is possible so to choose the number of roots of f_2 at infinity that S_2 shall be the region exterior to a circle which entirely contains S_1. Then S_1 and S_2 have no common point and z is not a point of the locus of the roots of the jacobian.

15. **Theorem XI; completion of the proof.** The case that the circles S of Theorem XI form a coaxial family of circles through two distinct points P and P' remains to be dealt with. Transform P' to infinity. The points P and P' are considered as null circles and hence allowed to be loci of a number of roots of f_1 or f_2 or both. As in § 8 we may assume that there is at least one circle C_i' or C_j'' distinct from P and P'.

If the circles $C_1', \cdots, C_{n''}''$ surround P, Theorem XI can be proved precisely as in §§ 8–10. If $C_1', \cdots, C_{n''}''$ do not surround P, these same methods show that no point z is on the boundary of its locus unless z is on one of the circles

* Coolidge, *A Treatise on the Circle and the Sphere*, p. 111, Theorem 217.

described in the theorem, provided that there is a circle S through z which actually cuts all the circles $C_1', \cdots, C_{n''}'''$. If all the finite regions $C_1', \cdots, C_{n''}'''$ are interior to their bounding circles, the theorem is Theorem VI and completely proved. If two or more of these regions are exterior to their bounding circles, every point z not on a circle S which cuts all the circles $C_1', \cdots, C_{n''}'''$ is a possible position of pseudo-equilibrium and hence a point of the locus. It remains to consider the case of such points z with merely one finite region, say C_1', exterior to its bounding circle. Let the line of centers of the circles $C_1', \cdots, C_{n''}'''$ be horizontal and denote by Σ_1 and Σ_2 the common tangents to $C_1', \cdots, C_{n''}'''$. Let C_1' lie to the left of P. We shall phrase the proof for $n' > 1, n'' > 1$.

The particles ζ_1 and ζ_2 whose loci are C_1' and C_2' are to be replaced by a particle ζ whose locus is a circular region S_1'. There are two circles through z tangent to C_1', C_2', S_1', one of which includes C_1' but not C_2', the other of which includes C_2' but not C_1'. If the locus S_1' is not the entire plane, it follows from a simple consideration of points ζ_1, ζ_2, ζ on the circle through z orthogonal to C_1' and C_2' that these two tangent circles include S_1' and exclude S_1' respectively. If the locus S_1' is the entire plane, the loci $S_2', S_3', \cdots, S_1$ are all the entire plane and z is a point of the locus.

These two tangent circles intersect on the line Pz,* and the circle S_1' lies to the left of Pz. The external center of similitude of C_1' and S_1' and the internal center of similitude of C_2' and S_1' lie on Pz. It is thus true that the external center of similitude of S_1' and any circle C_j'' lies on Pz and that the internal center of similitude of S_1' and any circle C_i' other than C_1' lies on Pz.

We now replace ζ and ζ_3, the particle whose locus is C_3', by a single equivalent particle ζ' whose locus is a circular region S_2'. If the locus S_2' is not the entire plane, there are two circles through z which intersect on Pz and which are tangent to S_1', C_3', S_2'; one of these tangent circles contains S_1' and S_2' but does not contain C_3', the other contains C_3' but neither S_1' nor S_2'. We continue in this way and finally reach a circle S_1 which bounds the locus of the point η_1 which represents all the roots of f_1; the locus of η_1 is exterior to S_1. The external center of similitude of S_1 and any of the finite circles C_j'' (and also of C_1') lies on Pz, and the internal center of similitude of S_1 and any of the finite circles C_i' except C_1' lies on Pz.

Similarly the particles representing the roots of f_2 are replaced by a single equivalent particle η_2 whose locus is the interior of a circle S_2 such that the external center of similitude of S_2 and any of the finite circles C_j' lies on Pz.

Hence the external center of similitude of S_1 and S_2 lies on Pz, from which it follows as before that there is at least one point common to S_1 and S_2, so z is a point of the locus. Likewise all points of Σ_1 and Σ_2 are points of the locus.

