ON THE OSCILLATION OF A CONTINUUM AT A POINT*

BY

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1. The concept of oscillation of a continuum at a given point was introduced by S. Mazurkiewicz in his paper on Jordan curves† and the idea has since been used frequently by other writers‡. Less attention has been given to the general concept, however, than to the special case where the oscillation is zero, a property which H. Hahn has shown to be equivalent to the property of connectedness im kleinen.

In this article several new properties of the oscillatory function are developed. In particular it is shown that the oscillation is an upper semi-continuous function and therefore has the well known characteristics of pointwise discontinuous functions. Furthermore, the behavior of the function in the neighborhood of points where its value is greater than zero is investigated. (See §§ 11–15.)

The results are obtained by the introduction of an auxiliary function \( \tau(a) \) (§ 3) and oscillatory sub-sets (§§ 7–10) of a continuum, both of which have interesting properties in themselves. The work is confined to continua, although several of the theorems can be readily extended to non-closed connected sets.

2. Notation. The ordinary notation of the theory of aggregates is employed, with the following modifications.

If \( A \) is the common part of a system of aggregates \( \{C\} \), we write \( A = Dv[C] \).

If \( A \) is a real part of \( B \), we write \( ACB \).

If \( A \) is a part of \( B \) and may be identical with \( B \), we write \( A \subseteq B \).

3. Definitions. Let \( A \) be any continuum and \( a \in A \). Let \( V_\delta(a) \) denote the set of points of \( A \) whose distance from \( a \) is less than \( \delta, \delta > 0 \). Let \( C_\delta \) denote a subcontinuum of \( A \) which contains all points of \( V_\delta(a) \). The lower bound of the diameters of all such sets \( C_\delta \), for all \( \delta > 0 \), is denoted by \( \tau_A(a) \), or simply \( \tau(a) \).

For convenience the previous sentence may be written

\[ \tau(a) = \text{Min Diam}[C_\delta]. \]

* Presented to the Society, September 10, 1925.
‡ E.g., Z. Janiszewski, C. Kuratowski, and B. Knaster.
If a set is unlimited, we call its diameter $\infty$. If for a point $a$ every subcontinuum $C_\delta$ is unlimited, we say that $\tau(a) = \infty$.

It is a comparatively simple matter to show the existence of the auxiliary function just defined for each point of any continuum. It is also evident that the above definition might well be employed as a definition of the oscillation of a continuum at a point. Without entering into details, it may be stated that $\tau(a)$ has all the properties of S. Mazurkiewicz' oscillatory function $\sigma(a)$ as given in the article referred to above, pp.170–178. It is not, however, precisely the same function for continua in general, as the next section will show.

4. The function $\sigma(a)$ is defined as follows (loc. cit., p. 170). If $x$ and $y$ are any two points of a continuum $A$, let $C(x,y)$ be any subcontinuum of $A$ containing $x$ and $y$. The number $\varrho_A(x,y) = \text{Min} \ \text{Diam} \ [C(x,y)]$ for all possible sets $C(x,y)$ is called the relative distance between $x$ and $y$ with respect to $A$. Then $\sigma(a) = \lim \varrho_A(x,y)$ as $x$ and $y$ approach $a$.

It is easy to construct continua for which $\tau(a) \neq \sigma(a)$ at certain points. However, as the following theorem shows, $\tau(a)$ and $\sigma(a)$ vanish at the same points. Hence the genre of a point is the same for both definitions of oscillation.

**Theorem.** If $A$ is a continuum and $a \in A$, then $\sigma(a) \leq \tau(a) \leq 2\sigma(a)$.

**Proof.** For any $\varepsilon > 0$ there exists a $\delta > 0$ and a sub-continuum $C_\delta$ of $A$ for which

\begin{equation}
V_\delta(a) \subseteq C_\delta \subseteq A,
\end{equation}

and

\begin{equation}
\text{Diam} \ C_\delta \leq \tau(a) + \varepsilon.
\end{equation}

