THREE-DIMENSIONAL MANIFOLDS OF STATES OF MOTION*

BY

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1. Professor Birkhoff has shown† that every dynamical problem with two degrees of freedom can be represented by the motion of a particle under a field of force on a characteristic surface which is either fixed (reversible case) or rotating at uniform velocity and carrying its field of force with it (irreversible case). In either type of problem the magnitude of the velocity is a function only of the position of the particle on the surface, so that the state of motion at any instant is given by three parameters, two giving the position and one the direction of motion. We shall for the most part consider the last named parameter as varying from 0 to 2π, regarding opposite directions at a point as distinct; this is essential in the irreversible though not in the reversible case.

The three parameters may be regarded as determining a point in a three-dimensional manifold. The manifolds thus obtained are considered in this paper from the point of view of analysis situs.

It should be noted that these manifolds are not in general, as might be expected, the "product manifolds" whose points are in $1-1$ continuous correspondence with pairs of points, one point of each pair lying on the characteristic surface and one on a circle. For example, if the characteristic surface is one-sided the product manifold is one-sided, but the manifold of states of motion is orientable. Only for a surface of the connectivity of the anchor ring are the two manifolds homeomorphic. The underlying reason is the impossibility of a non-singular coördinate system or continuous distribution of vectors on any other surface.

Each orbit is represented by a curve in the manifold. These curves are called "stream lines" by Birkhoff (loc. cit.) because of a hydrodynamical analogy. The stream lines constitute a non-singular congruence, one through each point of the manifold. To a periodic orbit corresponds a closed stream line. This relation and the importance of periodic orbits add interest to the problem of finding the properties of the manifolds of states of motion. Some special manifolds of this class are described, from a point of view slightly different from that here adopted, by Birkhoff in the paper above

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† *Dynamical systems with two degrees of freedom,* these Transactions, vol. 18 (1917), pp. 212 et seq.

cited and in a paper on *The restricted problem of three bodies* in the Rendiconti del Circolo Matematico di Palermo, vol. 39 (1915), pp. 265–334.

We limit our attention to cases in which the functions appearing in the differential equations of motion are finite throughout the region considered.

2. We shall make use of the *Heegaard diagram.* This method of representing a 3-dimensional manifold supposes it divided by a surface of genus p into *canonical regions,** each of which is characterized by the fact that it can be represented by the interior of a sphere with p non-linking and knotless handles in euclidean 3-space. A canonical region can be made simply connected by the removal of p 2-cells, one severing each handle. We select a particular set of 2-cells having this property and call them *canonical cuts* and their boundaries *canonical curves.* The regions are sometimes designated as the "red" and "green" regions and their respective canonical curves as the "red" and "green" curves. The canonical curves are evidently not deformable to a point on the common surface of the two regions. If the red curves are traced on the surface of the representation in euclidean space of the green region we have a diagram which completely characterizes the manifold.

The *group* of the manifold may be obtained from the group of the surface as follows. Let the generators of the latter group be a set of p green curves $g_1, g_2, \cdots, g_p$ and p other curves $a_1, a_2, \cdots, a_p$ such that a_i and g_i $(i = 1, 2, \cdots, p)$ intersect once. Relations (which may be redundant) among the a's are found by expressing each of the red curves $r_1, r_2, \cdots, r_p$ in terms of the g's and a's and then putting each g and each r equal to the identity. The group of the 3-dimensional manifold is generated by $a_1, \cdots, a_p$, subject to the relations just found.

3. **The sphere.** To obtain the manifold of states of motion on a surface of genus 0, represent the surface by a sphere, divide the latter by a great circle, and map each hemisphere conformally on the interior of a circle. The part of the manifold corresponding to each of these regions will be homeomorphic with the interior of a tore, since a vector at the point (x, y) of the circle or its interior making an angle φ $(0 \leqq \varphi < 2\pi)$ with the positive x-axis may be represented by the point (x, y, φ) in space; the cylinder thus obtained is to be deformed so that the face $\varphi = 0$ coincides with the face $\varphi = 2\pi$. We choose $\varphi = 0$ as canonical cut in each

* This and the other terms introduced in this section have been used in a work as yet unpublished by Professor J. W. Alexander, to whom is due also this method of exposition of the Heegaard diagram. A proof that a division of the manifold into two canonical regions is always possible will be found in the 1916 *Cambridge Colloquium Lectures,* by O. Veblen, p. 148 (New York, 1922).

case. Let us denote points on one circle by the coördinate θ measuring the angle made by the radius at the point with the positive x-axis, and let us denote the corresponding points on the other circle by θ'. Let the mapping be such that θ' is the angle which the radius of the second circle makes with the positive x-axis. Since a vector at the common boundary of the two parts of the surface makes the same angle ψ with the inward-drawn normal in one case as with the outward-drawn normal in the other, the equations expressing the transformation of one tore into the other are (Fig. 1)

$$\varphi' = 2\theta - \varphi + \pi, \qquad (1)$$
$$\theta' = \theta.$$

Along the canonical curve for one region—the red region, say—φ is zero and θ varies through 2π. Hence, by (1), φ' varies through 4π and