* Coolidge, loc. cit.

As in § 14, cases actually arise here where all the regions $C'_1, \cdots, C''_{n''}$ are not within their finite bounding circles and yet the locus of z is not the entire plane; the proof is as in § 14.

The number of roots of the jacobian in a region C_i which has no point in common with any other region C_i which is a part of the locus of the roots of the jacobian can be determined as in § 10 for Theorem VI; the proof of Theorem XI is now complete.

The determination in Theorem XI of whether or not a given circle C_i is actually a part of the boundary of the locus of the roots of the jacobian, and if so whether the circular region corresponding lies interior or exterior to C_i, can be made in any given case by the methods developed in the present chapter.

Theorem XI has obvious applications which will easily be made by the reader to the study of the location of the roots of the derivative of a polynomial and of the derivative of a rational function.

CHAPTER III: ON CENTERS OF GRAVITY

16. **The loci of certain centers of gravity.** There is a striking analogy between some of our results concerning the location of the roots of the jacobian of two binary forms and results which are easily proved concerning the location of the center of gravity of a number of particles. Thus, the fact that if a number of positive particles lie in a circle their center of gravity also lies in that circle is analogous to Lemma I (II, p. 102) and was used in the proof of that lemma, and is also analogous to the theorem of Lucas. From this fact and Theorem VIII of III we prove the analogue of a theorem given in II, p. 115 (= Theorem I of S) precisely as that theorem was proved:

THEOREM XII. *If the interiors and boundaries of two circles C_1 and C_2 of centers α_1 and α_2 and radii r_1 and r_2 are the loci respectively of m_1 and m_2 unit positive particles, then the locus of the center of gravity of these particles is the interior and boundary of the circle C whose center is*

$$\frac{m_1\alpha_1 + m_2\alpha_2}{m_1 + m_2}$$

and whose radius is

$$\frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}$$

The three circles C_1, C_2, C have as common external center of similitude the point

$$\frac{r_1\alpha_2 - r_2\alpha_1}{r_1 - r_2}.$$

Theorem XII can be largely extended by the method of proof used for III, Theorem VIII in S:

THEOREM XIII. *If the interiors and boundaries of n circles C_i, whose centers are α_i and radii r_i, are the loci respectively of m_i unit positive particles, then the locus of the center of gravity of these particles is the interior and boundary of the circle C whose center is*

$$\alpha = \frac{m_1 \alpha_1 + m_2 \alpha_2 + \cdots + m_n \alpha_n}{m_1 + m_2 + \cdots + m_n}$$

and whose radius is

$$r = \frac{m_1 r_1 + m_2 r_2 + \cdots + m_n r_n}{m_1 + m_2 + \cdots + m_n}.$$

Denote the m_i particles in or on C_i by $z_1^{(i)}, z_2^{(i)}, \cdots, z_{m_i}^{(i)}$, so that $|z_j^{(i)} - \alpha_i| \leqq r_i$ for every i and j. The center of gravity of all the $m_1 + m_2 + \cdots + m_n$ particles is

$$z = \frac{(z_1^{(1)} + z_2^{(1)} + \cdots + z_{m_1}^{(1)}) + \cdots + (z_1^{(n)} + z_2^{(n)} + \cdots + z_{m_n}^{(n)})}{m_1 + m_2 + \cdots + m_n},$$

so that we have

$$z - \alpha = \frac{\begin{array}{l}[(z_1^{(1)} - \alpha_1) + (z_2^{(1)} - \alpha_1) + \cdots + (z_{m_1}^{(1)} - \alpha_1)] + \cdots \\ \qquad + [(z_1^{(n)} - \alpha_n) + (z_2^{(n)} - \alpha_n) + \cdots + (z_{m_n}^{(n)} - \alpha_n)]\end{array}}{m_1 + m_2 + \cdots + m_n}$$

and hence z is on or within C.

