SYMMETRIC TENSORS OF THE SECOND ORDER
WHOSE FIRST COVARIANT DERIVATIVES ARE ZERO*

BY

LUTHER PFAHLER EISENHART

1. Consider a Riemann space of the nth order, whose fundamental quadratic form, assumed to be positive definite, is written

\[ ds^s = g_{rs} \, dx^r \, dx^s \]  
\[ (g_{rs} = g_{sr}), \]

where \( r \) and \( s \) are summed from 1 to \( n \) in accordance with the usual convention which will be followed throughout this paper. It is well known that the first covariant derivatives \( g_{rs/t} \) are zero, where

\[ g_{rs/t} = \frac{\partial g_{rs}}{\partial x^t} - g_{ra} \, r^a_s - g_{as} \, r^a_r \]

and

\[ R^a_{st} = \frac{1}{2} \, g^{ap} \left( \frac{\partial g_{sp}}{\partial x^t} + \frac{\partial g_{tp}}{\partial x^s} - \frac{\partial g_{st}}{\partial x^p} \right), \]

the function \( g^{ap} \) being the cofactor of \( g_{ap} \) in the determinant

\[ g = |g_{rs}| \]

divided by \( g \). It is the purpose of this paper to determine the necessary and sufficient conditions that there exist a symmetric covariant tensor \( \alpha_{rs} \) such that the first covariant derivatives \( \alpha_{rs/t} \) are zero, or more than one such tensor.

2. Let \( \alpha_{rs} \) denote the covariant components of any symmetric tensor of the second order. If \( \xi_h \) is a root of the equation

\[ |\alpha_{rs} - \xi g_{rs}| = 0, \]

the functions \( \lambda^r_h \) \((r = 1, \ldots, n)\) defined by

\[ (\alpha_{rs} - \xi_h g_{rs}) \lambda^r_h = 0 \quad (s = 1, \ldots, n) \]

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are the contravariant components of a vector. It is well known that the roots of (5) are real, and that if they are simple, the \( n \) corresponding vectors at a point are mutually orthogonal.* Moreover, if a root is of order \( m \), equations (6) admit \( m \) sets of independent solutions, and any linear combination of them is also a solution. It is possible to choose \( m \) solutions so that the corresponding vectors at a point are mutually orthogonal, and thus from (6) obtain \( n \) sets of solutions so that the corresponding vectors at a point are orthogonal; that is,

\[
g_{rs} \lambda^r_h \lambda^s_k = 0 \quad (h, k = 1, \ldots, n; \ h \neq k).
\]

Moreover, the components may be chosen so that

\[
g_{rs} \lambda^r_h \lambda^s_h = 1 \quad (h = 1, \ldots, n),
\]

that is, the vectors are unit vectors.

The curves in space whose direction at each point is defined by \( \lambda^r_h \) form a congruence of curves \( C_h \). Thus equations (6) define an \( n \)-uple of congruences of curves, such that the curves of the \( n \)-uple through a point are mutually orthogonal.

The covariant components \( \lambda_{h,r} \) of the vector \( h \) are given by

\[
\lambda_{h,r} = g_{rs} \lambda^s_h, \quad \lambda^s_h = g^{rs} \lambda_{h,r},
\]

and hence (7) and (8) are equivalent to

\[
\lambda_{h,r} \lambda^r_k = \delta_{hk},
\]

where

\[
\delta_{hk} = 1 \text{ for } h = k; = 0 \text{ for } h \neq k.
\]

The functions \( \gamma_{hij} \) defined by

\[
\gamma_{hij} = \lambda_{h,rj} \lambda^r_i \lambda^s_j,
\]

where \( \lambda_{h,rj} \) is the covariant derivative of \( \lambda_{h,r} \) with respect to \( x^s \), are invariants; they are called rotations by Ricci and Levi-Civita.† They have shown that

\[
\gamma_{hij} + \gamma_{ihi} = 0, \quad \gamma_{hhi} = 0 \quad (h, i, j = 1, \ldots, n).
\]

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* Cf. these Transactions, vol. 25 (1923), p. 259.
† Mathematische Annalen, vol. 54 (1901), p. 148; also, Wright, Invariants of Quadratic Differential Forms, Cambridge Tract, No. 9, p. 68.
From (12) we have

\[ \lambda_{h_1 r_1 s} = \sum_{i,j} \gamma_{h_1 i} \lambda_{i_1 r_1} \lambda_{j_1 s}, \]

and since \( g_{r_1 l_1} = 0 \), it follows from (9) that

\[ \lambda_{h_1 l_1}^{r_1} = \sum_{i,j} \gamma_{h_1 i} \lambda_{i_1}^{r_1} \lambda_{j_1 s}. \]

3. If all the roots of (5) are equal, we must have \( \alpha_{r_1} = \varphi_{r_1} \). Differentiating covariantly with respect to \( x^l \), and making use of the fact that \( g_{r_1 l_1} = 0 \) and the assumption that \( \alpha_{r_1 l_1} = 0 \), we have that \( \varphi \) is constant. Consequently \( \alpha_{r_1} \) is essentially the same as \( \varphi_{r_1} \). We exclude this case from further consideration.

