# AN EXISTENCE THEOREM*

BY

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1. In an earlier paper† the author has considered a certain singular integral equation of Volterra's type, namely

$$(1) \qquad w(z) = w_0(z) + \int_z^{\infty} \sin(t-z)\, \Phi(t)\, w(t)\, dt$$

where

$$(2) \qquad w_0''(z) + w_0(z) = 0.$$

The path of integration is the ray $\arg(t-z) = 0$. The function $\Phi(t)$ is single-valued and analytic at every finite point of the sector $S$ defined by

$$(3) \qquad -\vartheta \leqq \arg z \leqq +\vartheta, \qquad |z| \geqq \varrho > 0$$

and satisfies the inequality

$$(4) \qquad |\Phi(z)| < \frac{M}{|z|^{1+\nu}}$$

in $S$, $M$ and $\nu$ being positive constants. We shall take up the question of the existence of a solution of this integral equation for renewed consideration in some detail.‡

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† *Oscillation theorems in the complex domain,* these Transactions, vol. 23, no. 4, pp. 350–385; June, 1922. The developments of the present paper are intended to complete the scanty discussion in § 4.2 of that paper.

‡ Integral equations of a similar type have been studied by Evans and Love for real variables. Love has used his results in researches concerning the behavior of solutions of linear differential equations for large positive values (see *On linear difference and differential equations,* American Journal of Mathematics, vol. 38 (1916), pp. 57–80, where further citations are to be found). Reference should also be made to the investigations of Horn (e. g. in Journal für die reine und angewandte Mathematik, vol. 133 (1908)) with the spirit of which the present paper has much in common.

2. We shall need approximate evaluations of the integral

$$I(z;\,a) = \int_z^\infty \frac{dt}{|t|^a} \tag{5}$$

where $z$ is a complex number which is not real and negative; $a$ is a real constant greater than $+1$, and the path of integration is $\arg(t - z) = 0$. Putting $t = z + u$ ($u$ real) we obtain

$$I(z;\,a) = \int_0^\infty \frac{du}{|z+u|^a}. \tag{6}$$

Using the inequality

$$|re^{i\theta} + u| = \sqrt{(r+u)^2\cos^2\frac{\theta}{2} + (r-u)^2\sin^2\frac{\theta}{2}}$$
$$> (r+u)\cos\frac{\theta}{2},$$

we find that

$$I(re^{i\theta};\,a) < \left[\sec\frac{\theta}{2}\right]^a \int_0^\infty \frac{du}{(u+r)^a} = \left[\sec\frac{\theta}{2}\right]^a \frac{r^{1-a}}{a-1}. \tag{7}$$

This evaluation, however, is not very good when $a$ is large. We can get a better one by actually computing the integral. We have

$$I(z;\,a) = r^{1-a}\int_0^\infty \frac{dv}{|v+e^{i\theta}|^a} = r^{1-a}\,J(\theta;\,a). \tag{8}$$

Further,

$$|v+e^{i\theta}|^{-a} = (1+v^2+2v\cos\theta)^{-a/2}$$
$$= (1+v)^{-a}\left[1 - \frac{4\sin^2\frac{\theta}{2}\,v}{(1+v)^2}\right]^{-a/2}.$$

If we assume $|\theta| < \pi$, the second factor in this expression can be expanded by means of the binomial theorem in a series which is uniformly convergent when $0 \leqq v \leqq +\infty$. Integrating this series term-wise, using the known formula

$$\int_0^\infty \frac{v^k\, dv}{(1+v)^{a+2k}} = \frac{\Gamma(k+1)\,\Gamma(a+k-1)}{\Gamma(a+2k)}$$

we obtain

$$J(\theta;\, a) = \frac{1}{\Gamma\left(\frac{a}{2}\right)} \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{a}{2}+k\right)\Gamma(a+k-1)}{\Gamma(a+2k)} \left(4 \sin^2 \frac{\theta}{2}\right)^k.$$