Conversely, if z is given on or within C, we determine $z_j^{(i)}$ by the relation

$$z_j^{(i)} - \alpha_i = (z - \alpha)\frac{r_i}{r},$$

and we have the $z_j^{(i)}$ satisfying the proper conditions. The proof is thus complete. It may be remarked that when the $z_j^{(i)}$ trace their proper circles in such a manner that $(z_j^{(i)} - \alpha_i)/r_i$ is the same for every i and j, then this common value is equal to $(z - \alpha)/r$ while z traces its circle C.

Theorem XIII can be extended without difficulty in various directions: to particles of negative or even complex mass; to space of any number of dimensions; to give a result which shall be invariant under linear transformation; to regions other than the interiors of circles, especially convex regions. In this last extension, use is made of the fact that if m_i particles lie in a convex region their center of gravity also lies in that region; hence such results as III, Theorem IX can be applied.

There is much more than a mere analogy between Theorems XII and XIII for centers of gravity and our previous results concerning the derivatives of polynomials. In fact, the only root of the $(n - 1)$st derivative of a polynomial of degree n lies at the center of gravity of the roots of that polynomial. When viewed in this light, Theorems XII and XIII are results relating to the location of the roots of the derivatives of a polynomial even if not of the jacobian of two binary forms, and are conceived in precisely the same spirit as is Theorem IX. Thus the entire discussion of § 5 holds practically without change if we consider the problem of determining the locus of the roots of the kth derivative of a polynomial of degree n whose roots have certain assigned circular regions as their loci. Theorem XIII gives the complete solution of that problem for $k = n - 1$ if the assigned circular regions are interior to their bounding circles.

As a particular case of Theorem XIII, the fact that if a number of particles lie in a convex region their center of gravity also lies in that region follows from the theorem of Lucas* as applied successively to the various derivatives of a polynomial.

17. **The center of gravity of the roots of the derivative of a rational function and of the jacobian of two binary forms.** The center of gravity of any set of points has interesting properties with reference to that point set. It furnishes, for example, an approximate idea of the location of those points. Any line through the center of gravity either passes through all the points of the set or separates at least two of them.† We shall now find some results connecting the centers of gravity of related polynomials of the sort we have been considering. A classical theorem of this nature follows from a remark previously made:

THEOREM XIV. *The center of gravity of the roots of a polynomial coincides with the center of gravity of the roots of the derived polynomial.*

We derive the corresponding result for a rational function, which we take in the form

$$f(x) = \frac{x^m + a_0 x^{m-1} + a_1 x^{m-2} + \cdots + a_{m-1}}{x^n + b_0 x^{n-1} + b_1 x^{n-2} + \cdots + b_{n-1}};$$

$$f'(x) = \frac{\begin{array}{r}(x^n + b_0 x^{n-1} + \cdots)(mx^{m-1} + (m-1) a_0 x^{m-2} + \cdots) \\ -(x^m + a_0 x^{m-1} + \cdots)(nx^{n-1} + (n-1) b_0 x^{n-2} + \cdots)\end{array}}{(x^n + b_0 x^{n-1} + \cdots)}.$$

If we denote by α the center of gravity of the finite roots of $f(x)$ and by β that of the finite poles of $f(x)$, if $m \neq n$ and if $f(x)$ has no finite multiple poles, we have for the center of gravity of the finite roots of $f'(x)$ the formula

$$\gamma = -\frac{(m-n-1) a_0 + (m-n+1) b_0}{(m+n-1)(m-n)}$$

$$= \frac{m(m-n-1)\alpha + n(m-n+1)\beta}{(m+n-1)(m-n)},$$

* On the other hand, Lemma I (II, p. 102) enables us to use this fact to give immediately a very simple proof of the theorem of Lucas.

