SYMMETRIC TENSORS OF THE SECOND ORDER
WHOSE FIRST COVARIANT DERIVATIVES ARE ZERO*

BY

LUTHER PFAHLER EISENHART

1. Consider a Riemann space of the $n$th order, whose fundamental quadratic form, assumed to be positive definite, is written

\[ ds^2 = g_{rs} \, dx^r \, dx^s \quad (g_{rs} = g_{sr}), \]

where $r$ and $s$ are summed from 1 to $n$ in accordance with the usual convention which will be followed throughout this paper. It is well known that the first covariant derivatives $g_{rs; t}$ are zero, where

\[ g_{rs; t} = \frac{\partial g_{rs}}{\partial x^t} - g_{rs} \frac{\partial g_{sr}}{\partial x^s} - g_{as} \frac{\partial g_{ra}}{\partial x^t} \]

and

\[ F_{st}^a = \frac{1}{2} g^{ap} (\frac{\partial g_{sp}}{\partial x^t} + \frac{\partial g_{tp}}{\partial x^s} - \frac{\partial g_{st}}{\partial x^p}), \]

the function $g^{ap}$ being the cofactor of $g_{ap}$ in the determinant

\[ g = |g_{rs}| \]

divided by $g$. It is the purpose of this paper to determine the necessary and sufficient conditions that there exist a symmetric covariant tensor $a_{rs}$ such that the first covariant derivatives $a_{rs; t}$ are zero, or more than one such tensor.

2. Let $a_{rs}$ denote the covariant components of any symmetric tensor of the second order. If $\varrho_h$ is a root of the equation

\[ |a_{rs} - \varrho_h g_{rs}| = 0, \]

the functions $\lambda_h^r (r = 1, \ldots, n)$ defined by

\[ (a_{rs} - \varrho_h g_{rs}) \lambda_h^r = 0 \quad (s = 1, \ldots, n) \]

* Presented to the Society, April 28, 1923.
are the contravariant components of a vector. It is well known that the roots of (5) are real, and that if they are simple, the $n$ corresponding vectors at a point are mutually orthogonal.* Moreover, if a root is of order $m$, equations (6) admit $m$ sets of independent solutions, and any linear combination of them is also a solution. It is possible to choose $m$ solutions so that the corresponding vectors at a point are mutually orthogonal, and thus from (6) obtain $n$ sets of solutions so that the corresponding vectors at a point are orthogonal; that is,

$$(7) \quad g_{rs} \lambda_h^r \lambda_k^s = 0 \quad (h, k = 1, \ldots, n; h \neq k).$$

Moreover, the components may be chosen so that

$$(8) \quad g_{rs} \lambda_h^r \lambda_h^s = 1 \quad (h = 1, \ldots, n),$$

that is, the vectors are unit vectors.

The curves in space whose direction at each point is defined by $\lambda_h^r$ form a congruence of curves $C_h$. Thus equations (6) define an $n$-uple of congruences of curves, such that the curves of the $n$-uple through a point are mutually orthogonal.

The covariant components $\lambda_{h,r}$ of the vector $\lambda_h$ are given by

$$(9) \quad \lambda_{h,r} = g_{rs} \lambda_h^s, \quad \lambda_h^s = g^{rs} \lambda_{h,r},$$

and hence (7) and (8) are equivalent to

$$(10) \quad \lambda_{h,r} \lambda_k^r = \delta_{hk},$$

where

$$(11) \quad \delta_{hk} = 1 \text{ for } h = k; = 0 \text{ for } h \neq k.$$

The functions $\gamma_{hij}$ defined by

$$(12) \quad \gamma_{hij} = \lambda_{h,ris} \lambda_i^r \lambda_j^s,$$

where $\lambda_{h,ris}$ is the covariant derivative of $\lambda_{h,r}$ with respect to $x^s$, are invariants; they are called rotations by Ricci and Levi-Civita.† They have shown that

$$(13) \quad \gamma_{hij} + \gamma_{dij} = 0, \quad \gamma_{hhi} = 0 \quad (h, i, j = 1, \ldots, n).$$

* Cf. these Transactions, vol. 25 (1923), p. 259.
† Mathematical Annalen, vol. 54 (1901), p. 148; also, Wright, Invariants of Quadratic Differential Forms, Cambridge Tract, No. 9, p. 68.
From (12) we have
\[ \lambda_{h,r,s} = \sum_{i,j}^{1 \ldots n} \gamma_{hij} \lambda_{i,r} \lambda_{j,s}, \]
and since \( g_{r/s} = 0 \), it follows from (9) that
\[ \lambda_{h,r,s} = \sum_{i,j}^{1 \ldots n} \gamma_{hij} \lambda_{i,r} \lambda_{j,s}. \]

3. If all the roots of (5) are equal, we must have \( \alpha_{rs} = \varepsilon g_{rs} \). Differentiating covariantly with respect to \( x^t \), and making use of the fact that \( g_{r/s} = 0 \) and the assumption that \( \alpha_{r/t} = 0 \), we have that \( \varepsilon \) is constant. Consequently \( \alpha_{rs} \) is essentially the same as \( g_{rs} \). We exclude this case from further consideration.