From (2) we have at once that $\varrho_A(x,y) \leq \tau(a) + \varepsilon$ for any pair of points $x$ and $y$ in $V_\delta(a)$. Therefore

\begin{equation}
\sigma(a) \leq \tau(a).
\end{equation}

On the other hand, for any $\varepsilon > 0$ there is a $\delta > 0$ such that

\begin{equation}
\varrho_A(x,y) \leq \sigma(a) + \frac{\varepsilon}{2}
\end{equation}

for any pair of points $x$ and $y$ in $V_\delta(a)$. By definition of $\varrho_A(x,y)$ there exists a sub-continuum $C(x,y)$ of $A$ such that

\begin{equation}
\text{Diam} \ C(x,y) \leq \varrho_A(x,y) + \frac{\varepsilon}{2} \leq \sigma(a) + \varepsilon.
\end{equation}

Now relations (4) and (5) hold for the particular case that $x = a$. Hence the union of all the sets $C(a,y)$ is a connected sub-set $C$ of $A$ which con-
tains \( V_\delta(a) \) and \( \overline{C} \) is a sub-continuum of \( A \) containing \( V_\delta(a) \). From (5) we have

\[
\text{Diam } \overline{C} = \text{Diam } C \leq 2\sigma(a) + 2\epsilon,
\]

whence

\[
(6) \quad \tau(a) \leq 2\sigma(a) + 2\epsilon.
\]

Since (6) holds for any \( \epsilon > 0 \), (3) and (6) give the theorem.

**Example.** The following is a case where \( \sigma(a) \neq \tau(a) \). Draw a circle of center \( a \) and radius \( r \) and three radii trisecting the circle. On the radii bisecting each of the sectors thus formed take a sequence of points whose distance from \( a \) is \( r/2^n \), \( n = 1, 2, \ldots \), and through each of these points draw two lines parallel to the radii bounding the sector and terminated by the arc of the sector. Let \( A \) be the continuum consisting of the circumference, the three radii, and the series of broken lines.

Two points lying in any \( V_\delta(a) \) and not lying on the radii \( aB, aC, \) or \( aD \) will lie in the same sector or in adjacent sectors, say those whose arcs are \( BC \) and \( CD \). They can be joined by a sub-continuum passing through \( C \) and of diameter not greater than \( r + 4\delta \) and not less than \( r \). Thus \( \sigma(a) = r \).

But \( V_\delta(a) \) contains points in all three sectors. Hence any sub-continuum containing \( V_\delta(a) \) must contain two of the points \( B, C, D \) and therefore has

\[
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\]
a diameter at least as great as the distance $BC = r\sqrt{3}$. Thus \( r(a) \geq r\sqrt{3} \) and it is easy to see that \( r(a) = r\sqrt{3} \). In this example, then, \( r(a) = V_3 \sigma(a) \).

5. Theorem. Let \( A = \{x\} \) be a continuum. Then \( r(x) \) and \( \sigma(x) \) are upper semi-continuous at all points of \( A \).

Proof. At points where \( r(x) \) is infinite the theorem obviously holds. Let \( a \) be a point of \( A \) at which \( x(a) = k \neq \infty \). Then for \( \varepsilon > 0 \), there is a \( \delta > 0 \) and a sub-continuum \( C_\delta \) of \( A \) for which

\[
(1) \quad V_{\delta}(a) \subseteq C_\delta \subseteq A,
\]

and

\[
(2) \quad \text{Diam } C_\delta \leq k + \varepsilon.
\]

Now for any \( x \) in \( V_\delta(a) \) there is an \( \eta > 0 \) such that \( V_\eta(x) \subseteq V_\delta(a) \). Hence

\[
(3) \quad V_\eta(x) \subseteq C_\delta.
\]

Relations (1), (2), and (3) show that

\[
r(x) \leq k + \varepsilon = r(a) + \varepsilon, \quad \text{for } x \text{ in } V_\delta(a),
\]

which is the definition of upper semi-continuity.

Likewise for any \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that any two points \( x \) and \( y \) in \( V_\delta(a) \) can be joined by a sub-continuum \( C(x,y) \) of \( A \) whose diameter is not greater than \( \sigma(a) + \varepsilon \). For any point \( x' \) in \( V_\delta(a) \) there is a \( \eta > 0 \) such that \( V_\eta(x') \subseteq V_\delta(a) \). Hence any two points of \( V_\eta(a') \) can be joined by a sub-continuum of \( A \) of diameter not greater than \( \sigma(a) + \varepsilon \).

Then we have \( \sigma(x') \leq \sigma(a) + \varepsilon \) for all \( x' \) in \( V_\delta(a) \), which is the requirement for upper semi-continuity.