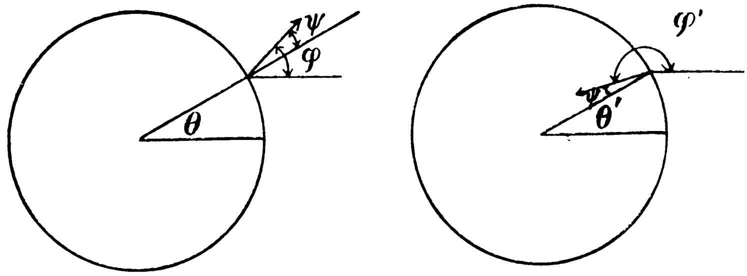

Fig. 1

θ' through 2π. This red curve cuts the green curve $\varphi' = 0$ twice; the Heegaard diagram may be taken as a tore and its interior, representing the green region, the red curve being traced on the surface. The red curve will go, so to speak, twice *around* the hole in the doughnut and once *through* the hole. Denoting a generating circle of this tore by g and a curve which cuts it once by a, the red curve is denoted by $r = a^2 g$. Hence the fundamental group of the manifold, obtained by putting $r = 1$, $g = 1$, is cyclic and of order 2.

This manifold is homeomorphic with the projective 3-space. Indeed the latter can be divided by a one-sheeted hyperboloid of revolution into two regions, each of which is homeomorphic with the interior of a tore. A plane through the axis of revolution cuts the surface in a hyperbola which bounds a 2-cell in the region not containing the center. This hyperbola has the same relation to the region containing the center that the curve on the Heegaard diagram described above bears to the region in the diagram. Having the same Heegaard diagram the two manifolds are homeomorphic.*

* O. Veblen, *Cambridge Colloquium Lectures,* p. 149.

The manifold is also homeomorphic with a two-sheeted Riemann space whose branch system consists of a pair of linking circles AGH and CDE (Fig. 2). This will appear if a pair of cuts separating the sheets is made along a strip generated in a non-singular manner by a straight segment joining a point of one circle to a point of the other, the segment moving in such a way that each end sweeps out one of the circles. The two strips form a tore. Each sheet is then a canonical region of genus 1, for it is bounded by this tore and is made simply connected by the removal of a 2-cell whose boundary is $ABCDEFGHA$ in one case, $AB'CDEF'GHA$ in the other; we suppose here that ABC and EFG lie in one sheet and $AB'C$ and $EF'G$ in the other. These curves may be deformed on the tore until they intersect in only two points, and this is the minimum number of intersections. The Heegaard diagram obtained in this way is the same as that found above for the manifold of states of motion.

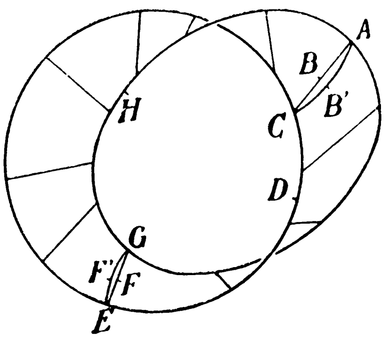

Fig. 2

4. If opposite directions are considered identical, φ will vary only from 0 to π. The cylinder mentioned in § 3 will now have altitude π. Equations (1) still hold, showing that as θ varies from 0 to 2π, φ being constant, φ' varies through 4π. Hence the Heegaard diagram will be a tore on which the characteric curve is $a^4 g$. The group is cyclic and of order 4.

The manifold is homeomorphic with a two-sheeted Riemann space whose branch system consists of a pair of knotless curves linking each other doubly in the manner indicated in Fig. 3. This is shown by an argument similar to that of the last paragraph of § 3.

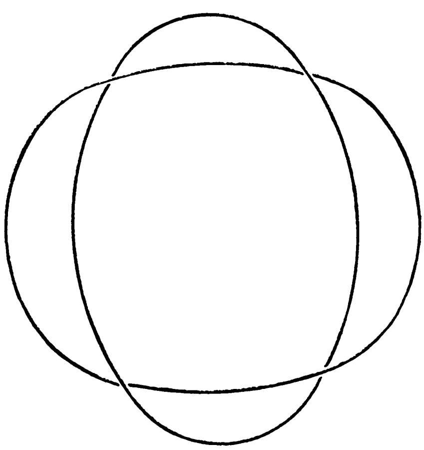
Fig. 3

5. **The projective plane.** A one-sided surface of connectivity 2 is representable by a circle and its interior, diametrically opposite points of the circle being considered identical. A vector on the boundary making an angle ψ with the outward-drawn normal is therefore to be identified with a vector at the diametrically opposite point which makes an angle $-\psi$ with the inward-drawn normal. Using the polar coördinate θ (pole at the

center) to represent points on the boundary, it is evident from principles of elementary geometry that a vector at θ making an angle φ with the direction of the polar axis is identical with a vector at $\theta+\pi$ making an angle $2\theta-\varphi$ with this direction.