Since (7) is satisfied whether the functions \( \lambda^h_1 \) and \( \lambda^h_2 \) correspond to different simple roots of (5), or to the same multiple root when such exists, we have from (6)
\[ \alpha_{rs} \lambda^r_h \lambda^s_h = 0 \quad (h, k = 1, \ldots, n; \ h \neq k). \]

Also from (6) we have
\[ \alpha_{rs} \lambda^r_h \lambda^s_h = \varepsilon_h. \]

From (17) we have by differentiating covariantly with respect to \( x^t \) and making use of (15), (16), and (17)
\[ \alpha_{r/s} \lambda^r_h \lambda^s_h = \frac{\partial \varepsilon_h}{\partial x^t}. \]

Also from (16) we have, because of (13), (14), (16) and (17),
\[ \alpha_{r/s} \lambda^r_h \lambda^s_h + \sum_j^{1 \ldots n} (e_k - e_h) \gamma_{hjk} \lambda_{j,s} = 0. \]

Multiplying by \( \lambda^r_t \) and summing for \( t \), we have
\[ \alpha_{r/s} \lambda^r_h \lambda^s_k \lambda^r_t + (e_k - e_h) \gamma_{h,k} = 0 \quad (h \neq k). \]
From (18) it follows that if \( \alpha_{rs} = 0 \) the roots \( \varphi \) are constant. And from (19) we have for two different roots

\[
\gamma_{hkl} = 0 \quad (h \neq k).
\]

Let \( \varphi_1 \) be a root of (5) which we assume to be a multiple root of order \( m \), and denote by \( \lambda_h^r (h = 1, \ldots, m) \) the components of the \( m \) mutually orthogonal vectors corresponding to it, and by \( \lambda_k^r (k = m + 1, \ldots, n) \) the components of the directions corresponding to the other roots of (5). From (20) we have

\[
\gamma_{hkl} = 0 \quad (h = 1, \ldots, m; k = m + 1, \ldots, n; l = 1, \ldots, n).
\]

Consider the system of equations

\[
X_k(f) = \lambda_k^r \frac{\partial f}{\partial x^r} = 0 \quad (k = m + 1, \ldots, n).
\]

If we introduce the notation

\[
\frac{\partial f}{\partial s^k} = \lambda_k^r \frac{\partial f}{\partial x^r},
\]

then, as Ricci and Levi-Civita have shown*, the relation

\[
\frac{\partial}{\partial s_j} \frac{\partial f}{\partial s_k} - \frac{\partial}{\partial s_k} \frac{\partial f}{\partial s_j} = \sum_{i=1}^{n} (\gamma_{ijk} - \delta_{ik}) \frac{\partial f}{\partial s_i}
\]

is satisfied for any function \( f \).

Applying this formula to equations (22) we have in consequence of (21)

\[
X_j X_k(f) - X_k X_j(f) = \sum_{i=1}^{n} (\gamma_{ijk} - \delta_{ik}) X_i(f) \quad (j, k = m + 1, \ldots, n).
\]

Hence the system (22) is complete and admits \( m \) independent solutions, say \( f^r_h (h = 1, \ldots, m) \).

* Loc. cit., p. 150; Wright, p. 69.
Let \( \xi \) be another root of (5), of order \( p \), and denote by \( \lambda_j^r \) \((j = m + 1, \ldots, m + p)\) the components of the corresponding vectors. In like manner we show that the equations

\[
\lambda_j^r \frac{\partial f}{\partial x^r} = 0 \quad (l = 1, \ldots, m, m + p + 1, \ldots, n)
\]

form a complete system and admit \( p \) independent solutions \( f_j \) \((j = m + 1, \ldots, m + p)\).