The above theorem shows that the oscillation of a continuum, if everywhere finite, is continuous or pointwise discontinuous, like the oscillation of a one-valued function of a real variable. Among the properties deducible from this fact may be mentioned the following.

Corollary 1. Let \( A = \{x\} \) be a continuum. Then \( r(x) \) and \( \sigma(x) \) are continuous at each point of \( A \) for which their value is zero.

Corollary 2. Let \( A = \{x\} \) be a continuum. Then the set of points for which \( r(x) \geq k \), or \( \sigma(x) \geq k \), any constant, is a closed set; the set of points for which either function is discontinuous is of the first category; and the set for which either function is continuous is of the second category with respect to \( A \).

Corollary 3. Let \( A = \{x\} \) be a continuum and \( B \) be a closed limited part of \( A \). If \( r(x) \) or \( \sigma(x) \) is finite for each point of \( B \), it is limited in \( B \) and there is at least one point of \( B \) at which it takes on its maximum value.
6. Generalized irreducible continua. We now proceed to extend the notion of a continuum irreducible between two points. Let $C$ be a continuum and let $A$ be any sub-set of $C$. The continuum $C$ is called irreducible about $A$ if there is no continuum $C'$ satisfying the relation $A \subseteq C' \subseteq C$.

**Theorem.** Let $C$ be a limited continuum and $A \subseteq C$. Then there exists at least one sub-continuum $D$ of $C$ which is irreducible about $A$.

**Proof.** In order to prove this theorem it is only necessary to show, as C. Kuratowski has demonstrated,* that if $C_i$ is a sequence of continua such that

\[ C \supseteq C_1 \supseteq C_2 \supseteq \cdots \supseteq C_i \supseteq C_{i+1} \supseteq \cdots \supseteq A \]

and $C_\omega = D [C_i]$, then $C_\omega$ is a continuum and contains $A$.

Since each $C_i$ is closed and limited, $C_\omega$ exists. By (1) $A \subseteq C_\omega$. Since all the aggregates $C_i$ form a part of the limited aggregate $C$ and $C_\omega \neq 0$, it follows† that $C_\omega$ is a continuum. Our theorem is therefore proved.

Without going into the properties of irreducible continua, it is perhaps well to call attention to the fact that under the conditions of the above theorem there may be many sub-continua of $C$ irreducible about $A$. If $C$ is unlimited, there may be no irreducible sub-continuum.

7. Oscillatory sets. Let $A$ be a continuum and $a \in A$. Let $\delta_1 > \delta_2 > \cdots$, $\delta_i \to 0$, and let $C_i$ denote a sub-continuum of $A$ irreducible about $V_{\delta_i}(a)$. If the sequence $\{C_i\}$ is monotone decreasing, i.e., $C_1 \supseteq C_2 \supseteq \cdots$, we call $C = D V [C_i]$ an oscillatory set of $A$ about $a$.

A continuum may have many oscillatory sets about one of its points. For instance, in the example given in § 4 each of the broken lines $BaC$, $CaD$, and $DaB$ is an oscillatory set of $A$ about the point $a$. If a continuum is unlimited, it may have no oscillatory set about one or more of its points. However, we have the following theorem.

**Theorem.** Let $A$ be a continuum, $a \in A$, and $\sigma(a)$ be finite. Then there exists at least one oscillatory set of $A$ about $a$.

**Proof.** Let $\delta_1 > \delta_2 > \cdots$ and $\delta_i \to 0$. Since $\sigma(a)$ is finite, so is $\tau(a)$. Hence there is a sub-continuum $C_1$ of $A$ which is limited and irreducible about $V_{\delta_1}(a)$. Likewise there is a sub-continuum $C_2$ of $C_1$ which is irreducible about $V_{\delta_2}(a)$; etc. Thus we have a monotone decreasing sequence

\[ C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots, \]

and $C = D V [C_i]$ is an oscillatory set of $A$ about $a$ by definition.

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8. Theorem. Let $A$ be a continuum and $a \in A$. If $\tau(a)$ is finite and $C$ denotes any oscillatory set of $A$ about $a$, then $\tau(a) = \text{Min Diam } [C]$. If $\tau(a) = \infty$, and any oscillatory set $C$ of $A$ about $a$ exists, $\text{Diam } C = \infty$.