The manifold of states of motion can therefore be represented by a cylinder and its interior, the ends of the cylinder being considered identical and the points of the curved surface being matched in such a way that (θ, φ) is identical with $(\theta+\pi, 2\theta-\varphi)$. Thus the points $(0, 0)$, $(\pi, 0)$, $(0, 2\pi)$ and $(\pi, 2\pi)$ all represent the same point A of the manifold (Fig. 4). Let α be a curve on the cylinder for which $\varphi = 0$ and $0 \leqq \theta \leqq \pi$. Then α represents a curve in the manifold which is also represented on the cylinder by $\varphi = 2\pi$, $0 \leqq \theta \leqq \pi$ and by $\varphi \equiv 2\theta \pmod{2\pi}$, $\pi \leqq \theta \leqq 2\pi$. Similarly, let β be a curve on the cylinder for which $\varphi = 0$ and $\pi \leqq \theta \leqq 2\pi$. It represents a curve in the manifold which is also represented by $\varphi = 2\pi$, $\pi \leqq \theta \leqq 2\pi$, and by $\varphi = 2\theta$, $0 \leqq \theta \leqq 2\pi$.

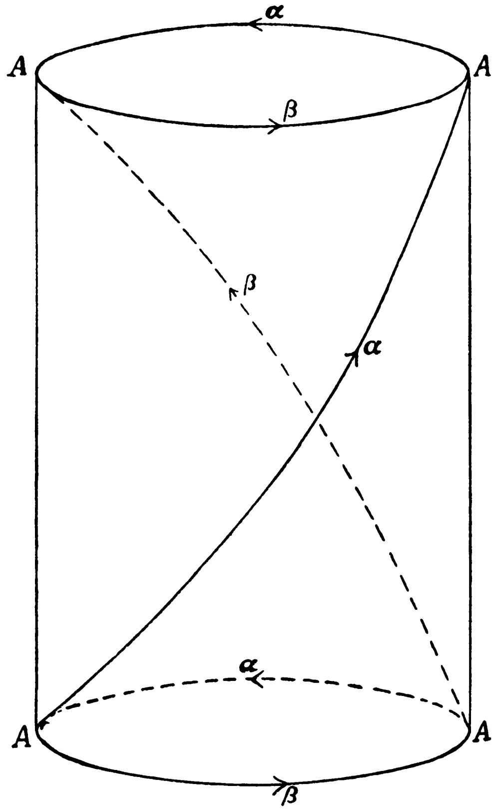

Fig. 4

To obtain a Heegaard diagram, surround* each of the curves in the manifold represented by α and β by a fine tube terminating in a ball surrounding A. On this ball with two handles are to be traced a pair of curves which bound 2-cells in the remainder of the manifold. For these 2-cells we may take the parts exterior to our fine tube of (1) an end of the cylinder and (2) a surface within the cylinder bounded by the semicircles $0 \leqq \theta \leqq \pi$ on the bases and by the curves $\varphi = 2\theta$. Thus we find for the group of the manifold two generators α and β connected by the two relations $\alpha\beta = 1$ and $\alpha^3\beta^{-1} = 1$, from which it follows that the group is cyclic and of order 4.

* Veblen, loc. cit., p. 148, § 41.

6. To show that this manifold is homeomorphic with that of § 4 we use a lemma due essentially to Heegaard:

If on a Heegaard diagram of genus 2 *a red curve* r *cuts a green curve* g *exactly once, the manifold may also be represented by a Heegaard diagram of genus* 1.

We may prove this lemma by representing the green region by a ball with two handles in such a way that g bounds a 2-cell severing one of the handles and any other curve g' which bounds a 2-cell in the interior but not on the surface and which does not intersect g and is not deformable into g bears the same relation to the other handle.

For the case in which r fails to meet g', so that it merely traverses the handle severed by the 2-cell bounded by g, the lemma may be proved simply by transferring from the red to the green region a 3-cell obtained by slightly thickening the 2-cell bounded by r.

A reduction to this case can always be effected on account of the freedom possible in choosing the second green curve g'. For we can always find a curve $g' = rgr^{-1}g^{-1}$ which is not deformable into g and which fails to bound on the surface but does bound a 2-cell in the green region. Such a 2-cell can be constructed by combining a strip bounded by a portion of r, counted twice, with a pair of 2-cells which are bounded by curves into which g is deformable on the surface. Since g' need not meet r this brings us to the former case.

7. Since one of the handles of the Heegaard diagram described in § 5 is traversed just once by either of the mentioned curves which bound in the other part of the manifold, it follows from the lemma of § 6 that the manifold has a Heegaard diagram consisting of a tore on which is traced one characteristic curve. Since the group is cyclic and of order 4 this curve must be expressible in the form $a^4 g^{2m+1}$. All the diagrams thus obtained are homeomorphic with a tore on which the characteristic curve is $a^4 g$.

The manifold of states of motion on a projective plane, opposite directions being considered distinct, is thus homeomorphic with the manifold for a sphere on which opposite directions are considered identical.

From this and from the theorem that on every smooth surface of genus 0 there is a closed geodesic it may be inferred that on every smooth surface of connectivity 2 there is a closed geodesic. This is otherwise evident because a two-sheeted covering surface of genus 0 may be placed upon a closed surface of connectivity 2.