From (22) and the equations

\[
\lambda_h^r \lambda_{h,r} = 0 \quad (h = 1, \ldots, m; k = m + 1, \ldots, n)
\]

it follows that there exist functions \( a_h^\sigma \) such that

\[
\frac{\partial f_h}{\partial x^r} = \sum_\sigma a_h^\sigma \lambda_{\sigma,r} \quad (h, \sigma = 1, \ldots, m).
\]

In like manner, we have

\[
\frac{\partial f_j}{\partial x^r} = \sum_\tau b_j^\tau \lambda_{\tau,r} \quad (j, \tau = m + 1, \ldots, m + p).
\]

Consequently we have

\[
g^{rs} \frac{\partial f_h}{\partial x^r} \frac{\partial f_j}{\partial x^s} = \sum_\sigma a_h^\sigma b_j^\tau g^{rs} \lambda_{\sigma,r} \lambda_{\tau,s} = 0,
\]

that is, any hypersurface \( f_h = \text{const.} \) is orthogonal to each of the hypersurfaces \( f_j = \text{const.} \).

Proceeding in this manner with the other roots of (5) we obtain a group of hypersurfaces corresponding to each distinct root of (5), the number of hypersurfaces in a group being equal to the order of the root. Any two hypersurfaces of two different groups are orthogonal to one another. If we take these \( n \) families of hypersurfaces for the parametric surfaces \( x^r = \text{const.} \), for the case where \( x^r = \text{const.} \) and \( x^s = \text{const.} \) are hypersurfaces of different groups; in this sense we say that \( r \) and \( s \) refer to different groups, or different roots of (5).

From the equations (22) for this choice of the variables \( x \), it follows that \( \lambda_k^r = 0 \), for \( r \) and \( k \) referring to different roots of (5). From (9) it follows also that \( \lambda_{k,r} = 0 \) for \( k \) and \( r \) referring to different roots.
Equations (6) may be replaced by

\[ \alpha_{rs} = \sum_{h} q_{h} \lambda_{h,r} \lambda_{h,s} \]

whether the roots of (5) are simple, or some are multiple. From (24) and the preceding observations it follows

\[ \alpha_{rs'} = g_{rs'} = 0, \]
\[ \alpha_{rs} = q_{h} g_{rs}, \]

where \( r \) and \( s' \) refer to any two different roots and \( r \) and \( s \) refer to the root \( q_{h}. \)

From (25) we have \( \alpha_{rs'} = 0 \), hence if \( \alpha_{rs'/l} = 0 \), we must have (cf. (2))

\[ \alpha_{rl} R_{st}^{l} + \alpha_{s'q} R_{rt}^{q} = 0 \quad (l, q = 1, \ldots, n), \]

that is

\[ \alpha_{rl} g^{ls} \left( \frac{\partial g_{s'l}}{\partial x^l} + \frac{\partial g_{s'q}}{\partial x^q} - \frac{\partial g_{s't}}{\partial x^t} \right) + \alpha_{s'q} g^{qs} \left( \frac{\partial g_{rs}}{\partial x^s} + \frac{\partial g_{tq}}{\partial x^q} - \frac{\partial g_{rt}}{\partial x^t} \right) = 0. \]

If \( r \) refers to the root \( q_{r} \) of (5), say \( r = 1, \ldots, m \) and \( s' \) to the root \( q_{s} \), say \( s' = m + 1, \ldots, m + p \), we have from (25)

\[ \alpha_{rl} = q_{r} g_{rl} \quad (l = 1, \ldots, m); \]
\[ \alpha_{rl} = 0 \quad (l = m + 1, \ldots, n); \]
\[ \alpha_{s'q} = q_{s} g_{s'q} \quad (q = m + 1, \ldots, m + p); \]
\[ \alpha_{s'q} = 0 \quad (q = 1, \ldots, m, m + p + 1, \ldots, n). \]

Hence the above equation reduces to

\[ q_{1} \left( \frac{\partial g_{s'r}}{\partial x^r} + \frac{\partial g_{tr}}{\partial x^r} - \frac{\partial g_{s't}}{\partial x^t} \right) + q_{s} \left( \frac{\partial g_{rs}}{\partial x^s} + \frac{\partial g_{tq}}{\partial x^q} - \frac{\partial g_{rt}}{\partial x^t} \right) = 0. \]

If now $t$ and $r$ refer to the same root, this equation reduces to

$$(q_1 - q_s) \frac{\partial g_{rs}}{\partial x^s} = 0,$$

and if $t$ and $s'$ refer to the same root, we have

$$(q_1 - q_s) \frac{\partial g_{s't}}{\partial x^{s'}} = 0.$$ 

If $r$, $s'$ and $t$ refer to three different roots, the equation vanishes identically.

