A UNIQUENESS THEOREM FOR THE LEGENDRE
AND HERMITE POLYNOMIALS*

BY

K. P. WILLIAMS

1. If we replace $y$ in the expansion of $(1 + y)^{-r}$ by $2xz + z^2$, the
coefficient of $z^n$ will, when $x$ is replaced by $-x$, be the generalized
polynomial $L^r_n(x)$ of Legendre. It is also easy to show that the Hermitian
polynomial $H_n(x)$, usually defined by

$$e^{z^2} \frac{d^n}{dx^n} e^{-z^2} = H_n(x),$$

is the coefficient of $z^n/n!$ in the series obtained on replacing $y$ in the
expansion of $e^{-y}$ by the same expression $2xz + z^2$. Furthermore, there
is a simple recursion formula between three successive Legrendre polynomials
and between three successive Hermitian polynomials. These facts suggest
the following problem.

Let

$$\varphi(y) = a_0 + a_1 y + \frac{a_2}{2!} y^2 + \frac{a_3}{3!} y^3 + \cdots$$

and put

$$\varphi(2xz + z^2) = P_0 + P_1(x) z + P_2(x) z^2 + \cdots.$$  

To what extent is the generating function $\varphi(y)$ determined if it is known
that a simple recursion relation exists between three of the successive
polynomials $P_0, P_1(x), P_2(x), \cdots$? We shall find that the generalized
Legendre polynomials and those of Hermite possess a certain uniqueness
in this regard.

2. We have

$$P_n(x) = - 1 \frac{d^n}{dz^n} \varphi(2xz + z^2) \bigg|_{z=0}.$$

When we make use of the formula for the $n$th derivative of a function
given by Faà de Bruno,† we find without difficulty

$$P_n(x) = \sum_{i,j} \frac{a_{n-i} a_{j}}{i! j!} (2x)^i.$$

* Presented to the Society, October 25, 1924.
where the summation extends to all values of $i$ and $j$ subject to the relation

$$i + 2j = n.$$  

When developed, the expression is

$$P_n(x) = \frac{a_n}{n!} (2x)^n + \frac{a_{n-1}}{(n-2)!} (2x)^{n-2} + \frac{a_{n-2}}{(n-4)! 2!} (2x)^{n-4} + \ldots.$$  

It is seen that while $P_n$ is an even or an odd function, the coefficients of the generating function that enter into it form a certain consecutive group, a fact which has important consequences.

3. Let us denote by $A_m^n$ the term in $P_n(x)$ that is of degree $m$ in $x$. We see that

$$A_n^{n-2j} = \frac{a_{n-j}}{(n-2j)! j!} (2x)^{n-2j},$$

$$A_{n+1}^{n-2j-1} = \frac{a_{n-j}}{(n-2j-1)! (j+1)!} (2x)^{n-2j-1},$$

$$A_{n+2}^{n-2j} = \frac{a_{n-j+1}}{(n-2j)! (j+1)!} (2x)^{n-2j},$$

the expressions being valid for $j = -1, 0, 1, 2, \ldots$ if we agree that $A_m^n = 0$, when $m > n$. The notable fact is that $A_n^{n-2j}$, $A_{n+1}^{n-2j-1}$ both contain $a_{n-j}$, but $A_{n+2}^{n-2j}$ contains $a_{n-j+1}$.

Let $k$ and $l$ be multipliers, which we shall assume to be polynomials in $n$ to be determined; then

$$2x l A_{n+1}^{n-2j-1} + k A_n^{n-2j} = [ln + (k-2l)j + k] \frac{a_{n-j}}{a_{n-j+1}} A_{n+2}^{n-2j},$$

a formula valid for $j = 0, 1, 2, \ldots$. Let

$$\psi(j) = ln + (k-2l)j + k.$$  

We see that

$$\psi(-1) = (n+2)l,$$

and when $n$ is even, that

$$\psi\left(\frac{n}{2}\right) = \left(\frac{n}{2} + 1\right)k.$$
This shows that, \( h \) being another polynomial in \( n \) to be determined,

\[
h P_{n+2} - 2xP_{n+1} - k P_n = \sum_{j=-1}^{n'} \left\{ h - \psi(j) \frac{a_{n-j}}{a_{n-j+1}} \right\} A_{n+2}^{n-2j},
\]

where \( n' = n/2 \), if \( n \) is even, and \( n' = (n-1)/2 \) if \( n \) is odd.