Proof. Let $\text{Min Diam } [C] = k$. Then there is at least one oscillatory set $C$ for which

$$k \leq \text{Diam } C \leq k + \frac{\varepsilon}{2},$$

for an arbitrarily chosen positive $\varepsilon$. Now suppose that $C = Dv[C_i]$, where each $C_i$ is a sub-continuum of $A$ irreducible about $V_{\delta_i}(a), \delta_i \to 0$. It follows at once that there is an $i_0$ such that for all $i > i_0$ every point of the set $C_i$ has a distance from $C$ not greater than $\varepsilon/4$. Hence

$$\text{Diam } C_i \leq \text{Diam } C + \frac{\varepsilon}{2} \leq k + \varepsilon.$$

Thus we have at least one sub-continuum $C_i$ of $A$ which contains a $V_{\delta}(a)$ and whose diameter is not greater than $k + \varepsilon$. Hence $\tau(a) \leq k + \varepsilon$, whatever $\varepsilon$ may be. Thus

$$\tau(a) \leq k.$$

On the other hand, if $\tau(a)$ is finite, there is a limited sub-continuum $B$ of $A$ containing a $V_{\delta}(a)$ and of diameter not greater than $\tau(a) + \varepsilon$, where $\varepsilon > 0$ is arbitrary. Now $B$ contains a sub-continuum $C$ irreducible about $V_{\delta_1}(a)$ and as in § 7 there is a monotone decreasing sequence $\{C_i\}$, whose divisor $C$ is an oscillatory set about $a$. Since $\text{Diam } B \leq \tau(a) + \varepsilon$, $\text{Diam } C \leq \tau(a) + \varepsilon$. Thus

$$\text{Min Diam } [C] \leq \tau(a).$$

Relations (3) and (4) give the theorem for the case that $\tau(a)$ is finite. The case that $\tau(a) = \infty$ is disposed of by (3).

9. The two kinds of oscillatory sets. We have at once as a corollary of the previous theorem that $\sigma(a) \leq \text{Min Diam } [C]$. It is also evident that these sets furnish an independent means of defining the function $\tau(a)$.

From the two previous sections it is clear that the oscillatory sets $C$ of a continuum $A$ may be of two kinds:

I. $C$ is the divisor of a descending sequence of sub-continua $\{C_i\}$, where each $C_i$ is irreducible about a $V_{\delta_i}(a), \delta_i \to 0$.

II. $C$ is irreducible about every $V_\delta(a)$ for every $\delta$ less than or equal to some $\delta_0$.

In particular $C$ is of the second kind if, in the monotone decreasing sequence $\{C_i\}$ determining $C$, all the sets $C_i$ from some $i_0$ on are identical.
That such sets exist follows at once from the fact that if $A$ is indecomposable there exist in any $V_{\lambda}(a)$ points $x$ such that $A$ is irreducible between $a$ and $x$. If, however, $C$ is an oscillatory set of the second kind, it is not necessarily indecomposable.

If there is only a finite number of oscillatory sets of $A$ about $a$, the diameter of one of them is $\tau(a)$. It is also easy to show that, if $\sigma(a) = 0$, one of the oscillatory sets of $A$ about $a$ is the point $a$ itself.

10. **Theorem.** Let $A$ be a continuum and $a \in A$. Let $C$ be an oscillatory set about $a$ with a finite diameter $k \neq 0$. Then for any $\varepsilon > 0$, however small, there exists a continuum of condensation of $A$ containing $a$ and forming a part of $C$ whose diameter is greater than or equal to $k - \varepsilon$.

Proof. Let $\delta_1 > \delta_2 > \ldots$ and $\delta_1 \to 0$, and let $\{C_I\}$ be a monotone decreasing sequence of sub-continua of $A$ with each $C_I$ irreducible about $V_{\delta_i}(a)$. Let $C = D_v[C_I]$. Since $C$ is limited, there is no loss in generality in assuming that every $C_I$ is also limited.*

Now let $x$ be any point of $C$ different from $a$, and let $h$ denote the distance from $a$ to $x$. Let $\varepsilon < h/4$, and let $\Gamma_{\varepsilon}$ be the interior of a sphere of center $x$ and of radius $\sigma < \varepsilon/2$. Then, for $i$ greater than some $i_0$, $\Gamma_{\varepsilon} \setminus V_{\delta_i}(a) = 0$. Let

$$D_i = C_i - C_i \cdot \Gamma_{\varepsilon}.$$ 

Each $D_i$ is closed and contains $V_{\delta_i}(a)$.