This expression can be simplified with the aid of the multiplication theorem of the $\Gamma$-function and becomes

$$J(\theta;\, a) = \frac{1}{a-1} F\left(a-1,\, 1,\, \frac{a+1}{2},\, \sin^2 \frac{\theta}{2}\right).$$

Consequently,

$$I(z;\, a) = \frac{1}{(a-1)\, r^{a-1}} F\left(a-1,\, 1,\, \frac{a+1}{2},\, \sin^2 \frac{\theta}{2}\right). \tag{9}$$

A particularly important case is that in which $a = 2$; we have*

$$I(r e^{i\theta};\, 2) = \frac{\theta}{r \sin \theta}. \tag{10}$$

In order to arrive at an approximate evaluation of $I(z;\, a)$ we use the expression of the hypergeometric series $F(\alpha,\, \beta,\, \gamma,\, x)$ in the neighborhood of $x = +1$. In the present case we find after some reduction

$$\begin{aligned} F\left(a-1,\, 1,\, \frac{a+1}{2},\, \sin^2 \frac{\theta}{2}\right) &= -F\left(a-1,\, 1,\, \frac{a+1}{2},\, \cos^2 \frac{\theta}{2}\right) \\ &\quad + 2\sqrt{\pi}\, \frac{\Gamma\left(\frac{a+1}{2}\right)}{\Gamma\left(\frac{a}{2}\right)} |\sin \theta|^{1-a}. \end{aligned} \tag{11}$$

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* Cf. Gauss, *Disquisitiones generales circa seriem infinitam* etc., *Werke,* vol. III, p. 127, formula XIV.

Since $a > +1$ the coefficients in the hypergeometric series in (9) are positive; consequently $J(\theta; a)$ is an increasing function of $|\theta|$, $0 \leqq |\theta| < \pi$. If $|\theta| \leqq \pi/2$ we get an upper limit for our function in $J(\pi/2; +a)$; from formula (11) we find

$$J\left(\frac{\pi}{2}; a\right) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{a+1}{2}\right)}{(c-1)\,\Gamma\left(\frac{a}{2}\right)}. \tag{12}$$

If $\pi/2 < |\theta| < \pi$, formula (11) tells us that

$$J(\theta; a) < 2\frac{\sqrt{\pi}\,\Gamma\left(\frac{a+1}{2}\right)}{(a-1)\,\Gamma\left(\frac{a}{2}\right)}|\sin\theta|^{1-a}. \tag{13}$$

Hence if we restrict $a$ by the assumption

$$a \geqq a_0 > 1$$

we can find a constant $C$ independent of $a$ and of $\theta$ such that

$$I(z; a) < C\frac{R^{1-a}}{\sqrt{a-1}} \tag{14}$$

where

$$R = \begin{cases} |z|, & \text{if } -\frac{\pi}{2} \leqq \arg z \leqq +\frac{\pi}{2}, \\ |y|, & \text{if } \frac{\pi}{2} < |\arg z| < \pi \end{cases} \tag{15}$$

with the understanding $z = x + iy$.

3. In order to show the existence of a solution of (1) we use the method of successive approximations. We put

$$K(z, t) = \sin(t - z)\,\Phi(t) \tag{16}$$

and define the sequence of functions

$$
(17)\qquad
\begin{aligned}
w_1(z) &= w_0(z) + \int_z^\infty K(z,t)\, w_0(t)\, dt,\\
w_2(z) &= w_0(z) + \int_z^\infty K(z,t)\, w_1(t)\, dt,\\
&\cdots\cdots\cdots\cdots\cdots\\
w_n(z) &= w_0(z) + \int_z^\infty K(z,t)\, w_{n-1}(t)\, dt,\\
&\cdots\cdots\cdots\cdots\cdots.
\end{aligned}
$$

