ISOMETRIC W-SURFACES*

BY

W. C. GRAUSTEIN

1. **Introduction.** An isometric W-surface is a surface whose total and mean curvatures are functionally dependent and whose lines of curvature form an isometric system. It is the purpose of this paper to give a complete classification of surfaces of this kind and to discuss the properties of the new types discovered.

Isometric W-surfaces are classified on the basis of the analytical analysis (Part I) into the three following types:

A. Surfaces of constant mean curvature.

B. Molding surfaces which have the isometric and Weingarten properties. These are the surfaces of revolution and the cylinders.

C. Special isometric W-surfaces, as follows:

C_1. A set of ∞^4 surfaces which are applicable to surfaces of revolution and can be arranged in one-parameter families so that every pair of surfaces of a family are applicable in a continuous infinity of ways with preservation of both the total and mean curvatures. To this set belong certain helicoidal surfaces.

C_2. A second set of ∞^4 surfaces. Each of these is symmetric in three mutually perpendicular planes. The lines of curvature of one family are plane curves lying in planes parallel to an axis of symmetry. For ∞^3 surfaces of the set these curves are cubics with a double point, whereas for the others they are transcendental. The ∞^4 surfaces can be arranged in three-parameter families so that every pair of surfaces of a family admit a map which preserves the lines of curvature and the principal radii of curvature.

C_3. The cones.

The surfaces A, B, and C_3 are well known as isometric W-surfaces. The non-helicoidal surfaces C_1 and the surfaces C_2 are new surfaces of this type. Their properties are established in Parts II and III of the paper.

With a complete tabulation of all the isometric W-surfaces at hand, it is not difficult to show that the surfaces of revolution of constant total curvature, together with the cylinders and the cones, are the only isometric surfaces of constant total curvature (§ 7).

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The new surfaces C_1 and C_2 result from the integration of the Gauss equation in case C. Though the elegant properties which these surfaces exhibit may prove of major interest to the reader, it was the differential equation itself which intrigued the writer. This equation, (8b) of § 3, is of peculiar form. It involves three unknown functions, $U(u)$, $V(v)$, $\varphi(u-v)$, is linear and of the first order in each of the functions U, V, but complicated and of the third order in φ. In its solution lies not only the crux but also the major difficulty of the entire problem. Previous writers, who believed they had solved it completely, fell into error and thereby failed to find all but the obvious solution, that in which U and V are constants. The present treatment (Part IV) aspires to the hope that it has escaped all pitfalls and that it may prove of interest in itself.

The literature relevant to the paper is discussed at the end of § 3.

I. Reduction of the problem

2. **Classification into types A, B, C.** Let S be an isometric surface referred to its lines of curvature and let the parameters be isometric. The linear element of S is then of the form,

(1) $$ds^2 = \lambda(du_1^2 + dv_1^2),$$

and the Codazzi equations become

(2) $$\frac{\partial e}{\partial v_1} = \frac{e+g}{2\lambda}\frac{\partial \lambda}{\partial v_1}, \qquad \frac{\partial g}{\partial u_1} = \frac{e+g}{2\lambda}\frac{\partial \lambda}{\partial u_1},$$

where e, f $(=0)$, g are the differential coefficients of the second order. Recalling that

$$\frac{1}{r_1} = \frac{e}{\lambda}, \qquad \frac{1}{r_2} = \frac{g}{\lambda},$$

setting

$$\frac{1}{r_1} + \frac{1}{r_2} = 2M, \qquad \frac{1}{r_1} - \frac{1}{r_2} = 2N,$$

and ruling out the trivial case of the sphere, we find that equations (2) can be replaced by

(3) $$\frac{\partial M}{\partial u_1} = N\frac{\partial \log \lambda N}{\partial u_1}, \qquad \frac{\partial M}{\partial v_1} = -N\frac{\partial \log \lambda N}{\partial v_1}.$$

On application of these equations, the condition that S be a W-surface becomes

$$\frac{\partial N}{\partial u_1}\frac{\partial \log \lambda N}{\partial v_1}+\frac{\partial N}{\partial v_1}\frac{\partial \log \lambda N}{\partial u_1}=0. \tag{4}$$

But this condition, by virtue of that for the compatibility of equations (3), is equivalent to

$$\frac{\partial^2 \log \lambda N}{\partial u_1\,\partial v_1}=0,$$

and hence to

$$\frac{1}{\lambda N}=m\,U_1(u_1)\,V_1(v_1), \tag{5}$$

where m is an arbitrary constant, not zero.

THEOREM 1. *A necessary and sufficient condition that an isometric surface be a W-surface is that, when it is referred to its lines of curvature and the parameters are isometric,* $\lambda(1/r_1-1/r_2)$ *is the product of a function of* u_1 *alone by a function of* v_1 *alone.*

Three cases arise, according to the nature of the functions $U_1(u_1)$, $V_1(v_1)$.

A. If both functions are constant, λN is constant and hence, by (3), S is a surface of constant mean curvature.

B. If just one of the functions is constant, λ, M, and N are functions of but one of the variables u_1, v_1. It follows that S is an isometric molding surface and hence either a surface of revolution or a cylinder.

C. The case in which neither U_1 nor V_1 is constant is that in which we are interested. Equations (4) and (3) can readily be solved for N and M. The resulting values of λ, M, and N, expressed in terms of the new parameters,

$$w=\log U_1-\log V_1,\qquad t=\log U_1+\log V_1,$$

are

$$\lambda=\frac{1}{m^2 e^t \varphi'(w)},\qquad M=-m\varphi(w),\qquad N=m\varphi'(w), \tag{6}$$

where φ is an unknown function.

It is to be noted that the constant m corresponds to a homothetic transformation of the surface and does not affect its shape.

3. The Gauss equation in Case *C*. The functions $U_1(u_1)$, $V_1(v_1)$, and $\varphi(w)$ are connected by the Gauss equation. If we set

$$\delta=\frac{d\log\varphi'}{dw},\qquad \gamma=\frac{\varphi^2-\varphi'^2}{\varphi'}, \tag{7}$$

this equation can be written in the condensed form

$$\frac{\partial}{\partial w}(P\delta)+\frac{\partial P}{\partial t}-P=2\gamma, \tag{8a}$$

where

$$P=e^{t}\left[\left(\frac{\partial w}{\partial u_1}\right)^2+\left(\frac{\partial w}{\partial v_1}\right)^2\right]. \tag{9}$$

Though this form of the Gauss equation is peculiarly adapted to certain purposes, a second form is more suitable in the discussion of its solutions. We introduce new independent and dependent variables in place of u_1, v_1, U_1, V_1, as follows:

$$\begin{gathered} u=\log U_1, \qquad v=\log V_1; \\ Ue^{-2u}=\left(\frac{d\log U_1}{du_1}\right)^2, \qquad Ve^{-2v}=\left(\frac{d\log V_1}{dv_1}\right)^2. \end{gathered} \tag{10}$$

It is to be noted that, since neither U_1 nor V_1 is constant, neither U nor V can be zero. Moreover, in terms of u and v, w and t have the simple forms

$$w=u-v, \qquad t=u+v. \tag{11}$$

If we now set

$$\alpha=\frac{1}{2}e^{-w}(1+\delta), \qquad \beta=\frac{1}{2}e^{w}(1-\delta), \tag{12}$$

the Gauss equation becomes

$$2U\alpha'+U'\alpha-2V\beta'+V'\beta=2\gamma, \tag{8b}$$

where the primes denote differentiation. Of the unknown functions, $U(u)$ and $V(v)$ enter only as explicitly shown, whereas $\varphi(w)$ is contained in α, β, and γ.