† An application of this fact to the more precise location of the roots of algebraic equations is given by Laguerre, *Œuvres*, vol. I, pp. 56, 133.

which is a point collinear with α and β. If α and β coincide, γ coincides with them; if $m = n + 1$, γ coincides with β; if $m = n - 1$, γ coincides with α. If $n = 0$, we have Theorem XIV. If $m = n$, γ cannot be expressed in terms of merely α and β, as is found simply by computing γ.

Let us inquire in what respect this work on centers of gravity can be made invariant under linear transformation and can be applied to the jacobian of two binary forms.

The concept *center of gravity* is surely not invariant under linear transformation. In fact, given any two distinct points of the plane ξ and η, any third point ζ of the plane can, by a suitable transformation, be made to correspond to the center of gravity of the transformed ξ and η. We need simply to transform to infinity the harmonic conjugate of ζ with respect to ξ and η.

We cannot expect to obtain results with the ordinary definition of center of gravity, so we introduce a new definition. The point G is said to be the *centroid of a set of points with respect to P* if when P is transformed to infinity G transformed into the center of gravity of the points corresponding to the original set. We suppose that P is not a point of that set. It should be noted by way of justification of the definition that the point G is uniquely defined, since the center of gravity is invariant under similarity transformation. The relation between the points P and G is not reciprocal.

The centroid with respect to a point of a set of points gives a rough indication of the distribution of that set of points, like the ordinary center of gravity. In particular, if P is external to a circular region containing the set of points, G is also in that circular region. In fact, examination of the proof of Lemma I (II, p. 102) will show that the force at a point P external to a circular region C due to k particles in C is equivalent to the force at P due to k particles which coincide at a point Q in C, and Q is the centroid of the k particles with respect to P. Thus we are studying the relation between P, Q, and the k particles, which is the same as Laguerre's relation set up between those points, referred to in § 2.

Let f_1 and f_2 be two binary forms, of respective degrees p_1 and p_2, and let the point P at infinity be a k-fold root of f_1. Let α, β, γ be the centroids with respect to P of the roots of f_1 other than P, the roots of f_2 (all of which are supposed finite), the finite roots of the jacobian of f_1 and f_2, respectively. We easily find that

$$\gamma = \frac{(p_1 - k)(k + 1)\alpha - (p_1 - kp_2)\beta}{k(p_1 + p_2 - k - 1)},$$

a point collinear with α and β. If α and β coincide, γ coincides with them; if $p_1 = kp_2$, $\gamma = \alpha$; if $p_1 = k$, $\gamma = \beta$, which is Theorem XIV. Always we shall have*

* It might seem at first sight that this cross ratio should be p_1 / k, since by Lemma II

$$(P, \alpha, \beta, \gamma) = \frac{(p_1 - k)(k+1)}{k(p_1 + p_2 - k - 1)},$$

which expresses the entire result in invariant form.

Chapter IV: On the roots of the jacobian of two real forms

18. **The locus of the roots of the derivatives of a polynomial whose roots are real.** The present chapter is devoted mainly to general theorems of the kind developed in Chapter II, but where we restrict ourselves to ground forms whose coefficients are real or can be made real by suitable linear transformation. We are placing additional restrictions on our ground forms, so it is to be expected that some additional properties will appear.

Any result concerning the location of the roots of the derivative of a polynomial is also a result concerning the roots of the jacobian of two binary forms. Thus all the facts proved in A can be given this interpretation and other results can be found by linear transformation.* The reader can easily formulate these new theorems. We now prove a new result concerning the derivatives of polynomials all of whose roots are real.