Let $D = C - C \cdot \Gamma_{\varepsilon}$ and $C_{\sigma}$ be that component of $D$ containing $a$; it also contains points of the frontier of $\Gamma_{\varepsilon}$, which we denote by $\text{Front } \Gamma_{\varepsilon}$. This follows from Janiszewski's theorem.† There is at least one point $z_i$ of $D_i$ in $V_{\delta_i}(a)$ not lying on $C_{\sigma}$, for otherwise $C_i$ would not be irreducible about $V_{\delta_i}(a)$.

Let $S_{z_i}$ be that component of $D_i$ which contains one of these points $z_i$. As in the case of $C_{\sigma}$ we see that $S_{z_i}$ has a point on $\text{Front } \Gamma_{\varepsilon}$. Now if $S_{z_i} \cdot C_{\sigma} \neq 0$, then $C_{\sigma} \subseteq S_{z_i}$ since $C \subseteq C_i$. If this were true for every $z_i$.

* For, since $C$ is closed and limited, there is a finite closed sphere $S$ such that every point of $C$ is an inner point of $S$. A set $C$, which is unlimited must contain points without $S$. Then, since it is a continuum and contains both inner and outer points of $S$, it must have points in common with $F = \text{Front } S$. Let $F_i = F \cdot C_i$. The sets $F_i$ are closed and limited; also the sequence $\{F_i\}$ is monotone decreasing. Then either $F_i = 0$ for all $i$ greater than some $i_0$ or $F_i \neq 0$ for all values of $i$. In the latter case $D_v[F_i] \neq 0$. But, since each $F_i \subseteq C_i$, $D_v[F_i] \subseteq C$, which makes $F \cdot C = 0$, contrary to our assumption regarding $S$.

† Janiszewski, *Sur les continus irréductibles entre deux points*, Journal de l'École Polytechnique, ser. 2, vol. 16 (1912), p. 100. "Let $C$ be a limited continuum and $a$ be one of its points. Let $A$ be closed and $a$ be an inner point of $A$. Then there is a sub-continuum of $C$ containing $a$ and a frontier point of $A$ (unless $C$ is contained in $A$) and contained in $A$."
in $V_{\delta_i}(a)$, the union of the sets $S_{\delta_i}$, which we denote by $U$, would be connected and $U$ would be a real sub-continuum of $C_i$ containing $V_{\delta_i}(a)$. This, however, is contrary to the hypothesis that $C_i$ is irreducible about $V_{\delta_i}(a)$. Hence in each $V_{\delta_i}(a)$ there is a point $z_i$ for which there is a sub-continuum $S_{\delta_i}$ of $C_i$ containing $z_i$ and a point of Front $\Gamma_\sigma$, but containing no point of $C_\sigma$.

As $i \to \infty$, $\delta_i \to 0$ and $z_i \to a$. Let $K$ be the aggregate of accumulation of $\{S_{\delta_i}\}$. $K$ is a continuum or reduces to the point $a$ by virtue of a theorem of Janiszewski*. As each $S_{\delta_i}$ has a point on Front $\Gamma_\sigma$, $K$ also has a point $x'$ on Front $\Gamma_\sigma$. Hence

\[(2) \quad \text{Diam } K \geq h - \varepsilon \geq h - \frac{\varepsilon}{2}.\]

Since $S_{\delta_i} \subseteq C_i$, $K$ is a part of every $C_i$. Hence $K \subseteq C$. Obviously $K$ has no points in $\Gamma_\sigma$. Thus

\[(3) \quad K \subseteq C = C \cdot \Gamma_\sigma \subseteq C_\sigma.\]

But $C_\sigma \cdot S_{\delta_i} = 0$; this with (3) shows that $K$ is a continuum of condensation. We have shown then that if $x$ is any point of $C$ different from $a$ there is a continuum of condensation of $A$ which forms a part of $C$ and which contains $a$ and at least one point $x'$ whose distance from $x$ is less than $\varepsilon/2$.