Since $q_1$ and $q_s$ are not equal by hypothesis, we have that each function $g_{rs}$ depends only on the coordinates referring to the same root as $r$ and $s$.

Consider again

$$\alpha_{rs} = q_1 g_{rs} \quad (r, s = 1, \ldots, m).$$

Now

$$\alpha_{rst} = q_1 \frac{\partial g_{rs}}{\partial x^l} - \alpha_{rl} F^l_{st} - \alpha_{sl} F^l_{rt} \quad (l = 1, \ldots, n),$$

which by (25) is reducible to

$$\alpha_{rst} = q_1 \left( \frac{\partial g_{rs}}{\partial x^l} - g_{rl} F^l_{st} - g_{sl} F^l_{rt} \right) = q_1 g_{rst} = 0.$$ 

Hence we have the following theorem:

A necessary and sufficient condition that a Riemann space admit a symmetric covariant tensor of the second order $\alpha_{rs}$ other than, with a positive definite fundamental form (1), $g_{rs}$, such that its first covariant derivative is zero, is that (1) be reducible to a sum of forms

$$\varphi^{(i)} = g_{r's'}^{(i)} dx^r dx^{s'},$$

where $g_{r's'}^{(i)}$ are functions at most of the $x'$s of that form; then

$$\alpha_{rs} dx^r dx^s = \sum_i q_i \varphi^{(i)},$$

where the $q_i$'s are arbitrary constants.
In particular, if all the roots of (5) are simple, the space is euclidean; if its fundamental form is taken in the form

\[ ds^2 = \sum_i dx^i \quad (i = 1, \ldots, n), \]

then

\[ \alpha_{rs} dx^r dx^s = \sum_i q_i dx^i \quad (i = 1, \ldots, n), \]

where the \( q \)'s are \( n \) different arbitrary constants.

When any one of the roots of (6) is simple, the corresponding congruence is normal, and the tangents to the congruence form a field of parallel vectors in the sense of Levi-Civita.*

5. In this section it will be shown that the problem of determining whether a given Riemann space admits one, or more, symmetric tensors whose first covariant derivatives are zero is a problem of algebra.†

We recall that if \( \alpha_{rs} \) is any symmetric tensor, then

\[ \alpha_{rsijk} - \alpha_{rs/kj} = \alpha_{rt} B_{ijk}^t + \alpha_{st} B_{rjk}^t, \]

where \( \alpha_{rsijk} \) is the second covariant derivative of \( \alpha_{rs} \), and \( B_{ijk}^t \) are the components of the Riemann tensor of the second kind formed with respect to (1). If then \( \alpha_{rsij} = 0 \), we must have

\[ (28) \quad \alpha_{rsijk} - \alpha_{rs/kj} = 0, \]

and consequently we have equations of the form

\[ (29) \quad \alpha_{rt} B_{ijk}^t + \alpha_{st} B_{rjk}^t = 0 \quad (j, k, r, s, t = 1, \ldots, n). \]

Differentiating these equations covariantly successively we have the sets of equations

\[ \alpha_{rt} B_{ijk/m_s}^t + \alpha_{st} B_{rjk/m_s}^t = 0, \]

\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]

\[ (30) \quad \alpha_{rt} B_{ijk/m_s \ldots m_t}^t + \alpha_{st} B_{rjk/m_s \ldots m_t}^t = 0, \]

\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]

Since $g_{rs}$ satisfies (28) the systems (29) and (30) are satisfied by $g_{rs}$ and consequently are algebraically consistent. From this it follows either that the functions $g_{rs}$ are the only solution of (29) and (30), or that (29) and the first $l \geq 0$ sets of (30) admit a complete system of solutions $g_{rs}$ and $\alpha_{rs}^{(1)}, \ldots, \alpha_{rs}^{(p)}$ which satisfy also the $(l+1)$th set of equations (30). In the latter case the general solution is of the form

$$\alpha_{rs} = \varphi^{(0)} g_{rs} + \varphi^{(1)} \alpha_{rs}^{(1)} + \cdots + \varphi^{(p)} \alpha_{rs}^{(p)}.$$  