The solutions of (8b), as found in Part IV and arranged so as to correspond to the subcases under C in the introduction, are as follows:

C_1. U and V have the values

$$U=ae^{2u}+a_0, \qquad V=-ae^{2v}+b_0,$$

whereas φ is the solution of the ordinary differential equation of the third order

$$a_0\,\alpha' - b_0\,\beta' = \gamma.$$

C_2. Two composite solutions:

$$U = k_1 a^2 e^{4u} + 2k_2 a e^{3u} + k_3 e^{2u} + a_0, \quad U = -k_1 a^2 e^{4u} + 2k_2 a e^{3u} - k_3 e^{2u},$$

$$V = -k_1 b^2 e^{4v} + 2k_2 b e^{3v} - k_3 e^{2v}, \quad V = k_1 b^2 e^{4v} + 2k_2 b e^{3v} + k_3 e^{2v} + b_0,$$

$$\varphi = \frac{a_0\,a}{a e^{w} + b}, \quad a_0\,a b \neq 0; \qquad \varphi = -\frac{b_0\,b}{a + b e^{-w}}, \quad b_0\,a b \neq 0.$$

C_3. Two solutions:

$$U = a e^{2u}, \quad V \not\equiv 0, \quad \varphi = b e^{w}, \quad ab \neq 0;$$

$$U \not\equiv 0, \quad V = a e^{2v}, \quad \varphi = b e^{-w}, \quad ab \neq 0.$$

It is a simple matter to show that either of the solutions C_3 yields all the cones. The solutions C_1 and C_2 are discussed in Parts II and III respectively.

Literature. In 1883, the year of Weingarten's fundamental paper on isometric surfaces, Willgrod* obtained the general classification of § 2, but did not discuss further case C. Five years later we find Knoblauch† maintaining that the surfaces A and B are the only ones with the isometric and Weingarten properties. About 1902, Demartres‡ and Wright§ published almost simultaneously solutions of the Gauss equation for the case C. Demartres' form of the equation is essentially the same as (8b), whereas Wright's is much less convenient. Both conclude, however, that the equation can be satisfied only when U and V are constant and thus obtain, of the above solutions, only the special solution C_1 for which $a = 0$.

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* Willgrod, *Über Flächen, welche sich durch ihre Krümmungslinien in unendlich kleine Quadrate teilen lassen,* Dissertation, Göttingen, 1883.

† J. Knoblauch, *Über die Bedingung der Isometrie der Krümmungskurven,* Journal für die reine und angewandte Mathematik, vol. 103 (1888), pp. 40–43.

‡ G. Demartres, *Détermination des surfaces (W) à lignes de courbure isothermes,* Annales de Toulouse, ser. 2, vol. 4 (1902), pp. 341–355.

§ J. E. Wright, *Note on Weingarten surfaces which have their lines of curvature forming an isothermal system,* Messenger of Mathematics, vol. 32 (1902-03), pp. 133–146.

II. THE SURFACES C_1

4. Surfaces C admitting continuous deformations into themselves. In discussing the surfaces defined by the solutions C_1 it is convenient to return to the original isometric parameters u_1, v_1 and the corresponding functions U_1, V_1 of them. The equations

$$U = ae^{2u} + a_0, \qquad V = -ae^{2v} + b_0$$

are equivalent to

$$U_1'' = aU_1, \qquad V_1'' = -aV_1, \tag{13}$$

and hence to the single equation

$$U_1''V_1 + U_1V_1'' = 0. \tag{14a}$$

We proceed to prove the following characteristic property of the surfaces C_1.

THEOREM 2. *The surfaces C_1 are applicable to surfaces of revolution and are the only isometric W-surfaces of type C which have this property.*

To determine all the surfaces C applicable to surfaces of revolution, we compute $\Delta_1 w$ and $\Delta_2 w$ with respect to the linear element (1). We find that

$$\Delta_1 w = m^2\varphi'(w)P,$$

where P is given by (9). Since

$$\frac{\partial P}{\partial w} = e^t\left(\frac{\partial^2 w}{\partial u_1^2} + \frac{\partial^2 w}{\partial v_1^2}\right),$$

it follows that

$$\Delta_2 w = m^2\varphi'(w)\frac{\partial P}{\partial w}.$$

Consequently, $\Delta_1 w$ and $\Delta_2 w$ are functions of w alone if and only if P depends merely on w. But

$$\frac{\partial P}{\partial t} = P + e^t\left(\frac{\partial^2 w}{\partial u_1^2} - \frac{\partial^2 w}{\partial v_1^2}\right) = e^t\left(\frac{U_1''}{U_1} + \frac{V_1''}{V_1}\right).$$

Hence $\partial P/\partial t = 0$ only when $(14a)$ is satisfied.

Incidentally we have obtained also the following theorem:

THEOREM 3. *An isometric W-surface of type C whose curves K =* const. *are geodesic parallels admits a continuous deformation into itself.*

In other words, the further stipulation that the curves K = const. form an isometric family is here unnecessary.

In light of (5), it is clear that equation (14a) definitive of the surfaces C_1 can be put into the form

$$\frac{\partial^2}{\partial u_1^2}\frac{1}{\lambda N}+\frac{\partial^2}{\partial v_1^2}\frac{1}{\lambda N}=0. \tag{14b}$$

THEOREM 4. *A necessary and sufficient condition that an isometric W-surface C be applicable to a surface of revolution is that, when the surface is referred to its lines of curvature and the parameters are isometric, the reciprocal of $\lambda(1/r_1-1/r_2)$ be a harmonic function.*

Theorems 2, 3, and 4 are stated for isometric W-surfaces C of variable total curvature. They remain valid when the curvature is constant, provided one replaces the condition that S be applicable to a surface of revolution by demanding that S admit a continuous deformation into itself in which the curves K' = const. are the path curves, where K' is the mean curvature. We shall consider this question in more detail in § 7.

5. **Relationship to surfaces of Bonnet.** We are now in a position to connect our results with a certain theorem of Bonnet,* namely that there exist no, one, or ∞^1 surfaces applicable to a given surface with preservation of both curvatures. In fact, it is proved elsewhere† that the condition that an isometric surface admit ∞^1 surfaces applicable to it with preservation of both curvatures is precisely the condition of Theorem 4.

THEOREM 5. *The surfaces C_1 are each applicable to ∞^1 surfaces with preservation of both curvatures and are the only isometric W-surfaces of type C with this property.*

It follows then that the surfaces C_1 arrange themselves in one-parameter families so that every pair of surfaces of a family are applicable with preservation of both curvatures. Moreover, this applicability is possible in a continuous infinity of ways, by Theorem 2. It is to be noted in this connection that the surfaces of constant mean curvature arrange themselves in similar one-parameter families,‡ except that in this case the applicability is not in general possible in a continuous infinity of ways.

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* *Mémoire sur la theorie des surfaces applicables sur une surface donnée,* Journal de l'École Polytechnique, vol. 42 (1867), pp. 72 ff.

† Author, *Applicability with preservation of both curvatures,* Bulletin of the American Mathematical Society, vol. 30 (1924), pp. 19–23.

‡ Cf. Bonnet, loc. cit.

6. The functions U_1, V_1, φ. Without loss of generality we can assume that the constant a in (13) is non-negative.

If $a=0$, then $U_1=a_1u_1+a_2$ and $V_1=b_1v_1+b_2$, where $a_1b_1\neq 0$. We can take $a_1=b_1$, because of the presence of the constant m in (5), and then change to new isometric parameters so that

$$\text{I}a \qquad U_1=u_1, \qquad V_1=v_1.$$

If $a\neq 0$, it is convenient to distinguish three cases, which can be defined without loss of generality by the following pairs of values for U_1 and V_1:

$$\text{I}b \qquad U_1=\sinh u_1, \qquad V_1=\sin v_1,$$

$$\text{II} \qquad U_1=\cosh u_1, \qquad V_1=\sin v_1,$$

$$\text{III} \qquad U_1=e^{u_1}, \qquad V_1=\sin v_1.$$

For each of these four pairs of values of U_1 and V_1, the Gauss equation ($8a$) reduces to

$$\frac{d}{dw}(P\delta)-P=2\gamma, \tag{15}$$

where P has in the several cases the values

$$\text{I: } P=2\cosh w, \qquad \text{II: } P=2\sinh w, \qquad \text{III: } P=e^{w}.$$

Since (15) is an ordinary differential equation of the third order in φ, and its solutions, in the several cases, yield all the surfaces C_1, these surfaces depend upon three parameters other than m.