8. If opposite directions on the projective plane are considered identical the manifold is represented by a cylinder whose bases correspond to each other and whose convex surface is divided by curves which correspond to

parts of the edges into four 2-cells which correspond to each other by pairs. By a method similar to that of § 5 it is easily found that the group of this manifold has two generators α and β connected by the two relations

$$\alpha\beta\alpha\beta^{-1} = 1 \qquad \text{and} \qquad \beta\alpha\beta\alpha^{-1} = 1 .$$

It follows that the connectivity is 3 and that there are two coefficients of torsion, each equal to 2.

9. **Surfaces of the connectivity of the anchor ring.** Both the anchor ring and the one-sided surface of the same connectivity are loci of points on a circle which is deformed into itself through a non-singular set of intermediate positions. If the generating circle returns with sense preserved the surface is orientable; if with sense reversed it is non-orientable. In either case we set up a system of coördinates u and v such that u is constant along a generating circle, v is constant along a curve which cuts each generating circle once, $0 \leqq u \leqq 2\pi$ and $-1 \leqq v \leqq 1$. The point $(u, 1)$ coincides with $(u, -1)$. Also $(0, v)$ coincides with $(2\pi, v)$ if the surface is orientable, otherwise with $(2\pi, -v)$.

A direction at a point on the surface may be uniquely specified by the angle $\varphi\,(-\pi \leqq \varphi \leqq \pi)$ which it makes with the curve $v =$ constant through the point, angles being measured positively from the direction of increasing u to that of increasing v. φ varies continuously with a moving vector except, in the one-sided case, at $u = 2\pi$, where φ and v change sign abruptly.

The portion of the manifold of states of motion corresponding to a generating circle (or any other closed curve) is a tore, since its points can be specified by the two cyclic coördinates v and φ. The manifold can thus be represented by a continuous one-parameter family of coaxial tores $u =$ constant, each within all the preceding.

In the orientable case the innermost and outermost tores of the family are matched together in the manner determined simply by swelling the inner one: $(-1, v, \varphi)$ is identical with $(1, v, \varphi)$. But if the characteristic surface is one-sided the tores are to be matched in such a way that $(-1, v, \varphi)$ is identical with $(1, -v, -\varphi)$.

The manifold for the one-sided surface is homeomorphic with a two-sheeted Riemann space whose branch system consists of two pairs of circles, each pair forming the boundary of a plane ring and each curve linking once each curve of the other pair (Fig. 5). (These plane rings we take as the surfaces on which the two sheets join.) For if we construct in each sheet a pair of non-intersecting tores, one surrounding each of the plane rings, we divide the space into four parts, each of which is homeomorphic with the region between two coaxial tores of which one is within the other. Since each part is the locus of points on a one-parameter family of non-intersecting tores, and since the four parts are arranged cyclically, the whole space is such a locus. The first and last tores of the family coincide with change of sign of both coördinates of a point, since one coördinate changes sign each time the moving tore passes from one sheet to the other and then covers a former position, and neither coördinate changes sign twice.

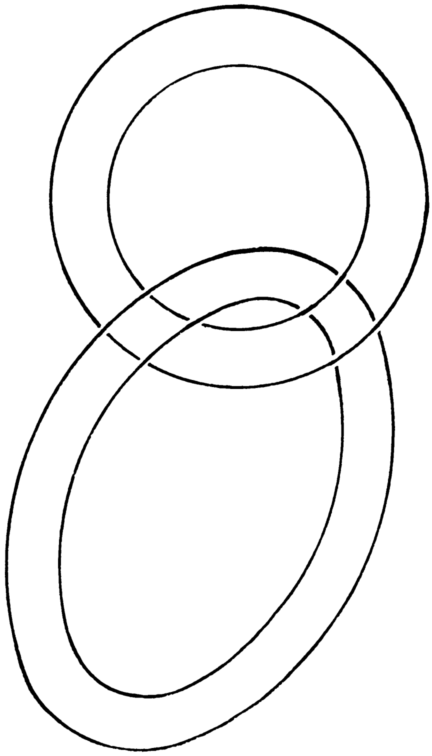

Fig. 5

In the same way the manifold of states of motion on a tore is seen to be homeomorphic with a four-sheeted space with the same branch curves if Sheet 1 permutes with Sheet 2 and Sheet 3 with Sheet 4 about each curve of one pair, while Sheet 1 permutes with Sheet 4 and Sheet 2 with Sheet 3 about each curve of the other pair.

10. **Closed surfaces in general.** We now pass to the general case of a closed surface of connectivity greater than that of the anchor ring.

The letter k is used in this paper, whether referring to manifolds of two or of three dimensions, to mean the number of independent non-bounding closed curves. Thus for an orientable surface of genus p, $k = 2p$. The (linear) connectivity is $k+1$.

