THE INTERSECTION NUMBERS*

BY

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1. In his first memoir on analysis situs Poincaré defined a number $N(\Gamma_k, \Gamma_{n-k})$ which had previously been considered, at least in special cases, by Kronecker. With certain conventions as to sign this number represents the excess of the number of positive over the number of negative intersections of a $k$-dimensional circuit $\Gamma_k$ with an $(n-k)$-dimensional circuit $\Gamma_{n-k}$ when both are immersed in an $n$-dimensional oriented manifold. The purpose of the present paper is to show how to calculate this number when the manifold is defined combinatorially as a collection of cells and the circuits are composed of sets of these cells; and to show how the matrices which represent the intersectional relations between the $k$-circuits and the $(n-k)$-circuits depend on the matrices of orientation of the manifold. We also define certain modulo 2 intersection numbers and discuss the matrices connected with them.

The terminology and notations of the Cambridge Colloquium Lectures on Analysis Situs (New York, 1922) will be used without further explanation, and the references not otherwise indicated will be to that book.

2. Let a manifold $M_n$ be given as the set of all points of a complex $C_n$. Let $C'_n$ be a complex dual to $C_n$ constructed as explained on page 88 by means of a complex $\overline{C}_n$ which is a regular subdivision both of $C_n$ and of $C'_n$. Every $k$-cell $a^k_j$ of $C_n$ has a single point $P_j$ (cf. p. 85) in common with a single $(n-k)$-cell $b^j_{n-k}$ of $C'_n$ which is called $b^j_{n-k}$. Our first problem will be to assign a positive or negative sign to the intersection of $a^k_j$ with $b^j_{n-k}$.

In order to do this, we suppose $M_n$ to be oriented as explained in Chapter IV and that all cells, circuits, etc., are oriented. Moreover, in the regular complex $\overline{C}_n$, in which each $i$-cell is uniquely determined by its $i+1$ vertices, the orientation of the $i$-cell will be denoted by the order in which its vertices are written, and the following two conventions will be followed: (1) if $A_0 \ A_1 \ \cdots \ A_k$ denotes a given oriented $k$-cell ($k = 1, 2, \ldots, n$) any even permutation of $A_0 \ A_1 \ \cdots \ A_k$ denotes the same oriented $k$-cell and any odd permutation denotes its negative; (2) the oriented $(k-1)$-cell $A_1 \ A_2 \ \cdots \ A_{k-1}$ is positively related to the oriented $k$-cell $A_0 \ A_1 \ \cdots \ A_k$.

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A simple argument by mathematical induction could, but will not here, be
given to prove that these notations and conventions are consistent with them-
selves and with the definition of oriented cells.

3. The \( k \)-cell \( a_j^k \) of \( C_n \) is made up of a number of \( k \)-cells of \( C_n \) having \( P_j^k \) as
their common vertex. Using the notation of page 86, let one of these be
denoted by

\[
P_a^0 P_b^1 \cdots P_i^{k-1} P_j^k,
\]

the points \( P \) being chosen, as is always possible, so that the orientation of
this \( k \)-cell agrees with that of \( a_j^k \). In like manner, \( b_j^{n-k} \) is made up of a number
of \( (n-k) \)-cells of \( C_n \) having \( P_j^k \) as their common vertex, and we let any one
of these be denoted by

\[
P_j^k P_{i+1}^{k+1} \cdots P_s^n,
\]

the points \( P \) being chosen this time so that the sense of the \( k \)-cell which they
represent agrees with that of \( b_j^{n-k} \). According as the oriented \( n \)-cell

\[
P_a^0 P_b^1 \cdots P_i^{k-1} P_j^k P_{i+1}^{k+1} \cdots P_s^n
\]

is positively or negatively oriented, we say that the intersection of \( a_j^k \) with
\( b_j^{n-k} \) is positive or negative. In the first case we write

\[
N(a_j^k, b_j^{n-k}) = 1
\]

and in the second case

\[
N(a_j^k, b_j^{n-k}) = -1.
\]

From the definition of the points \( P \) it follows directly that this definition
is independent of the particular cells of \( C_n \) which it employs. It also follows
that the function \( N \) is such that

\[
N(a_j^k, b_j^{n-k}) = -N(-a_j^k, b_j^{n-k})
\]

\[
= -N(a_j^k, -b_j^{n-k}).
\]