Now since $C$ is a closed set it contains two points, $x$ and $y$, whose distance apart is $h$. If one of these is $a$, the previous paragraph together with relations (2) and (3) give the theorem, for then $h = k$. If not, $C$ contains a continuum of condensation $K_1$ of $A$ joining $a$ and $x'$, a point whose distance from $x$ is less than $\varepsilon/2$, and a $K_2$ joining $a$ and $y'$, a point whose distance from $y$ is less than $\varepsilon/2$. Let $K = K_1 + K_2$. Since $K_1 \cdot K_2 \neq 0$, $K$ is a continuum of condensation. Moreover,

\[\text{Diam } K \geq \text{Dist } (x', y') \geq \text{Dist } (x, y) - \varepsilon \geq k - \varepsilon.\]

11. One of the fundamental theorems in S. Mazurkiewicz' paper† is to the effect that if a point of a continuum is of the second genre, it lies on a continuum of condensation. By means of § 10 we are able to show a relation between the size of this continuum of condensation and the oscillation of the given continuum at the point.

**Theorem.** Let $A$ be a continuum, $a \in A$, and $\sigma(a) = k$ be finite and different from zero. Then for any $\varepsilon > 0$ there exists a continuum of condensation of $A$ containing $a$ and of diameter greater than or equal to $k - \varepsilon$.

* Loc. cit., p. 97.
† Loc. cit., p. 176.
Proof. This is a corollary of § 10. For \( \tau(a) \) has a value between \( k \) and \( 2k \), and hence there is an oscillatory set of \( A \) about \( a \) whose diameter is greater than or equal to \( k \).

12. Theorem. Let \( A \) be a continuum, \( a \in A \), and \( \sigma(a) = \infty \). Then for every \( G > 0 \) there exists a continuum of condensation of \( A \) containing \( a \) and of diameter greater than or equal to \( G \).

Proof. Since \( \sigma(a) = \infty \), \( A \) is not limited. Let \( I_r \) be a closed sphere of center \( a \) and radius \( G \). Let \( D = A \cdot I_r \).

Let \( D_a \) be that component of \( D \) containing \( a \). In any \( V_{\delta}(a) \) there are points \( z \) of \( D \) not lying on \( D_a \). For otherwise \( \sigma(a) \leq \text{Diam } D_a \leq 2G \).

Let \( D_z \) be that component of \( D \) containing one of these points. It is easily seen that \( D_a \) and each \( D_z \) have points on \( \text{Front } I_r \). Obviously \( D_a \cdot D_z = 0 \) for every \( z \).

Since there is a \( z \) in every \( V_{\delta}(a) \), there is a sequence \( \{z_i\} \) converging to \( a \). Let \( K \) be the aggregate of accumulation of \( \{D_z\} \). It contains \( a \) and a point on \( \text{Front } I_r \). Hence

\[
(1) \quad \text{Diam } K \geq G.
\]

Since \( K \cdot D_a \) contains \( a \),

\[
(2) \quad K \subseteq D_a.
\]

But \( D_a \cdot D_{z_i} = 0 \); hence \( K \) is a continuum of condensation. As its diameter is greater than or equal to \( G \) by (1), the theorem is proved.

13. Theorem. Let \( A \) be a continuum, \( a \in A \), and \( \sigma(a) = k \) be finite and not zero. Then for any \( \varepsilon > 0 \) there exists a continuum of condensation of \( A \) of diameter not less than \( k/2 - \varepsilon \) containing \( a \) and contained in an oscillatory set \( C \) of \( A \) about \( a \), and having no point where the oscillation is zero.

Proof. Let \( \sigma = k/2 - \varepsilon/2 \) and \( \varrho = k/2 - \varepsilon \). Let \( I_\sigma \) and \( I_\varrho \) be closed spheres of center \( a \) and radii \( \sigma \) and \( \varrho \) respectively. Let \( A_a \) be that component of \( A \cdot I_\sigma \) which contains \( a \). For any \( \delta > 0 \) there are points \( z \) of \( V_{\delta}(a) \) which do not lie on \( A_a \). For otherwise \( \sigma(a) \leq \text{Diam } A_a \leq k - \varepsilon \).