Let $\Delta_0$ be a strip of finite width in $S$ defined by the inequalities $x \geqq A \geqq 0$, $B_1 \geqq y \geqq B_2$, and let $L$ be the maximum of the absolute value of $w_0(z)$ in $\Delta_0$. We have

$$
\begin{aligned}
w_{n+1}(z) - w_n(z) &= \int_z^\infty K(z,t)\,[w_n(t) - w_{n-1}(t)]\, dt\\
&= \int_0^\infty K(z, z+u)\,[w_n(z+u) - w_{n-1}(z+u)]\, du.
\end{aligned}
$$

Suppose that we have shown that for some value of $n$

$$
(18)\qquad |w_n(z) - w_{n-1}(z)| < \left(\frac{CM}{\sqrt{\nu}}\right)^n \frac{L}{\sqrt{n!}\,|z|^{n\nu}}
$$

when $z$ lies in $\Delta_0$. Then, using (4) and (14), we find

$$
\begin{aligned}
|w_{n+1}(z) - w_n(z)| &< \left(\frac{CM}{\sqrt{\nu}}\right)^n \frac{L}{\sqrt{n!}} \int_0^\infty \frac{|K(z, z+u)|}{|z+u|^{n\nu}}\, du\\
&< \left(\frac{CM}{\sqrt{\nu}}\right)^n \frac{LM}{\sqrt{n!}} \int_0^\infty \frac{du}{|z+u|^{(n+1)\nu+1}}\\
&< \left(\frac{CM}{\sqrt{\nu}}\right)^{n+1} \frac{L}{\sqrt{(n+1)!}\,|z|^{(n+1)\nu}}.
\end{aligned}
$$

But for $n = 1$ we have

$$w_1(z) - w_0(z) = \int_0^\infty K(z, z+u)\, w_0(z+u)\, du$$

and

$$|w_1(z) - w_0(z)| \leqq \int_0^\infty |K(z, z+u)\, w_0(z+u)|\, du$$

$$< LM \int_0^\infty \frac{du}{|z+u|^{1+\nu}} < \frac{CM}{\sqrt{\nu}} \cdot \frac{L}{|z|^\nu}.$$

Hence (18) follows by complete induction. Consequently $w_n(z)$ converges uniformly in $\Delta_0$ toward a single-valued and analytic function. On account of the uniform convergence the limiting function $w(z) = \lim_{n\to\infty} w_n(z)$ is a solution of the integral equation.

This is the only bounded solution. In fact, if a second bounded solution should exist, the difference, $D(z)$, of the two solutions would satisfy the integral equation

$$D(z) = \int_z^\infty K(z, t)\, D(t)\, dt.$$

Let $\Delta_X$ be the part of $\Delta_0$ in which $x \geqq X$ where $X$ is to be determined later, and let $\mu_X$ stand for the maximum of $|D(z)|$ in $\Delta_X$. Then using formula (14) we conclude that

$$\mu_X \leqq \frac{CM}{\sqrt{\nu}} \cdot \frac{1}{X^\nu}\, \mu_X.$$

But $X$ is at our disposal; if we make $X^\nu > CM/\sqrt{\nu}$, this inequality leads to a contradiction provided $\mu^X > 0$. Hence $D(z) \equiv 0$.

Since the width of the strip $\Delta_0$ is arbitrary we have shown that (1) has a unique analytic solution in that portion of $S$ which lies in the right half-plane. If the angle $\vartheta$ in formula (3) exceeds $\pi/2$ we can show that the solution exists also in the left half-plane in the following manner. Let $b$ be an arbitrarily large but fixed positive number; then we can find a positive constant $M_b$ such that

$$|\Phi(z)| < \frac{M_b}{|z+b|^{1+\nu}}$$

in $S$. If we go over the calculations again with this new majorant for $\Phi(z)$ we find that $w_n(z)$ converges to $w(z)$ provided the point $z=x+iy$ lies in $S$, $x>-b$, and (if $x<0$) $|y|\geqq\varrho$. The convergence is uniform in any portion of this region in which $y$ is bounded.