THEOREM 6. *There are ∞^4 isometric W-surfaces C_1.*

In Case Ia, when $a=0$ and U and V are constants, the surfaces are helicoidal, as has been shown by Demartres (loc. cit.). It can readily be proved that these are the only helicoidal surfaces of type C_1.

An isometric parameter $\bar{u}$ for the isometric family $w=$ const. is readily found from the values of $\Delta_1 w$ and $\Delta_2 w$ of § 4:

$$\bar{u}=\int\frac{dw}{P}.$$

Referred to $\bar{u}$ and a corresponding isometric parameter $\bar{v}$ for the orthogonal trajectories of the curves $w =$ const., the linear element of S takes on the form

$$(16) \qquad ds^2 = \frac{P}{m^2\varphi'}(d\bar{u}^2 + d\bar{v}^2).$$

7. **Isometric surfaces of constant curvature.** The total curvature of a surface C is

$$K = m^2(\varphi^2 - \varphi'^2).$$

Simple calculation shows that this is never constant for a surface of type C_2. In the case C_1 we have to solve (15) for K constant. It is found that the only solution occurs when $K = 0$ and $P = e^w$ (Case III). The surfaces C_1 of constant curvature are then isometric developables, not cylinders, and therefore cones.

THEOREM 7. *The only isometric surfaces of constant curvature are the cylinders, the cones, and the surfaces of revolution of constant curvature.*

We seek finally the isometric surfaces of constant curvature which admit continuous deformations into themselves in which the curves $K' =$ const. are the path curves; cf. end of § 4. The surfaces of revolution of constant curvature and the cylinders enjoy this property. It remains then to consider merely the cones.

An arbitrary cone, vertex at the origin, is represented by the equations

$$x_i = r\eta_i(s) \qquad (i = 1, 2, 3),$$

where $\eta = \eta(s)$ is a curve on the unit sphere referred to its arc s. Assuming that the cone is not a cone of revolution, we find that the curves $K' =$ const. on it are geodesic parallels if and only if the intrinsic equation of the curve η can be put into the form

$$(17) \qquad \frac{1}{R^2} = 1 + c^2 \csc^2 s, \qquad c \neq 0,$$

by measuring the arc s from a suitable point. Consequently, there is but a one-parameter family of non-congruent cones having the property in question.

Solving the Gauss equation (15) when $K = 0$ and $P = e^w$, we find that φ' is a constant multiple of P. Thus ds^2, as given by (16), is a constant multiple of $d\bar{u}^2 + d\bar{v}^2$. In other words, the geodesic parallels $K' =$ const. on one of the cones not only form an isometric family but are also geodesics; they are carried into straight lines when the cone is developed on a plane.

Each cone, according to Theorem 5, admits ∞^1 surfaces applicable to it with preservation of both curvatures. To ascertain whether any of these surfaces are cylinders, we apply to the general cylinder the condition of Theorem 4, which, as has been noted, is also the condition that an isometric surface admit ∞^1 others applicable to it with preservation of both curvatures. If we take $\lambda = 1$ and $1/r_1 = 0$, then $1/r_2$ is equal to the curvature, $1/R$, of the directrix of the cylinder, and the reciprocal of λN is proportional to R. Consequently, the condition is fulfilled if and only if the intrinsic equation of the directrix can be written in the form

$$R = cs, \qquad c \neq 0. \tag{18}$$

But the directrix is then a logarithmic spiral.

We now have ∞^1 cones and ∞^1 cylinders which are to be arranged in one-parameter families so that every pair of surfaces of a family are applicable with preservation of the mean curvature. It can be shown that there are ∞^1 of these families, corresponding to the ∞^1 values of the parameter c in (17) and (18). Each family contains a single cylinder and ∞^1 cones; the cones, however, are all congruent. We have thus an example, which is in all probability unique, of a Bonnet family which reduces essentially to two non-congruent surfaces.

III. The surfaces C_2

8. Differential coefficients. The surfaces defined by the two solutions C_2 of § 3 are identical, as is readily shown. We discuss those defined by the first, namely

$$U = k_1 a^2 e^{4u} + 2k_2 a e^{3u} + k_3 e^{2u} + a_0, \qquad V = -k_1 b^2 e^{4v} + 2k_2 b e^{3v} - k_3 e^{2v},$$

$$\varphi = \frac{a_0 a}{a e^w + b}, \qquad a_0 a b \neq 0.$$

In terms of the parameters u and v, the linear element (1) becomes

$$ds^2 = \lambda \left(\frac{e^{2u}}{U} du^2 + \frac{e^{2v}}{V} dv^2 \right).$$

For the case in hand, in accordance with (6),

$$\lambda = -\frac{(ae^w + b)^2}{m^2 a_0 a^2 e^{2u}}, \qquad M = -\frac{m a_0 a}{ae^w + b}, \qquad N = -\frac{m a_0 a^2 e^w}{(ae^w + b)^2}.$$

We now introduce the new parameters, x, y,

$$ae^u = \frac{1}{x}, \qquad be^v = \frac{1}{y};$$

and the new constants, c, c_1, c_2, c_3,

$$\frac{m a_0 a}{b} = c, \qquad c \neq 0,$$

$$\frac{k_1}{k} = c_1, \ \frac{k_2}{k} = c_2, \ \frac{k_3}{k} = c_3, \qquad \text{where } k = -a_0 a^2.$$

Then

$$M = -c\frac{x}{x+y}, \qquad N = -c\frac{xy}{(x+y)^2},$$

$$ds^2 = \frac{1}{c^2}(x+y)^2\left(\frac{dx^2}{\xi(x)} + \frac{dy^2}{\eta(y)}\right),$$

where

$$\xi(x) = c_1 + 2c_2 x + c_3 x^2 - x^4, \qquad \eta(y) = -c_1 + 2c_2 y - c_3 y^2.$$

Evidently the constant c corresponds to a homothetic transformation of the surface and does not affect its shape. We set $c = 1$ and obtain as our working formulas

$$\text{(19)} \qquad \begin{aligned} E &= \frac{(x+y)^2}{\xi(x)}, \qquad F = 0, \qquad G = \frac{(x+y)^2}{\eta(y)}, \\ \frac{1}{r_1} &= -\frac{x(x+2y)}{(x+y)^2}, \qquad \frac{1}{r_2} = -\frac{x^2}{(x+y)^2}. \end{aligned}$$

Inasmuch as the lines of curvature are still parametric, we can readily compute the coefficients in the linear element of the spherical representation:

$$\text{(20)} \qquad \mathfrak{E} = \frac{x^2(x+2y)^2}{(x+y)^2\xi}, \qquad \mathfrak{F} = 0, \qquad \mathfrak{G} = \frac{x^4}{(x+y)^2\eta}.$$

The geodesic curvature of the curves $x=$ const. on the sphere is $\sqrt{\xi}/x^2$. These curves are, therefore, circles, and the corresponding lines of curvature on the surface are plane curves.

On the other hand, the geodesic curvature, $\sqrt{\eta}/(x^2+2xy)$, of the curves $y=$ const. on the sphere is constant only if $\eta(y)=0$. But the curves $\eta(y)=0$ on the surface are singular and hence none of the regular lines of curvature $y=$ const. are plane curves.