(3.1)

Since the relation between \( C_n \) and \( C_n' \) is reciprocal, the definition given here
determines the meaning of \( N(b_j^{n-k}, a_j^k) \), and a simple count of transpositions
in the notation gives the formula

\[
N(b_j^{n-k}, a_j^k) = (-1)^{k(n-k)} N(a_j^k, b_j^{n-k}).
\]

(3.2)
4. The cells of $C_n$ and $\overline{C}_n$ are so oriented (cf. p. 123) that

$$E'_k = \overline{E}_{n-k+1},$$

which means that $a^j_i$ is positively or negatively related to $a^{k-1}_i$ according as $b^{n-k+1}_i$ is positively or negatively related to $b^{n-k}_j$. Now the points $P$ may be so chosen that $P^0_a P^1_b \cdots P^{k-1}_i$ represents an oriented cell on $a^{k-1}_i$ and $P^{k-1}_i P^k_j P^{k+1}_l \cdots P^n_u$ represents an oriented cell on $b^{n-k+1}_i$. By the definition in § 2 above, the oriented cell $P^0_a P^1_b \cdots P^{k-1}_i$ is positively or negatively related to $P^0_a P^1_b \cdots P^{k-1}_i P^k_j$, and therefore to $b^{n-k+1}_i$, according as $(-1)^k$ is positive or negative. On the other hand, $P^{k-1}_i P^k_j P^{k+1}_l \cdots P^n_u$ is positively related to $P^k_j P^{k+1}_l \cdots P^n_u$, and therefore to $b^{n-k}_j$. Hence if $b^{n-k+1}_i$ is positively related to $b^{n-k}_j$, $a^{k-1}_i$ is positively related to $a^k_i$ and $(-1)^k N(a^{k-1}_i, b^{n-k+1}_i)$ is positive or negative according as

$$P^0_a P^1_b \cdots P^{k-1}_i P^k_j P^{k+1}_l \cdots P^n_u$$

is positively or negatively oriented. A similar result holds if $b^{n-k+1}_i$ is negatively related to $b^{n-k}_j$. Hence

$$N(a^k_i, b^{n-k}_j) = (-1)^k N(a^{k-1}_i, b^{n-k+1}_i).$$

By repeated application of this formula we obtain

$$N(a^k_i, b^{n-k}_j) = (-1)^{k(k+1)/2} N(a^0_a, b^u_u).$$

But all the $n$-cells $b^u_u$ are similarly oriented. Hence the value of $N(a^0_a, b^u_u)$ is the same for all zero cells $a^0_a$, and consequently the value of $N(a^k_i, b^{n-k}_j)$ is independent of $j$. Hence if the notation is so chosen that $b^n_i$ is positively oriented, *

$$N(a^0_a, b^n_i) = 1,$$

$$N(a^1_a, b^{n-1}_i) = -1,$$

$$N(a^2_a, b^{n-2}_i) = 1,$$

$$N(a^3_a, b^{n-3}_i) = -1,$$

and all these equations are independent of $i$.

5. An oriented complex \( \Gamma_k \) composed of the oriented \( k \)-cells \( a^1_k, a^2_k, \ldots, a^n_k \) counted \( x^1 \) times, \( x^2 \) times, \ldots, \( x^n \) times, respectively, is represented by the notation

\[
(5.1) \quad \Gamma_k = (x^1, x^2, \ldots, x^n).
\]

Let \( \Gamma'_{n-k} \) be an arbitrary oriented complex of \( C_k \), so that

\[
(5.2) \quad \Gamma'_{n-k} = (y^1, y^2, \ldots, y^n).
\]

By the number of intersections of \( \Gamma_k \) with \( \Gamma'_{n-k} \), having regard to sign, we shall mean the number \( N(\Gamma_k, \Gamma'_{n-k}) \) defined by means of the equation

\[
N(\Gamma_k, \Gamma'_{n-k}) = \sum_{j=1}^{a_k} x^j y^j N(a^j_k b^{n-k}_j)
\]

\[
(5.3) \quad = (-1)^{k(k+1)/2} \sum_{j=1}^{a_k} x^j y^j.
\]

If we recall that there are no intersections of cells of \( \Gamma_k \) of dimensionality less than \( k \) with cells of \( \Gamma'_{n-k} \) and that no cell \( a^j_k \) intersects a cell \( b^{n-k}_j \) unless \( i = j \), it is clear that this definition is in accordance with geometric intuition.