The surface may be represented* by a plane polygon, with its interior, having $2k$ directed edges which correspond by pairs, all the vertices representing the same point of the surface. In passing around the polygon the edges occur in the order

$$a_1 \ b_1 \ a_1^{-1} \ b_1^{-1} a_2 \ \cdots \ a_p \ b_p \ a_p^{-1} \ b_p^{-1} \qquad (p = k/2)$$

if the surface is orientable; in the order

$$a_1 \ b_1 \ a_1^{-1} \ b_1^{-1} a_2 \ \cdots \ a_p \ b_p \ a_p^{-1} \ b_p^{-1} \ d_1 \ d_1 \qquad (p = (k-1)/2)$$

* Veblen, loc. cit., pp. 70–72, 137.

if the surface is one-sided and k is odd; or in the order

$$a_1\ b_1\ a_1^{-1}\ b_1^{-1}\ a_2\ \cdots\ a_p\ b_p\ a_p^{-1}\ b_p^{-1}\ d_1\ d_1\ d_2\ d_2 \quad (p=(k-2)/2)$$

if the surface is one-sided and k is even.

First suppose $k \geqq 3$. We use a curvilinear polygon whose edges are congruent circular arcs making interior angles π/k at the vertices. Then since the sum of the interior angles is 2π the surface can be mapped conformally on the interior and boundary of the polygon. The angle subtended by each edge at its center is $m = \pi(1-2/k)$.

The possible states of motion at points within the polygon are represented by the points within a fluted prism of height 2π (regarding opposite directions as distinct) erected upon the polygon as base, the height of a point being equal to the angle φ which the corresponding vector makes with a fixed direction in the plane. The bases of the prism correspond to each other in the manner determined by a translation of one into the other. The edges perpendicular to the bases all represent the same closed curve in the manifold of states of motion.

To see how the lateral faces correspond, observe that a vector on an edge of the plane polygon making an angle ψ with the outward-drawn normal corresponds to a vector at the corresponding point of the corresponding edge which makes with the inward-drawn normal an angle $-\psi$ or ψ according as the corresponding edges in question are described in the same or opposite senses in going around the polygon. From this it follows by means of elementary plane geometry that the inclination φ' of a vector on the edge a_i^{-1} is related to the inclination φ of the corresponding vector on a_i, both vectors being at points having angular distance θ from the initial points of their respective edges, by the formula

$$\varphi' = \varphi + 2\theta - 2m; \tag{2}$$

and similarly for the sides b_i and b_i^{-1}. For a pair of adjacent corresponding edges d_i the relation is

$$\varphi' = -\varphi - 2\theta + \frac{\pi}{k} + 2\varphi_0, \tag{3}$$

φ_0 being a constant representing the inclination of the outward-drawn normal to the first of the two sides d_i when $\theta = 0$.

On the vertical edges of the fluted prism select a set of points $A_1, A_2, \cdots, A_{2k}$, all representing the same point A of the manifold, such that A_1 is on the edge passing through the initial point of a_1, A_2 is on the edge passing through the terminal point of a_1, and the other points lie successively, one on each edge, in the order thus determined. Let α_i be an arbitrary directed curve running from A_{4i-3} to A_{4i-2} and lying on that lateral face of the prism which is bounded in part by a_i. Let β_i be an arbitrary curve running from A_{4i-2} to A_{4i-1} and lying on that lateral face of the prism which is bounded in part by b_i. Let γ be the curve in the manifold beginning and ending at A which is represented by the vertical edges of the prism.

On that lateral face of the prism which is bounded in part by a_i^{-1} lies a curve α_i' representing the same curve in the manifold as α_i. Recalling that θ varies along each side between 0 and m, it will be seen that (2) shows that in going along α_i from A_{4i-3} to A_{4i-2} and then along α_i' from A_{4i-1} to A_{4i} the sum of the variations of φ is $-2m$. Similarly the sum of

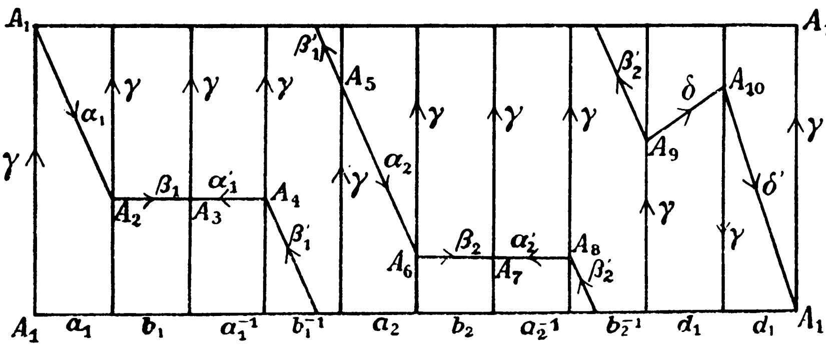

Fig. 6

the variations of φ in going along β_i from A_{4i-2} to A_{4i-1} and then along the corresponding curve β_i' from A_{4i} to A_{4i+1} is $-2m$. Thus in passing continuously around the prism the total variation of φ for each group of faces incident with the group of edges $a_i\, b_i\, a_i^{-1}\, b_i^{-1}$ is $-4m$.

For a pair of lateral faces incident with a pair of corresponding edges d_i the total variation of φ in going along a curve δ_i on the first face and then along the corresponding curve δ_i' on the second face is, according to (3), $-2m$.