Theorem XV. *Let intervals I_i $(i = 1, 2, \cdots, m)$ of the axis of reals, whose end points are $\alpha_i, \beta_i, \alpha_i \leqq \beta_i$, be the respective loci of m_i roots of a polynomial $f(z)$ which has no other roots. Then the locus of the roots of $f^{(k)}(z)$ is composed of a number of intervals $I_i^{(k)}$ of the axis of the reals. The left-hand end points of the intervals $I_i^{(k)}$ are the roots of $f^{(k)}(z)$ when the roots of $f(z)$ are concentrated at the points α_i; the right-hand end points are the corresponding roots of $f^{(k)}(z)$ when the roots of $f(z)$ are concentrated at the points β_i. Any interval $I_i^{(k)}$ which has no point in common with any other interval $I_j^{(k)}$ contains a number of roots of $f^{(k)}(z)$ equal to the multiplicity of its left-hand end point as a root of $f^{(k)}(z)$ when the roots of $f(z)$ are the points α_i. If the intervals I_i are all of equal length, the intervals $I_i^{(k)}$ are of this same length. If there is a point P which is a center of similitude for every pair of the intervals I_i (which is always true if $m = 2$), P is also a center of similitude for every pair of intervals $I_i^{(k)}, I_j^{(k)}$.*†

We prove this theorem under the assumption that no interval I_i reduces to

(II, p. 102) when the $p_1 - k$ finite roots of f_1 coalesce at α and the p_2 finite roots of f_2 coalesce at β there is but one position of equilibrium, namely, at the point γ' such that $(P, \alpha, \beta, \gamma') = p_1/k$. However, the jacobian vanishes not only at γ' but also at α and β if $p_1 - k$ and p_2 are greater than unity. It is the centroid with respect to P of *all* the finite roots of the jacobian that we have denoted by γ. The two formulas are the same when $p_1 - k = 1$, $p_2 = 1$.

*(Added in proof): There is an error in the statement of the italicized theorem of A, p. 133, as has been pointed out by Nagy, Jahresbericht der Deutschen Mathematiker-Vereinigung. vol. 31 (1922), pp. 245, 246. That theorem has no meaning as it appears at present, but becomes correct if the word *exterior* is replaced by the word *other*. The theorem is correctly stated in the abstract of A, Bulletin of the American Mathematical Society, vol. 26 (1919–20), p. 259.

† Some special cases of this theorem are given by Nagy, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 27 (1918), pp. 37–43; 44–48. The special case $m = 2$, $k = 1$ is Theorem II of S.

a point; to include this more general case requires merely a slight change in phraseology. We prove the theorem first for the case $k = 1$. In the theorem the intervals are assumed to be finite, but the theorem can be extended to include infinite intervals.

Let us denote by $\alpha_i^{(k)}$ the roots of $f^{(k)}(z)$ when the roots of $f(z)$ are concentrated at the points α_i, $\alpha_i^{(k)} \leqq \alpha_j^{(k)}$ when $i < j$, and similarly by $\beta_i^{(k)}$ the roots of $f^{(k)}(z)$ when the roots of $f(z)$ are concentrated at the points β_i, $\beta_i^{(k)} \leqq \beta_j^{(k)}$ when $i < j$. The intervals $I_i^{(k)} : (\alpha_i^{(k)}, \beta_i^{(k)})$ are then to be proved to form the locus of the roots of $f^{(k)}(z)$.

Let us start with the roots of $f(z)$ concentrated at the points α_i, and move these roots continuously to the right until they reach the points β_i. The roots of $f'(z)$ also vary continuously in their totality; they start at the points α_i' and reach the points β_i'. If we number these roots, commencing at the left, we can even say that the nth root z_n' of $f'(z)$ varies continuously. We now prove that z_n' moves always to the right.

The equation determining z_n' is of the form

$$F = \frac{m_1}{z_n' - \gamma_1} + \frac{m_2}{z_n' - \gamma_2} + \cdots + \frac{m_m}{z_n' - \gamma_m} = 0, \tag{3}$$

where the γ_i are the roots of $f(z)$, coinciding in any multiplicities m_i desired. We compute the values

$$\frac{\partial F}{\partial z_n'} = -\frac{m_1}{(z_n' - \gamma_1)^2} - \frac{m_2}{(z_n' - \gamma_2)^2} - \cdots - \frac{m_m}{(z_n' - \gamma_m)^2},$$
$$\frac{\partial F}{\partial \gamma_i} = \frac{m_i}{(z_n' - \gamma_i)^2}.$$

It is always true that $\partial z_n'/\partial \gamma_i$ is positive, so z_n' always increases with γ_i.