Let \( A_z \) be that component of \( A \cdot I_\varrho \) which contains \( z \). It is obvious that \( A_a \) and each \( A_z \) contains one or more points on \( \text{Front } I_\varrho \) and that

\[
(1) \quad A_a \cdot A_z = 0.
\]

Since \( \sigma(a) \) is finite, there is a monotone decreasing sequence of limited sub-continua \( \{C_i\} \) of \( A \), each \( C_i \) irreducible about a \( V_{\delta_i}(a) \), \( \delta_i \to 0 \), and \( C = Dv[C_i] \) is an oscillatory set of \( A \) about \( a \) of diameter not less than \( k \). Let \( C_{1,a} \) be that component of \( C \cdot I_\varrho \) containing \( a \), and \( C_{1,z} \) be that com-
ponent containing \( z \), where \( z \) is one of the points of the previous paragraph. Since \( \text{Diam } C_1 \geq k \), each of these components has points on \( \text{Front } \Gamma_\sigma \). Also, since \( C_1 \subseteq A \), we have

\[
(2) \quad C_{1,a} \subseteq A_a \quad \text{and} \quad C_{1,z} \subseteq A_z.
\]

Now let \( \{z_i\} \) be a sequence of the points \( z \) converging to \( a \). Let \( K' \) be the aggregate of accumulation of the sets \( \{C_{1,z_i}\} \). Then \( K' \) is a continuum containing \( a \) and a point on \( \text{Front } \Gamma_\sigma \). Since \( C_1 \cdot \Gamma_\sigma \) is closed, \( K' \subseteq C_1 \cdot \Gamma_\sigma \). Since \( K' \cdot C_{1,a} \supseteq C_{1,a} \), \( K' \subseteq C_{1,a} \). But \( C_{1,a} \cdot C_{1,z_i} = 0 \). Hence \( K' \) is a continuum of condensation.

Now let \( K_1 \) be the saturated sub-continuum of \( K' \) contained in \( \Gamma_\varphi \) and containing \( a \). It has a point on \( \text{Front } \Gamma_\varphi \), but no points of \( \Gamma_\sigma - \Gamma_\varphi \). Hence \( \text{Diam } K_1 \geq q = k/2 - \varepsilon \). If \( \sigma(x) = 0 \) for some point of \( K_1 \), there is an \( \eta > 0 \) such that \( V_\eta(x) \) lies in a sub-continuum \( C(x) \) of \( A \) of diameter not greater than \( \varepsilon/4 \). Since \( \sigma - \varphi = \varepsilon/2 \), \( C(x) \subseteq A \cdot \Gamma_\sigma \) and hence

\[
(3) \quad V_\eta(x) \subseteq C(x) \subseteq A_a.
\]

But each point of \( K_1 \) is a point of accumulation of the sets \( \{C_{1,z_i}\} \) and hence every \( V_\eta(x) \) contains a point of some \( C_{1,z_i} \). Thus we have \( A_a \cdot C_{1,z_i} \neq 0 \), and \( C_{1,z_i} \subseteq A_z \). This, however, contradicts relations (1) and (2). Hence \( \sigma(x) \neq 0 \) for every point in \( K_1 \).

Since \( d_2 < d_1 \) and \( C_2 \subseteq C_1 \), the same argument applied to \( C_2 \) will yield a continuum of condensation \( K_2 \), which is a part of \( K_1 \) and of \( C_{2,a} \), contains \( a \) and a point of \( \text{Front } \Gamma_\varphi \), and has no point for which \( \sigma(x) = 0 \).

Let this be continued and set \( K \) equal to the divisor of the monotone decreasing sequence \( \{K_i\} \). Since every \( K_i \subseteq C_i \), \( K \subseteq C \). Since \( \sigma(x) \neq 0 \) in every \( K_i \), \( \sigma(x) \neq 0 \) in \( K \). Since every continuum \( K_i \) contains \( a \) and a point on \( \text{Front } \Gamma_\varphi \), and is limited, the divisor \( K \) is a continuum of diameter not less than \( k/2 - \varepsilon \). Since every point of \( K_1 \) is a limit point of points not in \( K_1 \) and \( K \subseteq K_1 \), this is also true of \( K \), and therefore \( K \) is a continuum of condensation.

It should be noted that the proof can be considerably shortened, if we do not attempt to prove that the continuum of condensation is a part of \( C \).

14. THEOREM. Let \( A \) be a continuum, \( a \in A \), and \( \sigma(a) = \infty \). Then for any \( G > 0 \) there is a continuum of condensation of \( A \) containing \( a \) and of diameter not less than \( G \), at each point of which \( \sigma(x) > 0 \).