6. The last equation has as obvious corollaries the equations

\[
(6.1) \quad N(\Gamma_k + A_k, \Gamma'_{n-k}) = N(\Gamma_k, \Gamma'_{n-k}) + N(A_k, \Gamma'_{n-k}),
\]

\[
(6.2) \quad N(\Gamma_k, \Gamma'_{n-k} + A_{n-k}) = N(\Gamma_k, \Gamma'_{n-k}) + N(\Gamma_k, A_{n-k}),
\]

from which it follows that if \( \Gamma_k (i = 1, 2, \ldots, a_k) \) is any set of \( k \)-dimensional complexes on which all \( k \)-dimensional complexes of \( C_n \) are linearly dependent and \( \Gamma'_{n-k} (i = 1, 2, \ldots, a_k) \) a set of \( (n-k) \)-dimensional complexes on which all \( (n-k) \)-dimensional complexes of \( C'_n \) are linearly dependent, then if

\[
(6.3) \quad \Gamma_k = \sum_{i=1}^{a_k} x_i \Gamma'_{n-k}^i
\]

and

\[
(6.4) \quad \Gamma'_{n-k} = \sum_{i=1}^{a_k} y_j \Gamma_{n-k}^j.
\]
where the $x$'s and $y$'s are integers, then

\[(6.5) \quad N(I_k, I_{n-k}) = \sum_{i=1}^{a_k} \sum_{j=1}^{a_k} x_i y_j N(I^i_k, I^j_{n-k}).\]

Hence the intersection numbers of all $k$-dimensional complexes with all $(n-k)$-dimensional complexes depend on the matrix of numbers $N(I^i_k, I^j_{n-k})$. By choosing the complexes $I^i_k$ and $I^j_{n-k}$ in the normal manner described in the Colloquium Lectures this matrix may be given a very simple form, which we shall determine in the next three sections.

7. As proved on page 116 of the Colloquium Lectures, a set of $k$-dimensional complexes upon which all the complexes formed from cells of $C_n$ are linearly dependent may be so chosen as to consist of (1) a set of $P_{k-1}$ non-bounding circuits which we shall denote by $I^i_k (i = 1, \ldots, P_{k-1})$, or in Poincaré's notation,

\[(7.1) \quad I^i_k \equiv 0;\]

(2) a set of $\tau_k$ circuits $A^i_k (i = 1, \ldots, \tau_k)$ which satisfy the homologies

\[(7.2) \quad t^i_k \cdot A^i_k \sim 0\]

in which $t^i_k$ represents a $k$-dimensional coefficient of torsion; (3) a set of $\tau_{k+1} - \tau_k$ bounding circuits $\Theta^i_k$

\[(7.3) \quad \Theta^i_k \sim 0;\]

and (4) and (5) two sets of complexes $\Phi^i_k$ and $W^i_k$ which are not circuits but satisfy the following congruences:

\[(7.4) \quad \Phi^i_k \equiv \Theta^i_{k-1}, \quad 0 < i < \tau_k - \tau_{k-1},\]

\[(7.5) \quad W^i_k \equiv t^i_{k-1} A^i_{k-1}, \quad 0 < i < \tau_{k-1},\]

in which $\Theta^i_{k-1}$ and $A^i_{k-1}$ are defined by replacing $k$ by $k-1$ in (7.3) and (7.2).

These relations are derived from the matrix equation

\[(7.6) \quad E_k \cdot D_k = C_{k-1} \cdot E^*_k\]
which arises in reducing (cf. p. 108) the orientation matrix $E_k$ to normal form. The matrix $E_k^*$ is one in which all elements are zero except the first $r_k$ elements of the main diagonal. The first $r_k - r_{k-1}$ of the non-zero elements are 1 and the remaining $r_{k-1}$ are the coefficients of torsion of dimensionality $k - 1$.