Equation (3) is no longer valid to determine z_n' if z_n' is located at a multiple root of $f(z)$. Under these circumstances, if γ_i does not coincide with z_n', the motion of γ_i does not change the position of z_n'. If γ_i does coincide with z_n' and if γ_i is moved to the right, z_n' is either unchanged or moved to the right; this follows immediately from the fact that a k-fold root of $f(z)$ is a $(k-1)$-fold root of $f'(z)$ and from the fact that every interval bounded by roots of $f(z)$ contains at least one root of $f'(z)$.

From the general fact, then, that the nth root z_n' of $f'(z)$ varies continuously and in one sense under the indicated variation in the roots of $f(z)$, it follows that z_n' traces the entire interval from α_n' to β_n'. It remains to be shown that z_n' can never be outside of the interval (α_n', β_n'). If we assume z_n' to lie outside of that interval, say for definiteness to the right, for some possible position of the roots of $f(z)$, motion of those roots of $f(z)$ to the right always within their proper loci would move z_n' to the right and when the roots of $f(z)$ reached the ends of their proper intervals z'_n would lie to the right of β'_n, which is impossible.

The determination of the locus in Theorem XV is now complete for $k = 1$; the statement relative to the number of roots of $f'(z)$ in the various intervals is readily proved by the continuity methods previously used.

For the case of $k = 2$, the continuity of the motion of the roots of $f''(z)$ due to the motion of the roots of $f(z)$ shows that every point of each of the intervals I''_k is a root of $f''(z)$ for some $f(z)$. No other point can be a root of $f''(z)$, for when the roots of $f(z)$ vary continuously in one sense, the roots of $f'(z)$ and therefore of $f''(z)$ vary continuously in that same sense. The number of roots of $f''(z)$ proper to the various intervals is as indicated. Continuance of the method of reasoning enables us to determine the locus for $k = 3$ and so on for the other values of k.

If all the intervals I_i are of the same length, the $f(z)$ whose roots are the β_i is obtained from the $f(z)$ whose roots are the α_i by a translation, so the β_i are obtained from the corresponding α_i by the same translation and the I'_i (and hence the $I_i^{(k)}$) are all of the same length as the I_i. If the β_i are obtained from the α_i by a similarity transformation, the $\beta_i^{(k)}$ are obtained from the $\alpha_i^{(k)}$ by the same transformation.

19. **The extension of theorems for the derivative of a polynomial to the roots of the jacobian.** Theorem XV cannot be immediately extended to the location of the roots of the jacobian of two binary forms, where the loci of the roots of both forms are intervals of the axis of reals. First, all the roots of both forms may coincide, so that the locus of the roots of the jacobian is not a number of intervals of the axis of reals. Second, the jacobian may have non-real roots even when it does not vanish identically.*

We can avoid this first possibility by requiring that the loci of the roots of f_1 and f_2 be so arranged that the two forms cannot be identically equal. We can avoid the second possibility by requiring that these loci be so arranged that no two roots of f_1 can separate two roots of f_2. Then all the roots of the jacobian are real, for on any interval bounded by roots of either form and containing no root of the other form there lies at least one root of the jacobian.

With these new restrictions, Theorem XV extends directly to the jacobian of two binary forms. If all the intervals which are the loci of the roots of both forms are finite, we consider the α_i (β_i) to be at the left-hand (right-hand) ends of those intervals which are loci of the roots of f_1 and at the right-hand (left-hand) ends of those intervals which are loci of the roots of f_2. For infinite intervals this notation is reversed. The locus of the roots of the jacobian is composed of the intervals whose end points are the corresponding

* This is shown by the simplest examples, such as $f_1 = z_1^2 - z_2^2$, $f_2 = z_1 z_2$, $J = 2(z_1^2 + z_2^2)$.

roots of the jacobian when the roots of the ground forms are respectively the α_i and the β_i.