9. Finite equations of the spherical representation. The point coördinates, ζ_1, ζ_2, ζ_3, of the spherical representation of an arbitrary surface C_2 are solutions of the differential equation

$$\frac{\partial^2\zeta}{\partial x\,\partial y}-\begin{Bmatrix}12\\1\end{Bmatrix}'\frac{\partial\zeta}{\partial x}-\begin{Bmatrix}12\\2\end{Bmatrix}'\frac{\partial\zeta}{\partial y}=0,$$

where the Christoffel symbols are formed with respect to the spherical representation. This equation becomes

$$\frac{\partial^2\zeta}{\partial x\,\partial y}-\frac{x}{(x+y)(x+2y)}\frac{\partial\zeta}{\partial x}-\frac{(x+2y)}{x(x+y)}\frac{\partial\zeta}{\partial y}=0,$$

and has as its general solution

$$\frac{x+y}{x^2}\zeta=(x+2y)X'-X+Y,$$

where $X=X(x)$ and $Y=Y(y)$. Consequently, the point coördinates of the spherical representation are of the form

$$\frac{x+y}{x^2}\zeta_i=(x+2y)X_i'-X_i+Y_i \qquad (i=1,2,3). \tag{21}$$

To determine the triples X_i and Y_i, we demand that $\mathfrak{E}$, $\mathfrak{F}$, $\mathfrak{G}$ have the values (20) and that $\left(\zeta\,\middle|\,\frac{\partial\zeta}{\partial x}\right)=0$, $\left(\zeta\,\middle|\,\frac{\partial\zeta}{\partial y}\right)=0$.* We thus get the following five equations:

$$\begin{aligned}
&(a)\quad x^6\xi(X''|X'')=\xi+x^4, \qquad (b)\ x^5(X''|2X'+Y')=-1,\\
(22)\quad &(c)\ x^2(2X'+Y'|\zeta)=1, \qquad (d)\quad x^3(X''|\zeta)=-1,\\
&(e)\quad x^4\eta(2X'+Y'|2X'+Y')=\eta+x^4.
\end{aligned}$$

* If a: a_1, a_2, a_3 and b: b_1, b_2, b_3 are two triples, $(a|b)=a_1b_1+a_2b_2+a_3b_3$.

From $(22b)$ follows the identity $(X''|Y'')=0$. The assumption that either of the triples X'', Y'' has zero components leads to a contradiction. Hence, of the two directions X'', Y'', one is always fixed and the other perpendicular to it. The attempt to make X'' fixed in direction fails. Thus we must have

$$\begin{aligned} X &= a(x)\,\alpha + b(x)\,\beta + (p_3 x + q_3)\,\gamma, \\ Y &= (p_1 y + q_1)\,\alpha + (p_2 y + q_2)\,\beta + c(y)\,\gamma, \end{aligned}$$

where α, β, γ are three fixed, mutually perpendicular, oriented directions. If we now set

$$\begin{aligned} A(x) &= a(x) + \frac{1}{2} p_1 x - q_1, \\ B(x) &= b(x) + \frac{1}{2} p_2 x - q_2, \\ C(y) &= c(y) + 2 p_3 y - q_3, \end{aligned}$$

we can write (21), dropping the subscript i, in the form

$$\text{(23)} \qquad \frac{x+y}{x^2}\,\zeta = ((x+2y)A' - A)\,\alpha + ((x+2y)\,B' - B)\,\beta + C\gamma.$$

Moreover,

$$\text{(24)} \qquad X'' = A''\alpha + B''\beta, \qquad 2X' + Y' = 2A'\alpha + 2B'\beta + C'\gamma.$$

Thus the constants p_i, q_i have disappeared and it remains merely to determine the functions $A(x)$, $B(x)$, $C(y)$ by substituting from (23) and (24) into (22).

By virtue of $(22b)$, $(22c)$ and the partial derivative with respect to y of $(22c)$ yield the following equations:

$$C'^2 = \frac{1}{\eta} - a, \qquad C''C = -\frac{1}{\eta},$$

where a is an undetermined constant.

It follows that

$$C^2 = d^2\eta, \qquad a = d^2 c_3,$$

where

$$d^2 = \frac{1}{c_2^2 - c_1 c_3},$$

provided that $c_2^2 - c_1 c_3 \neq 0$.*

Equations (b), (c), and (d) of (22) now become

$$A'^2 + B'^2 = \frac{1}{4x^4} + \frac{d^2 c_3}{4},$$

$$2A'(xA' - A) + 2B'(xB' - B) = \frac{1}{x^3} - d^2 c_2,$$

$$(xA' - A)^2 + (xB' - B)^2 = \frac{1}{x^2} + b,$$

where b is a constant to be determined. Multiplying these equations respectively by x^2, $-x$, and 1, and adding, we find

$$4x^2(A^2 + B^2) = 1 + 4bx^2 + 4d^2 c_2 x^3 + d^2 c_3 x^4.$$

By proper application of ($22a$), the constant b can be shown to have the value $d^2 c_1$;† herewith conditions (22) are completely satisfied.

In stating the result in final form we can take as the directions α, β, γ in (23) those of the three axes, ζ_1, ζ_2, ζ_3.

The parametric equations of the spherical representation of an arbitrary surface C_2 can be written in the form

$$(25) \qquad \begin{aligned} \zeta_1 &= \frac{x^2}{x+y}((2x+y)A' - A), \\ \zeta_2 &= \frac{x^2}{x+y}((2x+y)B' - B), \\ \zeta_3 &= \frac{x^2}{x+y}C, \end{aligned}$$

* When $c_2^2 - c_1 c_3 = 0$, the surfaces are always imaginary, as can be readily shown. We exclude this case henceforth.

† Consider the triple θ with the components A, B, 0. Of the elements in the determinant which is the square of the determinant $(\theta\,\theta'\,\theta'')$, $(\theta|\theta)$ and $(\theta'|\theta')$ are given above, $(\theta|\theta')$, $(\theta|\theta'')$, and $(\theta'|\theta'')$ are readily computed from them, $4x^3(\theta|\theta') = -1 + 2d^2 c_2 x^3 + d^2 c_3 x^4$, $2x^4(\theta|\theta'') = 1$, $2x^5(\theta'|\theta'') = -1$, and $(\theta''|\theta'')$ is found from ($22a$), $x^6\xi(\theta''|\theta'') = \xi + x^4$. Substitution of these values into the identity $(\theta\,\theta'\,\theta'')^2 = 0$ leads to the determination of the constant b.

where $A(x)$ and $B(x)$ are defined by the equations

$$
(26)\qquad \begin{aligned} 4x^2(A^2+B^2) &= T, & T &= 1+d^2(4c_1x^2+4c_2x^3+c_3x^4),\\ 4x^4(A'^2+B'^2) &= 1+d^2c_3x^4, & d^2 &= 1/(c_2^2-c_1c_3), \end{aligned}
$$

and

$$
(27)\qquad C^2(y) = d^2\eta, \qquad \eta = -c_1+2c_2y-c_3y^2.
$$

10. **Finite equations of the surfaces C_2.** The point coördinates, x_1, x_2, x_3, of the surface C_2 are given by

$$
x_i = -\int r_1\frac{\partial\zeta_i}{\partial x}\,dx + r_2\frac{\partial\zeta_i}{\partial y}\,dy \qquad (i = 1, 2, 3).
$$

Substituting the values of r_1 and r_2 from (19) and those of $\partial\zeta_i/\partial x$ and $\partial\zeta_i/\partial y$ as computed from (25), and then integrating, we obtain

$$
(28)\qquad \begin{aligned} x_1 &= (xA'+A)y+(xA'-A)x,\\ x_2 &= (xB'+B)y+(xB'-B)x,\\ x_3 &= (x+y)C-2\int C\,dy. \end{aligned}
$$

The isometric W-surfaces C_2 are represented by the equations (28), *where* $A(x)$, $B(x)$, *and* $C(y)$ *are defined by* (26) *and* (27).

For complete generality, the expressions for x_1, x_2, x_3 in (28) should each be multiplied by an arbitrary constant, not zero. Hence the surfaces C_2 depend on four arbitrary constants.

Since equations (26) and (27) leave the signs of A, B, C undetermined, the surface (28) is symmetric in each of the three coördinate planes.