The first $r_k - r_{k-1}$ columns of $D_k$ represent the complexes $\Phi_k^*$, the next $r_{k-1}$ columns represent the complexes $\Psi_k^*$, the next $P_k - 1$ columns represent the circuits $I_k^*$, the next $r_{k+1} - r_k$ columns represent the circuits $\Theta_k^*$, the next $r_k$ columns represent the circuits $\Delta_k^*$. Thus, for example, if the $j$th column of $D_k$ ($0 < j < r_k - r_{k-1}$) is $(x_{1j}, x_{2j}, \ldots, x_{akj})$ we have

$$\Phi_k^* = (x_{1j}, x_{2j}, \ldots, x_{akj}).$$

The columns of the matrix $C_{k-1}$ are the same as the columns of $D_{k-1}$ in a different order, and each complex represented by a column of $D_k$ is bounded by the circuit represented by the corresponding column of the matrix $C_{k-1} \cdot E_k^*$. It is from this fact that the congruences (7.4) and (7.5) are derived. The fact that $r_k, A_k, \Theta_k$ are circuits is a consequence of the fact that all elements of $E_k^*$ subsequent to the $r_k$th column are zero.

The homologies (7.2) and (7.3) arise by similar reasoning from the matrix equation

$$E_{k+1} \cdot D_{k+1} = C_k \cdot E_{k+1}^*$$

in which it is to be remembered that the columns of $C_k$ are the same as those of $D_k$ in a different order.

8. The $(n-k)$-dimensional complexes required in the formulas of § 6 may be determined by the same process as described in § 7, from the matrices of the dual complex $C'_n$. The matrices of the dual complex are related to those of $C_n$ by the equation (cf. p. 123).

$$E_{n-k} = E_{k+1}'$$

in which $E_{n-k}$ is the matrix of the relations between $(n-k-1)$-cells and $(n-k)$-cells of $C'_n$ and $E_{k+1}'$ is the matrix obtained by interchanging rows and columns of $E_{k+1}$. The equation (7.8) gives the following:

$$C_k^{-1} \cdot E_{k+1} \cdot D_{k+1} = E_{k+1}^*,$$

$$D_{k+1}' \cdot E_{k+1}' \cdot C_k^{-1} = E_{k+1}^*,$$

$$\bar{E}_{n-k} \cdot C_k^{-1} = \bar{E}_{n-k}^* \cdot D_{k+1}^*.$$
The columns of $C_k^{-1}$ determine a linearly independent set of complexes analogous to those determined by the columns of $D_k$. They are described by the following homologies and congruences, written in the order of the columns of $C_k^{-1}$:

\[(8.2) \quad \overline{\omega}_{n-k}^j \equiv \overline{\theta}_{n-k-1}^j, \quad 0 < j \leq r_{k+1} - r_k;\]

\[(8.3) \quad \overline{\psi}_{n-k}^j \equiv t_{j}^{n-k-1} \overline{\theta}_{n-k-1}^j, \quad 0 < j \leq r_k;\]

\[(8.4) \quad \overline{\tau}_{n-k}^j \equiv 0, \quad 0 < j \leq P_k - 1;\]

\[(8.5) \quad \overline{\theta}_{n-k}^j \sim 0, \quad 0 < j \leq r_k - r_{k-1};\]

\[(8.6) \quad t_{j}^{n-k} \overline{\tau}_{n-k}^j \sim 0, \quad 0 < j \leq r_{k-1}.\]

9. Since the columns of $D_k$ are the same as those of $C_k$ in a different order, and the columns of $C_k^{-1}$ are the same as the rows of $C_k^{-1}$, the matrix equation

\[(9.1) \quad C_k^{-1} \cdot C_k = 1\]

implies the relations

\[(9.2) \quad \sum_{i=1}^{a_k} x_{ij} x_{ip} = \begin{cases} 1 & \text{if } j = p \\ 0 & \text{if } j \neq p \end{cases}\]

between the columns $(x_{1j}, x_{2j}, \ldots, x_{aj})$ of $D_k$ and the columns $(x_{1p}, x_{2p}, \ldots, x_{ap})$ of $C_k^{-1}$. But by (5.3) this implies that the intersection numbers of $\Gamma_k^i, \Lambda_k^i$, etc., with $\overline{\tau}_{n-k}^j, \overline{\theta}_{n-k}^j$, etc., are zero except in the following $a_k$ cases, written in the order of the columns of $C_k^{-1}$:

\[(9.3) \quad N(\Theta_k^j, \overline{\omega}_{n-k}^j) = (-1)^{k(k+1)/2}, \quad 0 < j \leq r_{k+1} - r_k;\]

\[(9.4) \quad N(\Lambda_k^j, \overline{\psi}_{n-k}^j) = (-1)^{k(k+1)/2}, \quad 0 < j \leq r_k;\]

\[(9.5) \quad N(\Gamma_k^j, \overline{\tau}_{n-k}^j) = (-1)^{k(k+1)/2}, \quad 0 < j \leq P_k - 1;\]

\[(9.6) \quad N(\Phi_k^j, \overline{\theta}_{n-k}^j) = (-1)^{k(k+1)/2}, \quad 0 < j \leq r_k - r_{k-1};\]

\[(9.7) \quad N(\Psi_k^j, \overline{\tau}_{n-k}^j) = (-1)^{k(k+1)/2}, \quad 0 < j \leq r_{k-1}.\]
Thus, each $k$-circuit $I^i_k$ intersects the corresponding $(n-k)$-circuit once and intersects no other of the fundamental $(n-k)$-dimensional complexes. None of the other $k$-circuits ($A^i_k$ or $\Theta^i_k$) intersects any $(n-k)$-circuits, but each $\Theta^i_k$ intersects a complex $\Phi^i_{n-k}$ which is bounded by $\Theta^i_{n-k-1}$; and each $A^i_k$ intersects a complex $\Phi^i_{n-k}$ which is bounded by $A^i_{n-k-1}$ counted $t^i_k$ times. Thus we may say that each $\Theta^i_k$ links one and only one $\Theta^i_{n-k-1}$ once and each $A^i_k$ links one $A^i_{n-k-1}$ in a manner which may be described as a fractional number of times, $\pm 1/t^i_k$. A further study of these linkages would carry us beyond the bounds of the present paper.

10. The matrix spoken of at the end of § 6 is now seen to consist entirely of zeros except for $\alpha_k$ elements whose value, 1 in every case, is given by equations (9.3), \ldots, (9.7). If we limit attention to circuits the only non-zero terms which remain are those given by the intersections of $I^i_k$, \ldots, $I^{P_k-1}_k$ with the corresponding non-bounding $(n-k)$-circuits. The matrix is therefore one which consists entirely of zeros except for the first $P_1-1$ terms of the main diagonal which are all 1’s. For any $k$-circuit $I^i_k$ of $C^i_n$ we have

\[
I^i_k = \sum_{i=1}^{P_k-1} x_i I^i_k + \sum_{i=1}^{\tau_k} y_i A^i_k + \sum_{i=1}^{r_{k-1}-r_k} z_i \Theta^i_k
\]

and for any $(n-k)$-circuit of $C^i_n$ we have

\[
I_{n-k} = \sum_{i=1}^{P_{n-k}} x^i I_{n-k} + \sum_{i=1}^{\tau_{n-k}} y^i A_{n-k} + \sum_{i=1}^{r_{k-1}-r_k} z^i \Theta_{n-k}
\]

When these expressions are substituted in (6.5) there results

\[
N(I^i_k, I_{n-k}) = (-1)^{k(k+1)/2} \sum_{i=1}^{P_k-1} x^i x^i' .
\]

Thus we have the theorem that if

\[
I^i_k \sim \sum_{i=1}^{P_k-1} x_i I^i_k + \sum_{i=1}^{\tau_k} y_i A^i_k
\]

and

\[
I_{n-k} \sim \sum_{i=1}^{P_{n-k}} x^i I_{n-k} + \sum_{i=1}^{\tau_{n-k}} y^i A_{n-k}
\]

then the intersection number of $I^i_k$ with $I_{n-k}$ is given by (10.3).
This theorem has the corollary that

$$N(I_k, I_{n-k}) = 0$$

if and only if at least one of the homologies

$$p I_k \sim 0 \text{ or } q I_{n-k} \sim 0$$

is satisfied for some integer value of $p$ or $q$. In other words, the statement $p I_k \sim 0$ is equivalent to the equation

$$N(I_k, I_{n-k}) = 0$$

for the one circuit $I_k$ and all circuits $I_{n-k}$.