A special case of this result is so similar to Theorem II that it deserves to be stated explicitly:

THEOREM XVI. *Let f_1 and f_2 be binary forms of degrees p_1 and p_2 respectively, and let arcs A_1, A_2, A_3 of a circle C be the respective loci of m roots of f_1, the remaining $p_1 - m$ roots of f_1, and all the roots of f_2. Suppose A_3 to have not more than one point in common with A_1 nor with A_2 and no point in common with both A_1 and A_2. Denote by A_4 the arc of C which is the locus of points z_4 such that*

$$(z_1, z_2, z_3, z_4) = \frac{p_1}{m},$$

when z_1, z_2, z_3 have the respective loci A_1, A_2, A_3. Then the locus of the roots of the jacobian of f_1 and f_2 is composed of the arc A_4 together with the arcs A_1, A_2, A_3, except that among the latter the corresponding arc is to be omitted if any of the numbers m, $p_1 - m$, p_2 is unity. If an arc A_i $(i = 1, 2, 3, 4)$ has no point in common with any other of those arcs which is a part of the locus of the jacobian, it contains precisely $m - 1$, $p_1 - m - 1$, $p_2 - 1$, or 1 of those roots according as $i = 1, 2, 3$, or 4.

Theorem XVI, as a special case of our more general result on the location of the roots of the jacobian, needs no separate proof, but it is interesting to notice that it can be proved in precisely the same manner as Theorem II was proved. Theorem I in the proof of Theorem II is replaced by III, Theorem IV, and Lemma I (II, p. 102) is replaced by the following

LEMMA. *The force at a point P on a circle C due to k unit positive particles lying on an arc A of C not containing P is equivalent to the force at P due to k coincident particles lying on A.*

We shall now obtain a result which has some relation to Theorem XVI as well as to Jensen's theorem, proved in A. We are dealing with pairs of points inverse with respect to a line, and as in A shall term circles whose diameters are the segments joining such pairs of points *Jensen circles*. Let f_1 and f_2 be two real forms which have not necessarily all their roots real. Let finite or infinite segments I_1, I_2, I_3 of the axis of reals either contain respectively m roots of f_1, the remaining $p_1 - m$ roots of f_1, and the p_2 roots of f_2, or contain some of these roots and the intercepts on the axis of reals of the Jensen circles of the remainder. Then any real root of the jacobian of f_1 and f_2 which is exterior to I_1, I_2, and I_3 lies in the interval I_4 which is the locus of the point z_4 defined by

$$(z_1, z_2, z_3, z_4) = \frac{p_1}{m}$$

when I_1, I_2, I_3 are the respective loci of z_1, z_2, z_3.

To prove this result we require the preceding lemma and the fact that the force at a point due to two particles is equivalent to the force at that point due to two coincident particles situated at the harmonic conjugate of that point with respect to the other two. If a point P is exterior to I_i, its harmonic conjugate with respect to two points the intersections of whose Jensen circle with the axis of reals lie in I_i, also lies in I_i.

This result is not expressed so as to be invariant under linear transformation, for if a real linear transformation is made and if the point at infinity is not invariant the Jensen circles are not invariant.

Our final result on the roots of the jacobian is similarly not invariant under linear transformation; it can be proved from the fact proved in A, § 2, that the force due to two positive particles at a point above the axis of reals but interior to their Jensen circle has a component vertically downward; at a point above the axis of reals but exterior to the Jensen circle the force has a component vertically upward.