For an orientable surface there are $p = k/2$ groups of sides $a_i\, b_i\, a_i^{-1}\, b_i^{-1}$ and no d's; the total variation of φ in going around the prism is thus $-4mp = -4\pi(1-2/k)\cdot k/2 = -2\pi(k-2)$.

For a one-sided surface of odd connectivity, $p = (k-1)/2$ and there is one pair of d's, so that the variation of φ is $-4m(k-1)/2-2m = -2\pi(k-2)$.

For a one-sided surface of even connectivity, $p = (k-2)/2$ and there are two pairs of d's, so that φ varies through $-4m(k-2)/2-4m = -2\pi(k-2)$.

A map of the lateral faces of the prism for the case $k = 5$, developed upon a plane, is shown in Fig. 6.

To obtain the Heegaard diagram, which is of genus $k+1$, each of the curves α_i, β_i, γ, δ_i in the manifold is surrounded by a fine tube beginning and ending in a ball surrounding A. The $k+1$ curves which bound in the remainder of the manifold are those which bound a base of the prism and the k pairs of ıaces. In this way we find that the group has $k+1$ generators connected by the following $k+1$ generating relations:

For the orientable case,

$$\begin{aligned} \gamma^{k-2}\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\cdots\alpha_p\beta_p\alpha_p^{-1}\beta_p^{-1} &= 1, \\ \alpha_i\gamma\alpha_i^{-1}\gamma^{-1} &= 1, \\ \beta_i\gamma\beta_i^{-1}\gamma^{-1} &= 1 \qquad \left(i = 1, 2, \cdots, p;\, p = \frac{k}{2}\right). \end{aligned}$$

For the one-sided case, k odd,

$$\begin{aligned} \gamma^{k-2}\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\cdots\alpha_p\beta_p\alpha_p^{-1}\beta_p^{-1}\delta^2 &= 1, \\ \alpha_i\gamma\alpha_i^{-1}\gamma^{-1} &= 1, \\ \beta_i\gamma\beta_i^{-1}\gamma^{-1} &= 1, \\ \gamma\delta\gamma\delta^{-1} &= 1 \qquad \left(i = 1, 2, \cdots, p;\, p = \frac{k-1}{2}\right). \end{aligned}$$

For the one-sided case, k even,

$$\begin{aligned} \gamma^{k-2}\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\cdots\alpha_p\beta_p\alpha_p^{-1}\beta_p^{-1}\delta_1^2\delta_2^2 &= 1, \\ \alpha_i\gamma\alpha_i^{-1}\gamma^{-1} &= 1, \\ \beta_i\gamma\beta_i^{-1}\gamma^{-1} &= 1, \\ \gamma\delta_1\gamma\delta_1^{-1} &= 1, \\ \gamma\delta_2\gamma\delta_2^{-1} &= 1 \qquad \left(i = 1, 2, \cdots, p;\, p = \frac{k-2}{2}\right). \end{aligned}$$

From the groups the connectivities and coefficients of torsion are easily found.* The connectivity of the manifold of states of motion on a closed surface is in every case greater by unity than the connectivity of the surface. If the surface is orientable there is one coefficient of torsion, equal to $k-2$. If the surface is non-orientable and k is even, there are

* Veblen, loc. cit., Chap. V.

two coefficients of torsion, each equal to 2; if k is odd, 4 is the only coefficient of torsion. In every case the 3-dimensional manifold is orientable.

The results of the present section are true for all values of k. If $k = 2$ the argument requires only slight verbal modifications made necessary by the fact that the sides of the polygon are straight, forming a rectangle. The formulas pertaining to the groups agree with the results found in §§ 3 and 5 for $k = 0$ and $k = 1$, respectively.

11. If opposite directions are considered identical the groups described in § 10 are changed only by the substitution of γ^2 for γ in the first of the generating relations. The connectivities are unchanged. The coefficients of torsion are $2k-4$ if the surface is orientable, 2 and 2 if it is one-sided. These results, for $k = 0$ or 1, agree with those already found.

The Riemann spaces described in § 9 are the manifolds for the surfaces of connectivity 3 in this case also.

12. **Another method.** The groups and Heegaard diagrams found in § 10 for orientable surfaces can also be found by a generalization of the method of § 3. For an orientable surface of genus p the manifold of states of motion can be represented by a pair of regions, each of which is bounded by $p+1$ coaxial tores, one enclosing all the others. Since a Heegaard diagram must have only one surface for each region, and must be representable in a non-singular manner in euclidean space, we connect the tores bounding each region by p holes, transferring the 3-cells removed to the other region, where they appear as handles running through the holes. The regions thus obtained are canonical and of genus $2p+1$, since they are made simply connected by the following $2p+1$ non-intersecting cuts: (1) a cut whose bounding curve runs along a generating circle of each tore; (2) a cut severing each of the p handles; (3) a cut which enlarges each of the p holes so as to extend all the way around the common axis of the tores. By tracing upon the surfaces of one region the curves which bound in the other region (determined as in § 3) the Heegaard diagram and group are obtained.