Various generalizations suggest themselves in connection with this proof. The function $w_0(z)$ need not satisfy the condition (2); all we have used in the proof is the property of $w_0(z)$ of being bounded in a strip where $y$ is bounded. We could also carry through the proof with a slightly more general majorant for $\Phi(z)$ than the one furnished by formula (4).

4. Let us consider a closed region $D$ in $S$ in which $y$ is bounded and $x$ is bounded below and the points whose abscissas are negative have ordinates which exceed $\varrho$ in absolute value. Let $K$ be the maximum of $|w(z)|$ in this region and let $z_1=x_1+iy_1$ be the point in $D$ where this maximum is taken on. Using (1) and (14) we find that

$$K<L+KM\int_0^\infty \frac{du}{|z_1+u|^{1+\nu}}<L+K\frac{CM}{\sqrt{\nu}\,R_1^\nu}$$

where $R_1=|z_1|$ or $|y_1|$ according as $x>0$ or $<0$. Let us choose $D$ in such a fashion that $R_1^\nu>2CM/\sqrt{\nu}$; then $K<2L$ and

$$|w(z)-w_0(z)|<\frac{2CLM}{\sqrt{\nu}\,R^\nu} \tag{19}$$

where $R=|z|$ or $|y|$ according as $x>0$ or $<0$. We can evidently drop the assumption that $x$ shall be bounded below in $D$. It is enough that $y$ shall be bounded in order that (19) shall be true. We notice that $L$ stands for the maximum of $|w_0(z)|$ in $D$.

We can arrive at a similar expression for $w(z)$ in the part of $S$ where $y>B_1$ by considering the integral equation

$$w^+(z) = w_0^+(z)+\int_z^\infty K^+(z,t)\,w^+(t)\,dt, \tag{20}$$

where

$$w_0^+(z) = e^{iz}\,w_0(z), \qquad K^+(z,t) = e^{i(z-t)}\,K(z,t) \tag{21}$$

which is satisfied by $w^+(z)=e^{iz}\,w(z)$.

It is an easy matter to show that $|w^+(z)|$ is bounded in the region $y>B_1$. If we choose $B_1$ properly we can make the maximum of $|w^+(z)|$

in the resulting region less than twice the maximum of $|w_0^+(z)|$ in the same region. Denoting the latter by $L^+$ we arrive at the expression

$$(22)\qquad |e^{iz}[w(z)-w_0(z)]| < \frac{2CL^+M}{\sqrt{\nu}\,R^\nu}$$

where $R = |z|$ or $|y|$ according as $x > 0$ or $< 0$. A similar formula can be obtained for the lower half-plane.

We have assumed that $w_0(z) = c_1 e^{iz} + c_2 e^{-iz}$. If either $c_1$ or $c_2 = 0$ we can continue the corresponding solution of (1) into a wider region. In order to fix ideas, let us assume $c_1 = 1$, $c_2 = 0$ and denote the solution of (1) by $T_1(z)$.

It can be shown* by a study of the integral equation

$$(23)\qquad u(z) = 1 + \frac{1}{2i}\int_z^\infty [e^{2i(t-z)} - 1]\,\Phi(t)\,u(t)\,dt,$$

which is satisfied by $e^{-iz}T_1(z)$, that $T_1(z)$ is analytic in the sector

$$-\pi + \varepsilon \leqq \arg z \leqq 2\pi - \varepsilon, \qquad |z| \geqq \varrho,$$

and satisfies the condition

$$(24)\qquad e^{-iz}T_1(z) = 1 + \frac{\Theta_1(z)}{z^\nu}$$

where $|\Theta_1(z)|$ is bounded in the sector in question. In fact,

$$(25)\qquad e^{-iz}T_1(z) \to 1$$

along any path in the sector $-\pi \leqq \arg z \leqq +2\pi$ whose distance from the bounding rays $\theta = -\pi$ and $\theta = 2\pi$ ultimately becomes infinite.

* For a proof valid in the case in which $\nu = 1$ see § 2.24 of *On the zeros of Mathieu functions,* Proceedings of the London Mathematical Society, vol. 23 (1924).

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