Since (7) is satisfied whether the functions \( \lambda_h^a \) and \( \lambda_h^r \) correspond to different simple roots of (5), or to the same multiple root when such exists, we have from (6)

\[ \alpha_{r_1} \lambda_h^a \lambda_h^a = 0 \quad (h, k = 1, \ldots, n; \ h \neq k). \]

Also from (6) we have

\[ \alpha_{r_1} \lambda_h^r \lambda_h^r = \varphi_h. \]

From (17) we have by differentiating covariantly with respect to \( x^l \) and making use of (15), (16), and (17)

\[ \alpha_{r_1 l_1} \lambda_h^a \lambda_h^a = \frac{\partial \varphi_h}{\partial x^l}. \]

Also from (16) we have, because of (13), (14), (16) and (17),

\[ \alpha_{r_1 l_1} \lambda_h^a \lambda_h^a + \sum_j (e_h - e_h) \gamma_{h_1 j} \lambda_{j_1 t} = 0. \]

Multiplying by \( \lambda_t^l \) and summing for \( t \), we have

\[ \alpha_{r_1 l_1} \lambda_h^a \lambda_h^a \lambda_t^l + (e_h - e_h) \gamma_{h_1 s} = 0 \quad (h \neq k). \]
From (18) it follows that if $\alpha_{rs}t = 0$ the roots $\varphi$ are constant. And from (19) we have for two different roots

\[(20) \quad r_{hkl} = 0 \quad (h \neq k).\]

Let $\psi_i$ be a root of (5) which we assume to be a multiple root of order $m$, and denote by $\lambda^r_h (h = 1, \ldots, m)$ the components of the $m$ mutually orthogonal vectors corresponding to it, and by $\lambda^r_k (k = m + 1, \ldots, n)$ the components of the directions corresponding to the other roots of (5). From (20) we have

\[(21) \quad r_{hkl} = 0 \quad (h = 1, \ldots, m; k = m + 1, \ldots, n; l = 1, \ldots, n).\]

Consider the system of equations

\[(22) \quad X_k (f) = \lambda^r_k \frac{\partial f}{\partial x^r} = 0 \quad (k = m + 1, \ldots, n).\]

If we introduce the notation

\[\frac{\partial f}{\partial s^k} = \lambda^r_k \frac{\partial f}{\partial x^r},\]

then, as Ricci and Levi-Civita have shown*, the relation

\[(23) \quad \frac{\partial}{\partial s_j} \frac{\partial f}{\partial s_k} - \frac{\partial}{\partial s_k} \frac{\partial f}{\partial s_j} = \sum_{i=1}^{n} (r_{ij} - r_{ik}) \frac{\partial f}{\partial s_i}\]

is satisfied for any function $f$.

Applying this formula to equations (22) we have in consequence of (21)

\[X_j X_k (f) - X_k X_j (f) = \sum_{i=1}^{n} (r_{ik} - r_{jk}) X_i (f) \quad (j, k = m + 1, \ldots, n).\]

Hence the system (22) is complete and admits $m$ independent solutions, say $f^s_h (h = 1, \ldots, m)$.

* Loc. cit., p. 150; Wright, p. 69.
Let \( q \) be another root of (5), of order \( p \), and denote by \( \lambda_j^r (j = m + 1, \ldots, m + p) \) the components of the corresponding vectors. In like manner we show that the equations

\[
\lambda^r_i \frac{\partial f}{\partial x^r} = 0 \quad (l = 1, \ldots, m, m + p + 1, \ldots, n)
\]

form a complete system and admit \( p \) independent solutions \( f_j (j = m + 1, \ldots, m + p) \).