If any one of the functions $\alpha_{rs}^{(\sigma)} (\sigma = 1, \ldots, p)$ is substituted in (29) and the first $l$ sets of (30), and these equations are differentiated covariantly, we have, in consequence of the above requirement, that the functions $\alpha_{rs/lm}^{(\sigma)} (\sigma = 1, \ldots, p; m = 1, \ldots, n)$ satisfy (29) and the first $l$ sets of (30). Consequently we have

$$\alpha_{rs/lm}^{(\sigma)} = \lambda_{m}^{(\sigma 0)} g_{rs} + \lambda_{m}^{(\sigma 1)} \alpha_{rs}^{(1)} + \cdots + \lambda_{m}^{(\sigma p)} \alpha_{rs}^{(p)},$$

where the $p(p+1)$ vectors $\lambda_{m}^{(\sigma \beta)} (\sigma = 1, \ldots, p; \beta = 0, 1, \ldots, p)$ must be such that the functions (32) shall satisfy (28). Substituting in these equations we find that the functions $\lambda$ must satisfy the system

$$\frac{\partial \lambda_{p}^{(\sigma \tau)}}{\partial x^{q}} - \frac{\partial \lambda_{q}^{(\sigma \tau)}}{\partial x^{p}} + \sum_{\omega} (\lambda_{p}^{(\omega \omega)} \lambda_{q}^{(\omega \tau)} - \lambda_{q}^{(\sigma \omega)} \lambda_{p}^{(\sigma \tau)}) = 0 \quad (\sigma, \omega = 1, \ldots, p; \tau = 0, 1, \ldots, p).$$

In order that $\alpha_{rs}$ given by (31) shall satisfy $\alpha_{rs/l} = 0$, it is necessary and sufficient that the functions $\varphi^{(\sigma)}$ satisfy

$$\frac{\partial \varphi^{(\sigma 0)}}{\partial x^{q}} + \sum_{\sigma} \varphi^{(\sigma)} \lambda_{q}^{(\sigma 0)} = 0 \quad (\sigma = 1, \ldots, p),$$

and

$$\frac{\partial \varphi^{(\sigma \tau)}}{\partial x^{q}} + \sum_{\sigma} \varphi^{(\sigma)} \lambda_{q}^{(\sigma \tau)} = 0 \quad (\sigma, \tau = 1, \ldots, p).$$

In consequence of (33) equations (35) are completely integrable and therefore admit solutions involving $p$ arbitrary constants. Because of (33) the conditions of integrability of (34) are satisfied; hence $\varphi^{(\sigma 0)}$ involves these $p$ arbitrary constants and an additive arbitrary constant which may be neglected.*

* If $\alpha_{rs}$ is a tensor whose first covariant derivative is zero, so also is $\alpha_{rs} + \lambda g_{rs}$, where $\lambda$ is an arbitrary constant.
In view of the above results we have the theorem:

If equations (29) and the first \( l (\geq 0) \) sets of equations (30) admit a complete system of solutions \( g_{rs} \) and \( \alpha_{rs}^{(a)} (a = 1, \ldots, p) \) which are also solutions of the \((l + 1)\)th set of equations (30), there exists a symmetric tensor of the second order, involving \( p \) arbitrary constants, whose first covariant derivative is zero.

6. Suppose that the fundamental form is the sum of \( j \) forms (26). By definition

\[ B_{pqrs}^a = g^{aq} B_{pqrs} \]

where \( B_{pqrs} \) is the covariant Riemann tensor of the fourth order, that is,

\[ B_{pqrs} = \frac{1}{2} \left( \frac{\partial^2 g_{ps}}{\partial x^q \partial x^r} + \frac{\partial^2 g_{qr}}{\partial x^p \partial x^s} - \frac{\partial^2 g_{pr}}{\partial x^q \partial x^s} - \frac{\partial^2 g_{qs}}{\partial x^p \partial x^r} \right) + g^{lm} (\Gamma_{ps,m} \Gamma_{qr,l} - \Gamma_{pr,m} \Gamma_{qs,l}), \]

where

\[ \Gamma_{ps,m} = \frac{1}{2} \left( \frac{\partial g_{pm}}{\partial x^s} + \frac{\partial g_{sm}}{\partial x^p} - \frac{\partial g_{ps}}{\partial x^m} \right). \]

For the case under consideration, namely (26), it is readily shown that the components \( B_{pqrs} \) are zero, unless \( p, q, r, s \) refer to the same root of (5); likewise \( B_{pqrs}^a \), and its first covariant derivatives \( B_{pqrs}^{al} \). Consequently equations (29) and the first set of (30) admit, in addition to \( g_{rs} \), the \( j \) sets of solutions of the form (25). If it is understood that each of the forms (26) is not further reducible to sums of such forms, we have a complete set of solutions of (29). Hence when the space is referred to the coordinates giving (25) the number \( l \) in the preceding theorem is zero.

PRINCETON UNIVERSITY,
PRINCETON, N. J.