THEOREM XVII. *If the forms f_1 and f_2 are both real and if f_1 has no finite real root, there is no root of the jacobian of f_1 and f_2 exterior to all the Jensen circles corresponding to the roots of f_2 but interior to all the Jensen circles corresponding to the roots of f_1.*

20. **Conclusion: extension of results to other types of polynomials.** We have considered in this paper generalizations of Theorem II in various directions. There is still another direction which we have not mentioned, namely, to the roots of polynomials other than the jacobian of two binary forms or the derivatives of a polynomial.

Thus the jacobian of two forms f_1 and f_2, of respective degrees p_1 and p_2, all of whose roots are finite and which correspond to two polynomials ϕ_1 and ϕ_2, has the same roots as the polynomial

$$p_2 \phi_1' \phi_2 - p_1 \phi_1 \phi_2'.$$

If we set ϕ_1 equal to the product of two polynomials ψ_1 and ψ_2 of respective degrees m and $p_1 - m$, Theorem II refers to the roots of the polynomial

$$p_2 \psi_1' \psi_2 \phi_2 + p_2 \psi_1 \psi_2' \phi_2 - p_1 \psi_1 \psi_2 \phi_2', \tag{4}$$

when the roots of ψ_1, ψ_2, ϕ_2 have the respective loci C_1, C_2, C_3.

We shall generalize Theorem II by considering three polynomials ω_1, ω_2, ω_3, of respective degrees μ_1, μ_2, μ_3, whose roots have the respective loci C_1, C_2, C_3. Our conclusion concerns the polynomial

$$\lambda_1 \omega_1' \omega_2 \omega_3 + \lambda_2 \omega_1 \omega_2' \omega_3 + \lambda_3 \omega_1 \omega_2 \omega_3', \tag{5}$$

where λ_1, λ_2, λ_3 are real* numbers not all zero such that

$$\lambda_1 \mu_1 + \lambda_2 \mu_2 + \lambda_3 \mu_3 = 0.$$

* Proof of Theorem I for complex λ enables us to remove this restriction of reality. See Walsh, Rendiconti del Circolo Matematico di Palermo, vol. 46 (1922), pp. 236–248.

It will be noticed that the polynomial (5) is indeed a generalization of (4), and has the additional advantage of being symmetric in ω_1, ω_2, ω_3.

If a point z is a root of (5) yet exterior to C_1, C_2, C_3, we must have

$$\lambda_1 \frac{\omega_1'}{\omega_1} + \lambda_2 \frac{\omega_2'}{\omega_2} + \lambda_3 \frac{\omega_3'}{\omega_3} = 0,$$

$$\frac{\lambda_1 \mu_1}{z - \alpha_1} + \frac{\lambda_2 \mu_2}{z - \alpha_2} + \frac{\lambda_3 \mu_3}{z - \alpha_3} = 0,$$

by Lemma I (II, p. 102), where α_1, α_2, α_3 lie in C_1, C_2, C_3 respectively. Hence z is given by the cross ratio

$$(\alpha_1, \alpha_2, \alpha_3, z) = -\frac{\lambda_3 \mu_3}{\lambda_1 \mu_1},$$

and lies in the region C_4 of Theorem I corresponding to the value

$$\lambda = -\frac{\lambda_3 \mu_3}{\lambda_1 \mu_1}.$$

We leave it to the reader to verify that the locus of the roots of (5) is composed of C_4 together with the regions C_1, C_2, C_3, except that among the latter the corresponding region is to be omitted if any of the degrees μ_1, μ_2, μ_3 is unity. If a region C_i $(i = 1, 2, 3, 4)$ has no point in common with any other of those regions which is a part of the locus of the roots of the jacobian, it contains precisely $\mu_1 - 1$, $\mu_2 - 1$, $\mu_3 - 1$, 1 of those roots according as $i = 1, 2, 3, 4$.

Many of the other theorems of the present paper, such as Theorems VI–XI, can similarly be extended to polynomials other than the jacobian of two binary forms or the derivative of a polynomial. Modifications of the methods used here can be made to apply to a still much broader type of polynomial about which the writer hopes to give some further results.

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