From (22) and the equations

\[
\lambda^r_h \lambda_{h,r} = 0 \quad (h = 1, \ldots, m; k = m + 1, \ldots, n)
\]

it follows that there exist functions \( a^r_h \) such that

\[
\frac{\partial f_h}{\partial x^r} = \sum_{\sigma} a^r_h \lambda_{\sigma,r} \quad (h, \sigma = 1, \ldots, m).
\]

In like manner, we have

\[
\frac{\partial f_j}{\partial x^r} = \sum_{\tau} b^r_j \lambda_{\tau,r} \quad (j, \tau = m + 1, \ldots, m + p).
\]

Consequently we have

\[
g^{rs} \frac{\partial f_h}{\partial x^r} \frac{\partial f_j}{\partial x^s} = \sum_{\sigma, \tau} a^r_h b^s_j g^{rs} \lambda_{\sigma,r} \lambda_{\tau,s} = 0,
\]

that is, any hypersurface \( f_h = \text{const.} \) is orthogonal to each of the hypersurfaces \( f_j = \text{const.} \).

Proceeding in this manner with the other roots of (5) we obtain a group of hypersurfaces corresponding to each distinct root of (5), the number of hypersurfaces in a group being equal to the order of the root. Any two hypersurfaces of two different groups are orthogonal to one another. If we take these \( n \) families of hypersurfaces for the parametric surfaces \( x^r = \text{const.} \) \((r = 1, \ldots, n)\), it follows that the functions \( g_{rs} \) are zero, for the case where \( x^r = \text{const.} \) and \( x^s = \text{const.} \) are hypersurfaces of different groups; in this sense we say that \( r \) and \( s \) refer to different groups, or different roots of (5).

From the equations (22) for this choice of the variables \( x \), it follows that \( \lambda_k^r = 0 \), for \( r \) and \( k \) referring to different roots of (5). From (9) it follows also that \( \lambda_{k,r} = 0 \) for \( k \) and \( r \) referring to different roots.
Equations (6) may be replaced by*

\[ (24) \quad \alpha_{rs} = \sum_{h} q_{h} \lambda_{h,r} \lambda_{h,s} \]

whether the roots of (5) are simple, or some are multiple. From (24) and the preceding observations it follows

\[ (25) \]

\[ \alpha_{rs'} = g_{rs'} = 0, \]
\[ \alpha_{rs} = q_{h} g_{rs}, \]

where \( r \) and \( s' \) refer to any two different roots and \( r \) and \( s \) refer to the root \( q_{h}. \)

From (25) we have \( \alpha_{rs'} = 0 \), hence if \( \alpha_{rs'j} = 0 \), we must have (cf. (2))

\[ \alpha_{rl} R_{s't}^{l} + \alpha_{s'q} R_{rl}^{q} = 0 \quad (l, q = 1, \ldots, n), \]

that is

\[ \left[ \frac{\partial g_{s'r}}{\partial x^{l}} + \frac{\partial g_{t'r}}{\partial x^{s'}} - \frac{\partial g_{s't}}{\partial x^{r}} \right] + \alpha_{s'q} g_{lq} \left[ \frac{\partial g_{rs}}{\partial x^{l}} + \frac{\partial g_{lt}}{\partial x^{s'}} - \frac{\partial g_{rl}}{\partial x^{r}} \right] = 0. \]

If \( r \) refers to the root \( q_{1} \) of (5), say \( r = 1, \ldots, m \) and \( s' \) to the root \( q_{s} \), say \( s' = m + 1, \ldots, m + p \), we have from (25)

\[ \alpha_{rl} = q_{1} g_{rl} \quad (l = 1, \ldots, m); \]
\[ \alpha_{rl} = 0 \quad (l = m + 1, \ldots, n); \]
\[ \alpha_{s'q} = q_{2} g_{s'q} \quad (q = m + 1, \ldots, m + p); \]
\[ \alpha_{s'q} = 0 \quad (q = 1, \ldots, m, m + p + 1, \ldots, n). \]

Hence the above equation reduces to

\[ q_{1} \left( \frac{\partial g_{s'r}}{\partial x^{l}} + \frac{\partial g_{t'r}}{\partial x^{s'}} - \frac{\partial g_{s't}}{\partial x^{r}} \right) + q_{2} \left( \frac{\partial g_{rs}}{\partial x^{l}} + \frac{\partial g_{lt}}{\partial x^{s'}} - \frac{\partial g_{rl}}{\partial x^{r}} \right) = 0. \]

If now \( t \) and \( r \) refer to the same root, this equation reduces to
\[
(e_1 - q_s) \frac{\partial g_{rs}}{\partial x^s} = 0,
\]
and if \( t \) and \( s' \) refer to the same root, we have
\[
(e_1 - q_s) \frac{\partial g_{s'r}}{\partial x^r} = 0.
\]
If \( r, s' \) and \( t \) refer to three different roots, the equation vanishes identically.