From (28) it is evident that the lines of curvature $x =$ const. lie in planes parallel to the x_3-axis. If $c_3 \neq 0$, these lines of curvature are transcendental, for the integral of C is transcendental. If $c_3 = 0$, this integral is algebraic, x_3 has the value

$$x_3 = \left(x + y - \frac{2\eta}{3c_2}\right) C,$$

and for x constant, x_3^2 is of the form

$$x_3^2 = a_1(y - a_2)(y - a_3)^2, \qquad a_1 \neq 0.$$

Consequently, a plane line of curvature in this case is a cubic with a loop, a cusp, or an isolated double point.

An exception to these statements could arise only if $xA' + A$ and $xB' + B$ can vanish simultaneously. But it is readily shown that this is impossible, except perhaps along singular lines of the surface.

THEOREM 8. *There are ∞^4 isometric W-surfaces of type C_2. Each of these surfaces is symmetric in three mutually perpendicular planes. The curves of one family of lines of curvature lie in planes parallel to an axis of symmetry. If $c_3 = 0$, these curves are cubics, each with a double point; if $c_3 \neq 0$, they are transcendental curves.*

The ∞^4 surfaces can be arrayed in ∞^1 three-parameter families, so that every pair of surfaces of a family admit a map in which the lines of curvature correspond and the principal radii of curvature are preserved; every pair of these families are homothetic.

The last part of the theorem follows from the fact that $1/r_1$ and $1/r_2$, as given by (19), are independent of c_1, c_2, c_3.

To determine $A(x)$ and $B(x)$ more precisely, we can, in light of the first equation of (26), set

$$2xA = \sqrt{T}\cos\psi, \qquad 2xB = \sqrt{T}\sin\psi,$$

where ψ is an undetermined function of x. Differentiating and substituting the values found for A' and B' in the second of equations (26), we find that

$$\psi = 2\int \frac{\sqrt{d^2\xi}}{T}\, dx.$$

It would appear, then, that ψ is in general an elliptic integral*.

* This is certainly not true if $c_1 = c_3 = 0$. For then

$$\psi = \tan^{-1}\frac{u}{3} - \frac{1}{3}\tan^{-1} u, \text{ where } u^2 = \frac{2c_2 - x^3}{x^3}.$$

IV. SOLUTION OF THE GAUSS EQUATION

11. General method of procedure. The solution C_1. We are to solve the differential equation (8b) of § 3, namely

$$\text{I} \qquad 2U\alpha' + U'\alpha - 2V\beta' + V'\beta = 2\gamma,$$

for

$$U = U(u), \qquad V = V(v), \qquad \varphi = \varphi(w),$$

where

$$\gamma = \frac{\varphi^2 - \varphi'^2}{\varphi'}, \qquad \delta = \frac{d \log \varphi'}{dw},$$

$$(29) \qquad \alpha = \frac{1}{2} e^{-w}(1+\delta), \qquad \beta = \frac{1}{2} e^{w}(1-\delta),$$

and

$$w = u - v.$$

The expressions α and β are connected by the important identities

$$(30) \qquad e^w \alpha + e^{-w}\beta \equiv 1,$$

$$(31) \qquad (\alpha' + \alpha)e^{2u} + (\beta' - \beta)e^{2v} \equiv 0.$$

When the partial derivatives of I with respect to u and v are added, the resulting equation is

$$\text{II} \qquad 2U'\alpha' + U''\alpha - 2V'\beta' + V''\beta = 0.$$

This process, when repeatedly applied, yields the following system of equations:

$$(32) \qquad \begin{aligned} 2U'\alpha' + U''\alpha &= 2V'\beta' - V''\beta, \\ 2U''\alpha' + U'''\alpha &= 2V''\beta' - V'''\beta, \\ 2U'''\alpha' + U^{\text{IV}}\alpha &= 2V'''\beta' - V^{\text{IV}}\beta, \\ 2U^{\text{IV}}\alpha' + U^{\text{V}}\alpha &= 2V^{\text{IV}}\beta' - V^{\text{V}}\beta, \text{ etc.} \end{aligned}$$

Differentiating

$$\varphi^2 - \varphi'^2 = \varphi'\gamma,$$

and eliminating φ from this and the resulting equation, we get

$$\text{III} \qquad 4(\delta^2-1)\varphi'^2+4(\eta\delta-\gamma)\varphi'+\eta^2 = 0,$$

where

$$\eta = \gamma\delta+\gamma'.$$

Equation II is a necessary condition for the satisfaction of I and enjoys the great advantage that it involves, besides U' and V', only δ or φ', and not φ itself. Moreover, it is evident that II, from the manner in which it was derived, is also a sufficient condition that the left hand side of I be a function of w alone. Consequently, when we consider γ in III as computed from I for solutions of II, equation III also does not involve φ itself.

We can now outline our general procedure. We shall first solve equation II for U, V, and φ'. The solutions can be tested immediately in I, provided the expression found for φ' can be integrated and thus the value of γ computed, from (29). Otherwise, the solutions of II have first to be tested in III, where now γ is given by I itself.

The principal solution, C_1. In light of the identity (31), an obvious solution of II is

$$U = ae^{2u}+a_0, \qquad V = -ae^{2v}+b_0,$$

φ remaining arbitrary. For these values of U and V, I becomes

$$a_0\alpha'-b_0\beta' = \gamma.$$

This solution, C_1, of I we shall call the *principal* solution.

12. **General case:** $(\alpha\alpha''-\alpha'^2)(\beta\beta''-\beta'^2) \neq 0$. In this and the three following sections, we assume that neither $\alpha\alpha''-\alpha'^2$ nor $\beta\beta''-\beta'^2$ vanishes identically.

We begin by establishing certain necessary conditions on U and V. When we divide I by β and differentiate partially with respect to u, the result is an equation of the form

$$B_0U''+B_1U'+B_2U+B_3V+B_4 = 0,$$

where the B's are functions of w and $B_3 \neq 0$. Division by B_3 and a second differentiation with respect to u then yields

$$A_0U'''+A_1U''+A_2U'+A_3U+A_4 = 0,$$

where, in particular,

$$A_0 = \frac{\alpha\beta}{\beta\beta''-\beta'^2}.$$

Giving to w a value for which $\alpha\beta \neq 0$ and $\beta\beta'' - \beta'^2 \neq 0$, we obtain the equation

$$U''' + a_1 U'' + a_2 U' + a_3 U + a_4 = 0,$$

where the a's are constants.

Consequently, U' must satisfy an equation of the form

$$U^{\text{IV}} + a_1 U''' + a_2 U'' + a_3 U' = 0.$$

That V' must satisfy a similar equation is obvious. As a matter of fact it must satisfy the same equation. For, if we multiply the first four equations of (32) by a_3, a_2, a_1, and I respectively and add, the resulting equation and its partial derivative with respect to u are

$$2(V^{\text{IV}} + a_1 V''' + a_2 V'' + a_3 V')\beta' - (V^{\text{V}} + a_1 V^{\text{IV}} + a_2 V''' + a_3 V'')\beta = 0,$$

$$2(V^{\text{IV}} + a_1 V''' + a_2 V'' + a_3 V')\beta'' - (V^{\text{V}} + a_1 V^{\text{IV}} + a_2 V''' + a_3 V'')\beta' = 0.$$

Since $\beta\beta'' - \beta'^2 \neq 0$, the contention follows.

THEOREM 9. *In order that* I *have a solution, U' and V' must satisfy the same linear homogeneous differential equation of the third order with constant coefficients*:

$$(33)\quad U^{\text{IV}} + a_1 U''' + a_2 U'' + a_3 U' = 0, \qquad V^{\text{IV}} + a_1 V''' + a_2 V'' + a_3 V' = 0.$$

The case $U' = V' = 0$ comes under that of the principal solution and can be laid aside. Moreover, neither of the two derivatives can vanish without the other vanishing also:

THEOREM 10. *If U is constant, V is constant, and vice versa.*

For, if $U' = 0$, equation II and its partial derivative with respect to u reduce to $2V'\beta' - V''\beta = 0$, $2V'\beta'' - V''\beta' = 0$. But then, since $\beta\beta'' - \beta'^2 \neq 0$, $V' = 0$. Similarly, $V' = 0$ implies $U' = 0$.