From this it follows that if $I'_k$ is any $k$-circuit composed of cells of $C_n$ and such that

$$I_k \sim I'_k$$

then

$$N(I_k, I_{n-k}) = N(I'_k, I_{n-k}).$$

11. Incidentally it may be remarked that (10.4) and (10.5) give rise to the following “homologies with division allowed”:

$$I_k \sim \sum_{i=1}^{P_1-1} x_i I_k^i, \quad I_{n-k} \sim \sum_{i=1}^{P_2-1} y_i I_{n-k}^i.$$

Whenever these homologies are satisfied the equation (10.3) is satisfied. As remarked by Poincaré, it is because the intersection numbers are more closely related to the homologies with division allowed than to the ordinary homologies that his attempt to prove the Euler theorem and the theorem about the duality of the Betti numbers by means of the intersection numbers was unsuccessful.
12. The fundamental sets of circuits which appear in the formulas of § 10
are chosen in a very special manner. A perfectly arbitrary fundamental set
of \( k \)-circuits is however related to this special set by homologies

\[
\mathcal{F}_k^i \sim \sum_{j=1}^{P_k-1} \alpha_j^i \mathcal{F}_k^j + \sum_{j=1}^{P_k-1+\tau_k} \alpha_j^i \mathcal{F}_k^{i-P_k+1}
\]

in which the \((P_k-1+\tau_k)\)-rowed determinant \(|\alpha_j^i| = \pm 1\). A general funda-
mental set of \((n-k)\)-circuits \(\mathcal{F}_{n-k}^i\) is related to the special set by an analogous
set of homologies. Hence the matrix of the intersection numbers

\[
N(\mathcal{F}_k^i, \mathcal{F}_{n-k}^i)
\]

is one of \(P_k-1+\tau_k\) rows and \(P_k-1+\tau_{k-1}\) columns, of rank \(P_k-1\) and
having all its invariant factors unity.

13. For some purposes it is desirable to introduce intersection numbers
which do not distinguish between positive and negative intersections. The
theory of these numbers is much simpler than that which we have been deve-
loping because all the determinations of algebraic sign in §§ 2, 3, 4, 5 can be
omitted. We simply replace the definitions of § 3 by the agreement that

\[
M(a_i^k, b_j^{n-k}) = 1 \text{ or } 0
\]

according as \(a_i^k\) and \(b_j^{n-k}\) have a common point or not. Then the definition in
§ 5 is replaced by

\[
M(\Gamma_k, \Gamma_{n-k}^i) = \sum_{j=1}^{\alpha} x^j y^j,
\]

the sum being taken modulo 2.

The determination of the intersection numbers of fundamental sets of
\( k \)-circuits and \((n-k)\)-circuits in §§ 7, 8, 9 is replaced by an analogous theory
based on the matrices \(A_{k-1}\) and \(B_k\) which arise in the reduction of the inci-
dence matrix \(H_k\) to normal form (cf. p. 79 and following pages). The result
obtained is that there exist a set of \( k \)-circuits \(\Gamma^1_k, \Gamma^2_k, \ldots, \Gamma^{P_k-1}_k\) and a set of
\((n-k)\)-circuits \(\Gamma^1_{n-k}, \Gamma^2_{n-k}, \ldots, \Gamma^{P_k-1}_{n-k}\) such that

\[
M(\Gamma^i_k, \Gamma^j_{n-k}) = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}
\]
and if

\[ I_k = \sum_{i=1}^{R_k-1} x_i I^i_k, \]

\[ I_{n-k} = \sum_{i=1}^{R_k-1} y_i I^i_{n-k}, \]

then

\[ M(I_k, I_{n-k}) = \sum_{i=1}^{R_k-1} x_i y_i \quad \text{(mod 2)}. \]

It should be observed that these formulas cannot be obtained by reducing the formulas of § 10, modulo 2, because the formulas of the present section take account of non-orientable circuits which do not enter into the theory of oriented intersections.

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