Proof. The theorem may be established by reasoning similar to that in \( \S \, 13 \), by taking \( \sigma = G \) and substituting for \( C_1 \) the set \( A \) itself.
15. **Theorem.** Let $A$ be a continuum, $a \in A$, and $\tau(a) = k$, finite and not zero. Then $a$ lies on a continuum of condensation $K$ of $A$ such that as $x \to a$ on $K$, $\lim \tau(x) \leq k$ and $\lim \tau(x) \geq k/2$.

**Proof.** The first statement holds by virtue of § 5. To prove the second let $K$ be the continuum of condensation obtained in § 13. Now if $\lim \tau(x) < k/2$ as $x \to a$ on $K$, there is in every $V_\delta(a)$ a point $x$ on $K$, such that for any $\varepsilon$ sufficiently small and positive

$$\tau(x) = \lambda < \frac{k}{2} - \varepsilon.$$  

Now there is an $\eta > 0$ so small that $V_\eta(x)$ lies in a sub-continuum $C(x)$ of $A$ of diameter not greater than $\lambda + \varepsilon/4$. If $x \in C(x)$,

$$\text{Dist} (a, z) \leq \text{Dist} (a, x) + \text{Dist} (x, z) \leq \delta + \lambda + \frac{\varepsilon}{4} \leq \lambda + \frac{\varepsilon}{2} \leq \frac{k}{2} - \frac{\varepsilon}{2},$$

if $\delta \leq \varepsilon/4$. This shows that $C(x)$ lies in the closed sphere $\Gamma_\sigma$ of § 13, since $\sigma = k/2 - \varepsilon/2$. Hence $C(x) \subseteq A \cdot \Gamma_\sigma$, and therefore

$$C(x) \subseteq A_a.$$  

But, since $x \in K$, every $V_\eta(x)$ contains a point of some $C_{1, \tau}$, which would give $A_a \cdot C_{1, \tau} \neq 0$, and as in § 13 we arrive immediately at the contradiction that $C_{1, \tau} \subseteq A_a$.

Hence the assumption that $\lim \tau(a)$ can be less than $k/2$ is false.

The above theorem can at once be expressed in terms of $\sigma(x)$ by means of § 4. That $\lim \tau(x)$ can equal $\frac{1}{2} \tau(a)$ is seen by considering the continuum defined as follows. For $x = 0$, $y$ has all the values between $\pm 1$; for $x \neq 0$, $y = \pm \sin^t (1/x)$ according as $x$ is positive or negative. If we set $a = (0, 0)$, the segment of the $y$-axis between $\pm 1$ is a continuum of condensation containing $a$ and $\tau(a) = 2$, but as $x \to a$ on this continuum of condensation, $\lim \tau(x) = 1$.

16. The example just given brings out clearly the analogy between the oscillation of a continuum and the oscillation of certain functions of a real variable. If in the definition of the continuum we set $y = 0$ when $x = 0$,
then \( y = f(x) \) is a one-valued function, continuous except at \( x = 0 \). At points of continuity the oscillation of the function is 0; at the origin it is 2. At points of the continuum where \( x \neq 0 \), \( \tau(x) = 0 \); at the point \((0,0)\) \( \tau(x) = 2 \). The oscillatory set at the point \((0,0)\) corresponds to the aggregate of limiting values of \( f(x) \) as \( x \to 0 \).

If we consider the oscillatory set as a tool in studying continua (as the oscillation of a function is used in studying discontinuous functions), the question at once arises as to whether it possesses any merits differentiating it from continua of condensation. Two are evident from the contents of this paper. First, the diameter of an oscillatory set is at least as great as the oscillation \( \sigma(x) \) or \( \tau(x) \); this is not in general true for continua of condensation, as can be shown by an example. Second, for such a continuum as the unit square the oscillation at every point is zero and the oscillatory set is the point itself, while any linear continuum containing the point is a continuum of condensation. The obvious defect is that, like the continuum of condensation, the oscillatory set at a point is not unique. This is to be expected on account of the generality of the continua considered here. With the proper restrictions this difficulty should disappear. For example, when a continuum is irreducible between two points and limited, the oscillatory set is unique, as will be shown in a paper at present under preparation.

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