According to Theorems 9 and 10, we can restrict U' and V' in II to be solutions of (33), neither identically zero. We can, in fact, since II is linear in U' and V', impose more rigid restrictions. For the sake of conciseness in the statement of them, let us call a solution of an equation (33) *fundamental* if it is not identically zero and depends merely on *one* of the roots of the characteristic equation,

$$(34)\qquad m^3 + a_1 m^2 + a_2 m + a_3 = 0,$$

and designate two solutions U', V' of (33) as *corresponding* if they both depend *actually* on the same roots of (34).

THEOREM 11. *In dealing with* II, U' *and* V' *can be restricted to be corresponding fundamental solutions of* (33).

For, if U' and V' are arbitrary solutions of (33), we can write $U' = \sum A_i U_i'$, $V' = \sum B_i V_i'$, where U_i' and V_i' are corresponding fundamental solutions and each of the constants A_i, B_i is either zero or unity. Substituting in II, we have

$$\sum [A_i(2 U_i' \alpha' + U_i'' \alpha) - B_i(2 V_i' \beta' - V_i'' \beta)] = 0.$$

If now we replace u by $v + w$, each bracket in the summation is a fundamental solution of the second of the equations (33), with coefficients dependent on w. But the brackets are then linearly independent and each must vanish:

$$A_i(2 U_i' \alpha' + U_i'' \alpha) - B_i(2 V_i' \beta' - V_i'' \beta) = 0.$$

But this is the result of substituting $A_i U_i'$ and $B_i V_i'$ directly into II, and since, by Theorem 10, A_i and B_i must be both unity or both zero, the desired result is established.

Suppose now that U' and V' are corresponding fundamental solutions of (33), formed for a root m, $\neq 0$, of (34). Then U and V satisfy equations of the form

$$U''' + a_1 U'' + a_2 U' + a_3 U - a_3 a_0 = 0, \quad V''' + a_1 V'' + a_2 V' + a_3 V - a_3 b_0 = 0,$$

where $a_3 \neq 0$ and a_0 and b_0 are the constants of integration in U and V. Assume further that U', V' (and a certain φ') satisfy II and hence (32). Multiplying I by a_3 and the first three equations of (32) by a_2, a_1, and 1 respectively, and adding, we get

$$a_0 \alpha' - b_0 \beta' = \gamma.$$

THEOREM 12. *If* U' *and* V' *are fundamental solutions of* (33) *which correspond to the same non-zero root of* (34) *and which with a certain* φ' *satisfy equation* II, *then for these values of* U', V', *and* φ', *equation* I *becomes*

$$a_0 \alpha' - b_0 \beta' = \gamma, \tag{35}$$

where a_0 *and* b_0 *are the absolute terms in* U *and* V.

13. **Solutions of II.** The most general corresponding fundamental solutions of (33) are

$$U' = (a_2 u^2 + a_1 u + a) e^{mu}, \qquad V' = -(b_2 v^2 + b_1 v + b) e^{mv}. \tag{36}$$

For these values of U' and V', II becomes

$$e^{mw/2}[(a_2u^2+a_1u+a)(2\alpha'+m\alpha)+(2a_2u+a_1)\alpha]$$
$$+e^{-mw/2}[(b_2v^2+b_1v+b)(2\beta'-m\beta)-(2b_2v+b_1)\beta] = 0.$$

If we set $u=v+w$, the left hand side of this equation is a function of the independent variables v and w, and is a quadratic polynomial in v. Hence its coefficients must vanish and we thus obtain three equations in w alone. The first of these is

$$a_2 e^{mw/2}\left(\alpha'+\frac{m}{2}\alpha\right)+b_2 e^{-mw/2}\left(\beta'-\frac{m}{2}\beta\right) = 0,$$

and is immediately integrable. The second can be rendered integrable by means of the first, and the third by means of the first and second. Thus in the end we have three ordinary linear equations in α and β, to which we adjoin the identity (30):

$$\begin{aligned} (4a+2a_1w+a_2w^2)e^{mw/2}\alpha+(4b-2b_1w+b_2w^2)e^{-mw/2}\beta &= 4c,\\ (a_1+a_2w)e^{mw/2}\alpha+(b_1-b_2w)e^{-mw/2}\beta &= c_1,\\ a_2e^{mw/2}\alpha+b_2e^{-mw/2}\beta &= c_2,\\ e^{w}\alpha+e^{-w}\beta &\equiv 1. \end{aligned}\tag{37}$$

When these equations are compatible, the value or values determined by them for (α, β, and hence for δ and) φ', together with the expressions (36) for U' and V', constitute the solutions of II sought.

It is readily shown that equations (37) are incompatible unless

(i) $\quad a_2=b_2=c_2=0, \quad a_1=b_1=c_1\neq 0, \quad m=2;$

(ii) $\quad a_2=b_2=c_2=0, \quad a_1=b_1=c_1=0.$

We proceed to show that in Case (i) the solutions of II can never satisfy I. Here

$$U'=(a_1u+a)e^{2u}, \quad V'=-(a_1v+b)e^{2v}, \quad a_1\neq 0,$$

and equations (37) become

$$\begin{aligned} (2a+a_1w)e^{w}\alpha+(2b-a_1w)e^{-w}\beta &= 2c,\\ e^{w}\alpha+e^{-w}\beta &\equiv 1. \end{aligned}$$

Hence α and β are of the forms $\alpha = e^{-w} R_1(w)$, $\beta = e^{w} R_2(w)$, where R_1 and R_2 are rational functions, neither zero. Consequently, α' and β' are respectively of the same forms, and γ, by Theorem 12, is of the form

$$\gamma = e^{-w} R_3(w) + e^{w} R_4(w),$$

where R_3 and R_4 are rational functions, not both zero. On the other hand, φ', as computed from δ, is of the form

$$\varphi' = h(a_2 w + l)^p \qquad h \neq 0,$$

and hence the second value of γ, computed from (29), can never be equal to the first.

14. **Corresponding solutions of I.** In Case (ii) U' and V', as given by (36), become

$$U' = a e^{mu}, \qquad V' = -b e^{mv}, \qquad ab \neq 0,$$

and equations (37) reduce to

$$(38) \qquad a e^{mw/2} \alpha + b e^{-mw/2} \beta = c, \qquad e^{w} \alpha + e^{-w} \beta \equiv 1.$$

Suppose first that $m = 2$. If $a = b = c$, equations (38) are identical and we are led to the principal solution, C_1, of § 11. Otherwise, a contradiction is readily established.

The assumption $m = 0$ leads to no solutions of I. For, in this case,

$$U = au + a_0, \qquad V = -bv + b_0,$$

and I becomes, after setting $u = v + w$ and applying the first of equations (38),

$$2\gamma = 2(aw + a_0)\alpha' + a\alpha - 2b_0\beta' - b\beta.$$

But α and β, as found from (38), are rational functions of e^{w}; hence α' and β' are also, and γ is of the form

$$\gamma = wR_1(e^{w}) + R_2(e^{w}), \qquad R_1 = a\alpha' \neq 0,$$

where R_1 and R_2 are rational functions. Moreover, since δ is a rational function of e^{w}, η and $\eta\delta - \gamma$ are of the same form as γ. Hence III can be written as

$$R_3 \varphi'^2 + (wR_4 + R_5)\varphi' + (wR_6 + R_7)^2 = 0,$$

where the R's are all rational functions of e^w. On the other hand, we find from the value of δ that φ' is of the form

$$\varphi' = R_8(e^w)\, e^{kf(R_9(e^w))},$$

where f is either a logarithm or an anti-tangent, and k a constant. Hence III can be satisfied only if $R_4 = R_6 = 0$. But this implies that $R_1 = 0$, a contradiction.