13. **Boundaries.** The field of force for a dynamical problem may be such that there exist on the characteristic surface points where the velocity must vanish. A continuous closed curve consisting of such points is called by Birkhoff (loc. cit.) an oval of zero velocity.

An orbit which meets an oval of zero velocity must be normal to it, since a particle starting from rest must move along a curve tangent to the line of force through the initial point, and since the lines of force are normal to the oval of zero velocity, as well as to other equipotentials.*

* This is obvious in the reversible case. The following analytic proof, for which I am indebted to Professor C. E. Hille, holds in general. *Contin. p. 341.*

Since there is only one possible direction of motion at each point, the part of the manifold of states of motion corresponding to an oval of zero velocity is a curve, though the part corresponding to any other curve is a surface.

An isolated point P of zero velocity implies that the states of motion cannot be represented by a true manifold of three dimensions, since every neighborhood of the point corresponding to P represents the states of motion on a family of closed curves such as concentric circles, and so must be homeomorphic with a family of coaxial anchor rings, one enclosing all the others. The states of motion will constitute a *generalized manifold* of the type described by Dehn and Heegaard.* Any singularity of an oval

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The differential equations of the motion, taken in Birkhoff's normal form

$$x'' + \lambda(x, y)y' = \frac{\partial}{\partial x} g(x, y), \qquad y'' - \lambda(x, y)x' = \frac{\partial}{\partial y} g(x, y),$$

where primes denote differentiation with regard to t and isothermal coördinates are used, reduce by the substitution

$$x_1 = x, \quad x_2 = y, \quad x_3 = x', \quad x_4 = y'$$

to the system of first-order equations

$$x_1' = x_3, \quad x_3' = -\lambda(x_1, x_2)x_4 + g_1(x_1, x_2),$$
$$x_2' = x_4, \quad x_4' = \lambda(x_1, x_2)x_3 + g_2(x_1, x_2).$$

From the general theory of differential equations it follows that x_1, x_2, x_3 and x_4 can be expressed as power series in t which are convergent for sufficiently small values of t and which satisfy the differential equations. Mapping the characteristic surface conformally upon the xy-plane and taking the oval of zero velocity $g = 0$ through the origin, we may write, if $x_1 = x_2 = x_3 = x_4 = 0$ when $t = 0$,

$$g = ax + by + \cdots,$$
$$\text{(i)} \qquad x = \alpha_2 t^2 + \cdots,$$
$$\text{(ii)} \qquad y = \beta_2 t^2 + \cdots,$$

where the terms neglected are of higher orders. Substituting in the differential equations and equating constant terms we see that $2\alpha_2 = a$ and $2\beta_2 = b$. The slope of the tangent to the orbit at the origin is $\beta_2/\alpha_2 = b/a$. But the slope of $g = 0$ is $-a/b$. Hence the two lines are perpendicular.

Further, the orbit has a cusp at the origin. For by means of (i) t may be expressed as a power series in $\sqrt{x}$, and by means of (ii) t may be expressed as a power series in $\sqrt{y}$. Equating these two series and dropping higher order terms yields the equation of a semicubical parabola. It is of course possible that the time may become infinite as the particle approaches the origin. In this case the above argument requires only slight modification.

* In the Encyclopädie article on analysis situs; or cf. Veblen, loc. cit., p. 92.

of zero velocity, or an oval of zero velocity which does not form a true boundary between a region of positive velocity and a region not of positive velocity, will be represented in the manifold of states of motion by a singularity. We shall hereafter consider only non-singular ovals of zero velocity which are true boundaries of the characteristic surface. Since the stream lines are to be free from singularities we shall stick to the case in which opposite directions are considered distinct.

A short segment having one end on such a boundary will be represented in the manifold of states of motion by the interior of a circle, the center representing the point on the boundary. If the segment moves along the boundary and sweeps out a band, the circle will move through the manifold and sweep out a tore. Let us call this tore T_1. Variation through 2π of direction at a point of the surface is represented in the manifold by a curve which is deformable into a generating circle of this tore and hence to a point. An orbit having a cusp on the oval of zero velocity is represented in the manifold by a smooth curve through the center of one of these generating circles, with no singularity.

We are now in a position to describe the three-dimensional manifold of the states of motion on a characteristic surface bounded by a number ovals of zero velocity. The characteristic surface may first be closed by filling in a number of 2-cells bounded by the ovals. The manifold of possible states of motion on such a closed surface is studied in § 10 and we shall call this manifold M. Now let us remove a 2-cell containing one of the ovals.

Referring to § 10, it will be seen that removal of a 2-cell from the characteristic surface is equivalent to removal from the prism there used to represent the manifold M of a cylinder with generators parallel to those of the prism. As the ends of the prism are supposed to be joined, this cylinder represents a tore, T_1', in the manifold. Its removal means that the first generating relation given in § 10 for the group of each manifold no longer holds good, since a curve surrounding a base of the prism no longer bounds. Let M_1' be the open manifold obtained from M by removing the tore T_1' and let M_1 be the manifold obtained by putting T_1 back in place of T_1'.* The curves γ on the surface of T_1' are the boundaries of a moving 2-cell which generates T_1. In the group, γ now becomes the identity. Hence the generating relations given in § 10 which are not false when applied to M_1 are redundant.