4. We see from the last expression what must be the character of the recursion relation,* and that for it to exist we must have

\[
a_{n+1} = \varphi(n) a_n,
\]

where \( \varphi(n) \) is a polynomial in \( n \). In order that the summation on the right vanish, it is necessary that

\[
\psi(j) = \varphi(n-j) \theta(n),
\]

\( \theta(n) \) being a polynomial in \( n \). The polynomial \( h(n) \) is then given at once by

\[
h(n) = \theta(n).
\]

It is easy to determine \( l \) and \( k \), so that \( \psi(j) \) will have the desired form. Since \( \varphi(n-j) \) is of the same degree in \( j \) that \( \varphi(n) \) is in \( n \), and since \( \psi(j) \) is linear in \( j \), we see that \( \varphi(n) \) must be linear in \( n \).

Put

\[
\varphi(n) = \alpha n + \beta.
\]

Then

\[
l n + (k - 2l) j + k = (\alpha n - \alpha j + \beta) \theta(n).
\]

This is to be an identity in both \( n \) and \( j \). Put \( j = -1 \), and we find

\[
(n+2) l = (\alpha n + \alpha + \beta) \theta(n).
\]

Since \( \alpha \) and \( \beta \) are arbitrary it follows that \( \theta(n) \) must contain \( n+2 \) as a factor, and

\[
l = (\alpha n + \alpha + \beta) \frac{\theta(n)}{n+2}.
\]

* It is evident that a linear recursion relation will not exist unless the factor \( 2x \) is introduced as in the middle term above.
It follows then at once that
\[ k = \left( \alpha n + 2 \beta \right) \frac{\theta(n)}{n + 2}. \]

No loss of generality results from putting
\[ h = \theta(n) = (n + 2), \quad l = (\alpha n + \alpha + \beta), \quad k = (\alpha n + 2 \beta). \]

The polynomials will therefore have the recursion relation
\[(n + 2) P_{n+2}(x) - 2x(\alpha n + \alpha + \beta) P_{n+1}(x) - (\alpha n + 2 \beta) P_n(x) = 0, \]
if
\[ a_{n+1} = (\alpha n + \beta) a_n. \]

Taking \( a_0 = 1 \), we have for generating function
\[ \varphi(y) = F\left( a, \frac{\beta}{\alpha}, a, \alpha y \right) = (1 - \alpha y)^{-\beta/\alpha}, \text{ if } \alpha \neq 0, \]
where \( F \) represents the hypergeometric function, and
\[ \varphi(y) = e^{\beta y}, \text{ if } \alpha = 0. \]

These then are the only types of generating function that will give a recursion relation, with the conditions that \( h, l, \) and \( k \) are polynomials in \( n \).*

5. A further remark might be made about the case \( \alpha \neq 0 \).

We have
\[ 2 \varphi'(2xz + z^2) = P_1'(x) + P_2'(x) z + P_3'(x) z^2 + \ldots. \]

Also we find
\[ \varphi'(y) (1 - \alpha y) = \beta \cdot \varphi(y), \]
and can then deduce
\[ P_{n+2}(x) - 2 \alpha x P_{n+1}'(x) - \alpha P_n'(x) = 2 \beta P_{n+1}(x). \]

When this is combined with the recursion formula we have
\[ x P_{n+1}'(x) + P_n'(x) = (n + 1) P_{n+1}(x), \]

* It would evidently be no more general to take \( h, l, k \) rational in \( n \).
a relation independent of $\alpha$ and $\beta$. From this and the recursion relation we can obtain

$$(1 + \alpha x^2) P'_n + (\alpha + 2\beta) x P_n' - n(\alpha n + 2\beta) P_n = 0.$$ 

Now the differential equation

$$(1 + \alpha x^2) \frac{d^2y}{dx^2} + (\alpha + 2\beta) x \frac{dy}{dx} - n(\alpha n + 2\beta) y = 0$$

is changed into

$$(n^2 - 1) \frac{d^2y}{du^2} + (1 + 2\gamma) n \frac{dy}{du} - n(n + 2\gamma) y = 0$$

by putting $u = \sqrt{1 - x^2}$, $\gamma = \beta/a$. But this is the differential equation satisfied by the generalized Legendre polynomials.

6. It is evident that we can now state the following theorem:

Let

$$\varphi(y) = a_0 + a_1 y + \frac{a_2}{2!} y^2 + \cdots,$$

and put

$$\varphi(2xz + z^2) = P_0 + P_1(x) z + P_2(x) z^2 + \cdots.$$ 

The only cases in which there will be a recursion relation of the form

$$h(n) P_{n+2}(x) - 2l(n) x P_{n+1}(x) - k(n) P_n(x) = 0,$$

where $h(n)$, $l(n)$, and $k(n)$ are polynomials, are essentially where we have the generalized polynomials of Legendre, and the polynomials of Hermite.

Indiana University
Bloomington, Ind.