15. **Continuation. The special solutions C_2.** There remains the general case, in which $m \neq 0, 2$. Here

$$(39) \qquad U = \frac{a}{m} e^{mu} + a_0, \qquad V = -\frac{b}{m} e^{mv} + b_0, \qquad ab \neq 0,$$

and I becomes, by Theorem 12,

$$(35) \qquad \gamma = a_0\, \alpha' - b_0\, \beta'.$$

Computing α and β from (39), we find that

$$\delta = \frac{2c - a\, e^{((m/2)-1)w} - b\, e^{(1-(m/2))w}}{a\, e^{((m/2)-1)w} - b\, e^{(1-(m/2))w}}.$$

For the sake of brevity, we set

$$y = e^w,$$

and

$$k = 1 - \frac{m}{2}, \qquad k \neq 0, 1.$$

Then

$$(40) \qquad \delta = \frac{2c - a y^{-k} - b y^k}{a y^{-k} - b y^k},$$

and

$$(41) \qquad d \log \frac{d\varphi}{dw} = \frac{1}{k}\, d \log (a y^{-k} - b y^k) + \frac{2c}{k}\, \frac{dy^k}{a - b y^{2k}}.$$

It is expedient to distinguish three cases.

Case 1: $c = 0$. Here (40) and (41) become

$$(42)\quad \delta = -\frac{ay^{-k}+by^{k}}{ay^{-k}-by^{k}}, \qquad \frac{d\varphi}{dw} = h(ay^{-k}-by^{k})^{1/k}, \qquad h \neq 0.$$

From the value of δ we compute α' and β', then γ, from (35), and finally the coefficients in III. Thus III reduces to

$$4abh^2(ay^{-k}-by^{k})^{4+(2/k)} + h(ay^{-k}-by^{k})^{2+(1/k)}(a_0A+b_0B) + a^2b^2(2k-1)^2(a_0C+b_0D)^2 = 0,$$

where

$$C = (1-k)ay^{-k-1}-(1+k)by^{k-1},$$
$$D = -(1+k)ay^{-k+1}+(1-k)by^{k+1},$$
$$A = c_2a^3by^{-2k-1}+c_1a^2b^2y^{-1}+c_0ab^3y^{2k-1}+b^4y^{4k-1},$$
$$B = a^4y^{-4k+1}+c_0a^3by^{-2k+1}+c_1a^2b^2y+c_2ab^3y^{2k+1},$$

and

$$c_0 = 4k^2+4k-5, \qquad c_1 = 8k^2-8k+3, \qquad c_2 = (2k-1)^2.$$

It is readily shown, since a_0 and b_0 cannot both be zero, that a_0A+b_0B can never vanish. Consequently, k must be the reciprocal of an integer and lies then in the interval $-1 \leqq k \leqq 1/2$. Direct computation shows that III cannot be satisfied when $k = -1$, $1/2$, or $1/3$. Hence this interval can be restricted further, to

$$-\frac{1}{2} \leqq k \leqq \frac{1}{4}.$$

Three cases arise, according as $2+1/k$ is zero, positive, or negative. *First special solutions* C_2: $k = -1/2$. In this case III reduces to

$$4abh^2+h(a_0A+b_0B)+4a^2b^2(a_0C+b_0D)^2 = 0,$$

where a_0A+b_0B and $(a_0C+b_0D)^2$ are both polynomials in integral powers of y ranging from -3 to $+3$. It is found that the equation is satisfied if

$$b_0 = 0, \qquad h+a_0a^2 = 0, \qquad a_0 \neq 0,$$

or

$$a_0 = 0, \qquad h+b_0b^2 = 0, \qquad b_0 \neq 0.$$

But then we have, from (39) and (42),

$$U = \frac{a}{3}e^{3u} + a_0, \qquad V = -\frac{b}{3}e^{3v}, \qquad \varphi = \frac{a_0 a}{ae^{w} - b} + l,$$

or

$$U = \frac{a}{3}e^{3u}, \qquad V = -\frac{b}{3}e^{3v} + b_0, \qquad \varphi = \frac{b_0 b}{a - be^{-w}} + l,$$

and these sets of values actually satisfy I if in each case $l = 0$. Replacing a by $3a$ and b by $-3b$, we get, as the final form of the *first special solutions* C_2,

$$(43)\qquad \begin{aligned} &U = ae^{3u} + a_0, \qquad V = be^{3v}, \qquad \varphi = \frac{a_0 a}{ae^{w} + b}, \qquad a_0 ab \neq 0,\\ &U = ae^{3u}, \qquad V = be^{3v} + b_0, \qquad \varphi = -\frac{b_0 b}{a + be^{w}}, \quad b_0 ab \neq 0. \end{aligned}$$

If $2 + 1/k > 0$, then $0 < k \leqq 1/4$. Inspection shows that each of the first two terms in III is a polynomial in powers of y ranging from $-(4k + 2)$ to $4k + 2$ by jumps of $2k$, whereas the powers of y in the third term range only from $-(2k + 2)$ to $2k + 2$. To show that there are no solutions possible in this case, it is sufficient to set the coefficients of $y^{-(4k+2)}$ and $y^{-(2k+2)}$ equal to zero. From the resulting equations there is readily deduced a cubic equation in k, having $k = 1/2$ as a double and $k = 1/3$ as a simple root. But these values are outside the interval for k.

If $2 + 1/k < 0$, then $-1/3 \leqq k < 0$. We multiply III by $(ay^{-k} - by^{k})^{-4-(2/k)}$. The three expressions in III are then polynomials in y of degrees 0, $-2k + 2$, and $2k + 4$ respectively. Moreover, there is but a single term in each of the extreme powers $2k + 4$ and $-(2k + 4)$, and the coefficients of these terms cannot vanish simultaneously.

It remains to consider the case $c \neq 0$. Here $ab > 0$, since otherwise φ' involves an anti-tangent and III can never be satisfied. If a and b were both negative, we could replace a, b, c by $-a, -b, -c$, without changing δ.* Hence we can assume that a and b are both positive.

We now replace a and b by a^2 and b^2 respectively. Since the signs of the new a and b are at our disposal, we can choose them so that ab is opposite in sign to c and then set

$$c = -rab, \qquad r > 0.$$

* Of course, the signs of U and V are thereby changed; account of this is taken in (44).

Formulas (39), (40), and (41) become:

$$(44)\qquad U = \pm\frac{a^2}{m}e^{mu} + a_0, \qquad V = \mp\frac{b^2}{m}e^{mv} + b_0, \qquad ab \neq 0,$$

$$(45)\qquad \delta = -\frac{a^2y^{-k} + 2rab + b^2y^k}{a^2y^{-k} - b^2y^k},$$

$$(46)\qquad \frac{d\varphi}{dw} = h(ay^{-k/2} - by^{k/2})^{(1+r)/k}(ay^{-k/2} + by^{k/2})^{(1-r)/k}, \qquad h \neq 0,$$

whereas formula (35) for γ remains unchanged.

Case 2: $r = 1$. *Second special solutions* C_2. When $r = 1$, (45) and (46) reduce to

$$\delta = -\frac{ay^{-k/2} + by^{k/2}}{ay^{-k/2} - by^{k/2}}, \qquad \varphi' = h(ay^{-k/2} - by^{k/2})^{2/k}.$$

But these are precisely the values (42) of δ and φ' in Case 1, except that here we have $k/2$ where before we had k. But solutions existed in Case 1 only when $k = -1/2$. Hence they exist in this case only when $k = -1$, or $m = 4$. We are thus led to the *second special solutions* C_2, which we can write in the forms

$$(47)\qquad \begin{array}{llll} U = \pm a^2e^{4u} + a_0, & V = \mp b^2e^{4v}, & \varphi = \dfrac{a_0a}{ae^w + b}, & a_0ab \neq 0, \\ U = \mp a^2e^{4u}, & V = \pm b^2e^{4v} + b_0, & \varphi = -\dfrac{b_0b}{a + be^{-w}}, & b_0ab \neq 0. \end{array}$$

Case 3: $r \neq 1$. In this, the general, case, δ and φ' are given by (45) and (46). The analysis proceeds as in Case 1, but the reduced equation III is of such proportions that the discussion of it may well be spared the reader, — though not the writer! Suffice it, then, to say that III is never satisfied in this case.