The group of the manifold for a surface of connectivity $k+1$ having one oval of zero velocity thus has k entirely independent generators.

* We may describe these operations in other words as removing the interior of a tore and putting it back with interchange of bounding and non-bounding curves. The resulting figure cannot lie in euclidean 3-space.

Consider now the effect upon the group of the manifold of introducing a second non-singular oval of zero velocity. Performing a sequence of operations similar to that described above, we first form an open manifold M_2' by removing from M_1 a tore T_2', and then replace T_2' by a tore T_2 analogous to T_1. Let M_2 be the resulting manifold. Let α_{k+1} be the curve of intersection of the base $\varphi = 0$ of the prism with the surface of T_2'. Then α_{k+1} bounds a 2-cell in M_1 and T_2' but not in M_2', T_2 or M_2.

The group of M_2 has as generators α_{k+1} and the generators $\alpha_1, \cdots, \alpha_k$ of the group of M_1, and contains no generating relation. For since the group of M_1 contains no generating relation, the opposite supposition would be equivalent to assuming that a curve C exists which bounds a 2-cell in T_2 but not in M_2'. But it is one of the elementary properties of the tore in euclidean 3-space that a curve which bounds a 2-cell in its interior must be a curve of the class of a generating circle—counted perhaps more than once. Therefore, since C bounds a 2-cell in T_2, C is a power of γ. Since γ bounds a 2-cell in M_2' and C by assumption does not, we have a contradiction.

By repetition of this argument it may be shown that introduction of new non-singular ovals of zero velocity has no other effect on the group than to introduce new and independent generators. Our general result, therefore, is that if m ovals of zero velocity are introduced into a surface of connectivity $k+1$ the group of the manifold of states of motion has $k+m-1$ generators and no generating relation. The connectivity is therefore $k+m$.

14. We now show that this manifold can be represented by a two-sheeted Riemann space having $k+m$ non-linking and knotless branch curves.

The group of a surface of connectivity $k+1$ from which m 2-cells are removed has $n = k+m-1$ independent generators. Let A be a point in which n independent closed curves intersect. Let each of the removed 2-cells be enlarged until nothing is left of the surface but a set of n narrow bands along each of which one of the generating curves runs and a simply connected region about A which adjoins both ends of each band.

For each generator g_i let there be a one-parameter continuous family C_i $(i = 1, 2, \cdots, n)$ of non-intersecting open curves on the surface having both ends on the boundary, each curve intersecting g_i once, and the first and

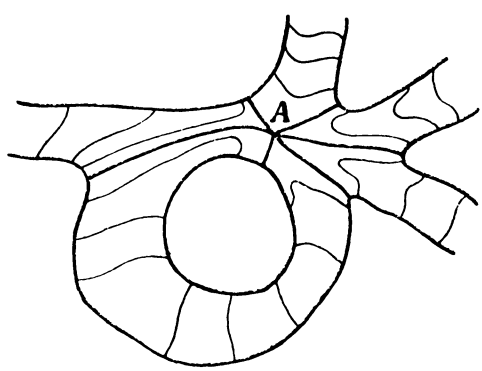

Fig. 7

last curves of C_i passing through A (Fig. 7). The boundaries being ovals of zero velocity, the part of the manifold of states of motion corresponding to each curve of C_i is topologically a sphere, since it can be described by a one-parameter family of circles of which the first and last are of zero radius, each circle representing the possible directions at one point of the curve. The manifold therefore consists of n one-parameter families of spheres, half of each end sphere being identified with half of some other end sphere.

Each of these families of spheres may be regarded as lying in and filling the portion of a space of inversion between two spheres. This region is homeomorphic with a 2-sheeted Riemann space from each sheet of which a sphere has been removed, the branch system being a single circle.

The manifold is therefore homeomorphic with a Riemann space of $2n$ sheets with one circular branch curve about which all the sheets permute in the same cyclic order as that in which the corresponding ends of the bands are arranged about A, and n other non-linking circular branch curves connecting the sheets in pairs in a manner corresponding to that in which the bands connect their ends.

By a reduction similar to that of Riemann surfaces by Clebsch and Lüroth this Riemann space may be shown to be homeomorphic with a two-sheeted space with non-linking and knotless branch curves. These curves must be $k+m$ in number because it was proved in § 13 that the connectivity of the manifold is $k+m$. This value of the connectivity may also be inferred from the present section without reference to § 13.

15. **Summary.** We have first found the simple manifolds of states of motion for the sphere and projective plane and then obtained general formulas for the groups and have shown how to construct the Heegaard diagrams for closed surfaces in general. The general method used applies with slight modifications to the sphere and projective plane. For surfaces of the connectivity of the anchor ring, Riemann spaces of simple type homeomorphic with the manifolds of states of motion were found.

Open surfaces, which appear in dynamical problems in which the potential functions are such as to delimit regions which the moving particle cannot enter, were considered in §§ 13 and 14 from different points of view. In § 13 the groups were deduced from the results already found for closed surfaces. In § 14 a method of attack yielding extremely simple Riemann spaces was used.

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