Since \( q_t \) and \( q_s \) are not equal by hypothesis, we have that each function \( g_{rs} \)
depends only on the coordinates referring to the same root as \( r \) and \( s \).

Consider again
\[
\alpha_{rs} = e_1 g_{rs} \quad (r, s = 1, \ldots, m).
\]
Now
\[
\alpha_{rs} t = e_1 \frac{\partial g_{rs}}{\partial x^t} - \alpha_{rt} f_{st}^l - \alpha_{st} f_{rt}^l \quad (l = 1, \ldots, n),
\]
which by (25) is reducible to
\[
\alpha_{rs} t = e_1 \left( \frac{\partial g_{rs}}{\partial x^t} - g_{rt} f_{st}^l - g_{st} f_{rt}^l \right) = e_1 g_{rst} = 0.
\]
Hence we have the following theorem:

A necessary and sufficient condition that a Riemann space admit a symmetric covariant tensor of the second order \( \alpha_{rs} \) other than, with a positive definite fundamental form (1), \( g_{rs} \), such that its first covariant derivative is zero, is that (1) be reducible to a sum of forms

\[
\varphi^{(0)} = g_{r's'} dx^{r'} dx^{s'},
\]
where \( g_{r's'} \) are functions at most of the \( x \)'s of that form; then

\[
\alpha_{rs} dx^r \ dx^s = \sum_i q_i \varphi^{(i)},
\]
where the \( q \)'s are arbitrary constants.
In particular, if all the roots of (5) are simple, the space is euclidean; if its fundamental form is taken in the form

\[ ds^2 = \sum_i dx^i \]  

\( (i = 1, \ldots, n) \),

then

\[ \alpha_{rs} dx^r dx^s = \sum_i \alpha_i \, dx^i \]  

\( (i = 1, \ldots, n) \),

where the \( \alpha_i \)'s are \( n \) different arbitrary constants.

When any one of the roots of (5) is simple, the corresponding congruence is normal, and the tangents to the congruence form a field of parallel vectors in the sense of Levi-Civita.*

5. In this section it will be shown that the problem of determining whether a given Riemann space admits one, or more, symmetric tensors whose first covariant derivatives are zero is a problem of algebra.†

We recall that if \( \alpha_{rs} \) is any symmetric tensor, then

\[ \alpha_{rsijk} - \alpha_{rs/kj} = \alpha_{rst} B_{sjkt} + \alpha_{st} B_{rjkt}, \]

where \( \alpha_{rsijk} \) is the second covariant derivative of \( \alpha_{rs} \), and \( B_{sjkt} \) are the components of the Riemann tensor of the second kind formed with respect to (1). If then \( \alpha_{rsijk} = 0 \), we must have

\[ \alpha_{rsijk} - \alpha_{rs/kj} = 0, \]

and consequently we have equations of the form

\[ \alpha_{rst} B_{sjkt} + \alpha_{st} B_{rjkt} = 0 \]  

\( (j, k, r, s, t = 1, \ldots, n) \).

Differentiating these equations covariantly successively we have the sets of equations

\[ \alpha_{rst} B_{sj/km} + \alpha_{st} B_{rj/km} = 0, \]

\[ \alpha_{rst} B_{sj/km_1} + \alpha_{st} B_{rj/km_2} = 0, \]

\( (j, k, r, s, t, m, m_1, m_2 = 1, \ldots, n) \).

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Since \( g_{rs} \) satisfies (28) the systems (29) and (30) are satisfied by \( g_{rs} \) and consequently are algebraically consistent. From this it follows either that the functions \( g_{rs} \) are the only solution of (29) and (30), or that (29) and the first \( l \geq 0 \) sets of (30) admit a complete system of solutions \( g_{rs} \) and \( \alpha^{(1)}_{rs}, \ldots, \alpha^{(p)}_{rs} \) which satisfy also the \((l+1)\)th set of equations (30). In the latter case the general solution is of the form

\[
\alpha_{rs} = q^{(0)}_{rs} q^{(1)}_{rs} + \cdots + q^{(p)}_{rs} \alpha^{(p)}_{rs}.
\]