16. **Composite solutions of** I. Thus far we have restricted ourselves to solutions U', V', δ of II in which U' and V' are corresponding fundamental solutions of (33), and to resulting solutions, U, V, φ, of I. If U_1', V_1', δ_1 and U_2', V_2', δ_2 are solutions of II of the type in question and $\delta_1 = \delta_2 = \delta$, then $k_1U_1' + k_2U_2'$, $k_1V_1' + k_2V_2'$, δ is also a solution of II. Can we then obtain from this solution by integration a set of functions, $k_1U_1 + k_2U_2$, $k_1V_1 + k_2V_2$, φ, which satisfy I?

It is only natural that in this connection the equation obtained from I by replacing γ by 0, namely

$$\text{IV} \qquad 2U\alpha' + U'\alpha - 2V\beta' + V'\beta = 0,$$

should play an important rôle.

THEOREM 13. *If* U, V, φ *is a solution of* I, *and if* U_0, V_0, φ, *where* U_0, V_0 *are of the forms* ae^{mu}, be^{mv}, $m \neq 0$, *satisfy* II, *then* $U + kU_0$, $V + kV_0$, φ *is a solution of* I.

For it is readily proved that, if U_0, V_0, φ, where U_0, V_0 are of the stated forms, satisfy II, they also satisfy the reduced equation IV. The theorem follows immediately by virtue of the linearity in U and V of I.

We obtain an important special case under the theorem when we recall that $U_0 = e^{2u}$, $V_0 = -e^{2v}$ satisfy II, no matter what the value of φ.

COROLLARY. *If* U, V, φ *is a solution of* I, $U + ke^{2u}$, $V - ke^{2v}$, φ *is also a solution of* I.

Consider now the first of each pair of special solutions C_2 of I, given by (47) and (43). The function φ is the same in both cases. Moreover, the constants a and b enter into φ only in their ratio. Consequently, we can replace, say, the solution (47) of I by $k_1 a^2 e^{4u} + a_0$, $-k_1 b^2 e^{4v}$, φ. In obtaining (43) we learned that ae^{3u}, be^{3v}, φ was a solution of II. Consequently, by Theorem 13, $k_1 a^2 e^{4u} + 2k_2 ae^{3u} + a_0$, $-k_1 b^2 e^{4v} + 2k_2 be^{3v}$, φ is a solution of I. Applying to it the corollary to the theorem, we obtain the first complete solution C_2 listed in § 3. The second is found in a similar fashion.

In proving that herewith we have exhausted composite solutions of I, let us recall, from § 13, that in solutions U', V', δ of II, where U', V' are corresponding fundamental solutions of (33), U', V' are of one of the three forms

$$\text{(i)} \qquad U' = (a_1 u + a)e^{2u}, \qquad V' = -(a_1 v + b)e^{2v}, \qquad a_1 \neq 0,$$

$$\text{(ii)} \qquad U' = a, \qquad V' = -b, \qquad ab \neq 0,$$

$$\text{(iii)} \qquad U' = ae^{mu}, \qquad V' = -be^{mv}, \qquad abm \neq 0.$$

Let

$$(48) \qquad k_1 U_1' + k_2 U_2', \qquad k_1 V_1' + k_2 V_2', \qquad \delta, \qquad k_1 k_2 \neq 0,$$

be a composite solution of II. The pairs of functions, U_1', V_1' and U_2', V_2', cannot be of the types (i) and (ii) respectively, since the values, δ_1 and δ_2, of δ corresponding to these two types are incompatible. Hence one pair of functions, say U_1', V_1', is of type (iii). Since U_1', V_1', δ satisfy II, $U_1 = U_1'/m$, $V_1 = V_1'/m$, δ satisfy the reduced equation IV. If then functions obtained from (48) by integration are to satisfy I, functions obtained from $k_2 U_2'$, $k_2 V_2'$, δ by integration must satisfy I. But the only solutions of I of this type are those from which we just formed the composite solutions C_2.

17. Exceptional case: $(\alpha\alpha''-\alpha'^2)(\beta\beta''-\beta'^2) = 0$. The solutions C_3. We first prove the following theorem:

THEOREM 14. *If* $(\alpha\alpha''-\alpha'^2)(\beta\beta''-\beta'^2) = 0$, *then* $\alpha\beta = 0$.

Suppose, first, that both $\alpha\alpha''-\alpha'^2$ and $\beta\beta''-\beta'^2$ vanish and assume that $\alpha\beta \neq 0$. It follows, then, when account is taken of (30), that

$$\alpha = ae^{-w}, \qquad \beta = be^{w}, \qquad a+b = 1, \quad ab \neq 0.$$

For these values of α and β, II is readily solved for U and V:

$$U = \left(\frac{h}{2a}u + a_0\right)e^{2u} + a_1, \qquad V = \left(-\frac{h}{2b}v + b_0\right)e^{2v} + b_1,$$

and I then reduces to

$$a a_1 e^{-w} + b b_1 e^{w} + \gamma = 0.$$

Now $\delta = a-b$. If $a-b = 0$, then $\varphi = c_1 w + c_2$, $c_1 \neq 0$; computing γ and substituting its value into the above equation leads to an immediate contradiction. Similarly, if $a - b \neq 0$.

Assume now that $\alpha\alpha''-\alpha'^2 = 0$, $\beta\beta''-\beta'^2 \neq 0$, and $\alpha\beta \neq 0$. In this case,

$$\alpha = ae^{kw}, \qquad \beta = e^{w} - a e^{(k+2)w}, \qquad a(k+1) \neq 0,$$

and II becomes

$$(U''+2kU')e^{(k-1)w} = (V''-2(k+2)V')e^{(k+1)w} - \frac{1}{a}(V''-2V').$$

Differentiating partially with respect to u, we get

$$(U'''+(3k-1)U''+2k(k-1)U')e^{-2u} = (k+1)(V''-2(k+2)V')e^{-2v}.$$

Setting each side of this equation equal to the constant $4b(k+1)^2$, solving the resulting equations for U', V', and substituting the values obtained in the reduced equation II, we obtain, finally,

$$U' = 2be^{2u} + 2ce^{-2ku}, \qquad V' = -2be^{2v}.$$

Two cases naturally arise, according as $k = 0$, or $k \neq 0$. But in both cases, γ as computed from I is a polynomial in powers of e^w; this is true also of δ:

$$\delta = 2\,a\,e^{(k+1)w} - 1,$$

and hence of all three coefficients in III. On the other hand, φ' as computed from δ has the value

$$\varphi' = c e^{-w} e^{(2a/(k+1))\,e^{(k+1)w}}, \qquad c \neq 0.$$

It follows, then, that the coefficients in III must vanish and in particular that $\delta^2 - 1 = 0$. But this is a contradiction, and the proof of the theorem is complete.

Since $\alpha\beta = 0$, either $\alpha = 0$ or $\beta = 0$. If $\beta = 0$, then $\delta = 1$ and φ is of the form

$$\varphi = ae^w + b, \qquad a \neq 0.$$

Equation I becomes

$$U' - 2\,U = 4\,be^w + \frac{2\,b^2}{a},$$

and is satisfied only if $b = 0$: $\varphi = ae^w$, and $U' - 2\,U = 0$; $U = be^{2u}$. We thus have the first of the solutions C_3. The second is obtained in a similar fassion, when $\alpha = 0$.

For both of these solutions, $K = 0$. Conversely, if $K = 0$, then $\gamma = 0$, $\varphi = ae^{\pm w}$, and $\delta = \pm 1$ or $\alpha\beta = 0$. Thus the solutions C_3 define the only isometric W-surfaces of type C which are developables.

Harvard University,
Cambridge, Mass.