If any one of the functions \( \alpha^{(\sigma)}_{rs} (\sigma = 1, \ldots, p) \) is substituted in (29) and the first \( l \) sets of (30), and these equations are differentiated covariantly, we have, in consequence of the above requirement, that the functions \( \alpha^{(\sigma)}_{rs/m} (\sigma = 1, \ldots, p; m = 1, \ldots, n) \) satisfy (29) and the first \( l \) sets of (30). Consequently we have

\[
\alpha^{(\sigma)}_{rs/m} = \lambda^{(\sigma \omega)}_{m} g_{rs} + \lambda^{(\sigma 1)}_{m} \alpha^{(1)}_{rs} + \cdots + \lambda^{(\sigma p)}_{m} \alpha^{(p)}_{rs},
\]

where the \( p(p+1) \) vectors \( \lambda^{(\sigma \beta)}_{m} (\sigma = 1, \ldots, p; \beta = 0, 1, \ldots, p) \) must be such that the functions (32) shall satisfy (28). Substituting in these equations we find that the functions \( \lambda \) must satisfy the system

\[
\frac{\partial \lambda^{(\sigma \tau)}}{\partial x^\rho} - \frac{\partial \lambda^{(\sigma \tau)}}{\partial x^\sigma} + \sum_{\omega} \left( \lambda^{(\sigma \omega)}_{p} \lambda^{(\omega \tau)}_{q} - \lambda^{(\sigma \omega)}_{q} \lambda^{(\omega \tau)}_{p} \right) = 0 \quad (\sigma, \omega, \tau = 0, 1, \ldots, p).
\]

In order that \( \alpha_{rs} \) given by (31) shall satisfy \( \alpha_{rs/t} = 0 \), it is necessary and sufficient that the functions \( q^{(0)} \) satisfy

\[
\frac{\partial q^{(0)}}{\partial x^t} + \sum_{\sigma} q^{(\sigma)} \lambda^{(\sigma 0)}_{t} = 0 \quad (\sigma = 1, \ldots, p),
\]

and

\[
\frac{\partial q^{(\tau)}}{\partial x^t} + \sum_{\sigma} q^{(\sigma)} \lambda^{(\sigma \tau)}_{t} = 0 \quad (\sigma, \tau = 1, \ldots, p).
\]

In consequence of (33) equations (35) are completely integrable and therefore admit solutions involving \( p \) arbitrary constants. Because of (33) the conditions of integrability of (34) are satisfied; hence \( q^{(0)} \) involves these \( p \) arbitrary constants and an additive arbitrary constant which may be neglected.*

* If \( \alpha_{rs} \) is a tensor whose first covariant derivative is zero, so also is \( \alpha_{rs} + \lambda g_{rs} \), where \( \lambda \) is an arbitrary constant.
In view of the above results we have the theorem:

If equations (29) and the first \( l (\geq 0) \) sets of equations (30) admit a complete system of solutions \( g_{rs} \) and \( a_{rs}^{(\sigma)} (\sigma = 1, \ldots, p) \) which are also solutions of the \((l + 1)\)th set of equations (30), there exists a symmetric tensor of the second order, involving \( p \) arbitrary constants, whose first covariant derivative is zero.

6. Suppose that the fundamental form is the sum of \( j \) forms (26). By definition

\[
B_{pqrs}^a = g^{aq} B_{pqrs},
\]

where \( B_{pqrs} \) is the covariant Riemann tensor of the fourth order, that is,

\[
B_{pqrs} = \frac{1}{2} \left( \frac{\partial^2 g_{ps}}{\partial x^p \partial x^r} + \frac{\partial^2 g_{qr}}{\partial x^q \partial x^p} - \frac{\partial^2 g_{pr}}{\partial x^p \partial x^q} - \frac{\partial^2 g_{qs}}{\partial x^q \partial x^s} \right)
+ g^{lm} \left( \Gamma_{pq,m} \Gamma_{qr,l} - \Gamma_{pr,m} \Gamma_{qs,l} \right),
\]

where

\[
\Gamma_{pq,m} = \frac{1}{2} \left( \frac{\partial g_{pm}}{\partial x^q} + \frac{\partial g_{sq}}{\partial x^p} - \frac{\partial g_{qs}}{\partial x^m} \right).
\]

For the case under consideration, namely (26), it is readily shown that the components \( B_{pqrs} \) are zero, unless \( p, q, r, s \) refer to the same root of (5); likewise \( B_{pqrs}^a \), and its first covariant derivatives \( B_{pqrs}^{a\mu} \). Consequently equations (29) and the first set of (30) admit, in addition to \( g_{rs} \), the \( j \) sets of solutions of the form (25). If it is understood that each of the forms (26) is not further reducible to sums of such forms, we have a complete set of solutions of (29). Hence when the space is referred to the coordinates giving (25) the number \( l \) in the preceding theorem is zero.

PRINCETON UNIVERSITY,
PRINCETON, N. J.