ON THE ZEROS OF THE FUNCTION $\beta(z)$ ASSOCIATED WITH THE GAMMA FUNCTION

BY

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1. Nielsen† has raised the question whether the function $\beta(z)$ defined by

$$\beta(z) = \frac{1}{2} \left[ \psi \left( \frac{1+z}{2} \right) - \psi \left( \frac{z}{2} \right) \right], \quad \left( \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \right),$$

has any zeros, and has shown that there are no real zeros, and that the complex zeros, if any, must have their real part less than $-\frac{1}{2}$.

This question will be answered completely in the following, by showing that

The zeros of $\beta(z)$ are all complex, and their real part less than $-\frac{1}{2}$. For $n = 1, 2, 3, \ldots$, each of the infinite strips

$$-2n - \frac{1}{2} < \text{real part of } z < -2n + \frac{3}{2}$$

contains exactly two zeros, and for $n$ sufficiently large, their asymptotic expression is

$$-2n + \frac{1}{2} + \frac{2 \log(8n+2)\pi}{(4n+1)^2} \pm \frac{i \log(8n+2)\pi}{\pi} + \frac{\xi_n \pm i \eta_n}{2\pi n},$$

$$\xi_n^2 + \eta_n^2 < 1.$$

2. From the definition of $\beta(z)$ it follows at once that

$$\beta(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{z+n} = -\log 2 + \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n \left( \frac{1}{z+n} - \frac{1}{n} \right),$$

both series converging uniformly (and the second also absolutely) in any finite region in the $z$-plane to which the poles $z=0, -1, -2, \ldots$ are exterior; and from (1) (or directly from the definition) we obtain the relation

$$\beta(z) + \beta(1-z) = \frac{\pi}{\sin \pi z}.$$ 

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Let us begin by proving Nielsen's results. For $z$ real and positive, the terms in (1) have alternating signs and decrease in absolute value, so that $\beta(z) > 0$. For $z = -m - \xi$, where $m$ is a positive integer or zero, and $0 < \xi < 1$, we may write (1) in the form

$$(-1)^{m+1}\beta(-m-\xi) = -\left[\frac{1}{1+\xi} - \frac{1}{2+\xi} + \cdots + (-1)^m \frac{1}{m+\xi}\right] + \frac{1}{\xi}$$

$$+ \frac{1}{1-\xi} - \left[\frac{1}{2-\xi} - \frac{1}{3-\xi} + \cdots \right];$$

the terms in brackets having alternating signs and decreasing in absolute value, we have

$$(-1)^{m+1}\beta(-m-\xi) > \frac{1}{\xi} + \frac{1}{1-\xi} - \frac{1}{1+\xi} - \frac{1}{2-\xi}$$

$$= \frac{1}{\xi(1+\xi)} + \frac{1}{(1-\xi)(2-\xi)} > 0.$$

Hence there are no real zeros. On the other hand, let $z = x + yi$, $y \neq 0$, be a complex zero; taking the imaginary part of (1) and dividing by $y$, we find

$$\frac{1}{x^2+y^2} - \frac{1}{(x+1)^2+y^2} + \left[\frac{1}{(x+2)^2+y^2} - \frac{1}{(x+3)^2+y^2} + \cdots \right] = 0,$$

and assuming $x > -2$, the terms in the bracket have alternating signs and decrease in absolute value, whence

$$\frac{1}{x^2+y^2} - \frac{1}{(x+1)^2+y^2} < 0 \text{ or } x < -\frac{1}{2}.$$

3. It is convenient to determine first the zeros of $\beta(1-z)$ and then replace $z$ by $1-z$. From the preceding results it follows that the zeros of $\beta(1-z)$ will be complex and have their real part greater than $\frac{3}{2}$. To obtain an asymptotic expression for $\beta(1-z)$, we observe that, for $\Re z > 0$ ($\Re z = x =$ real part of $z$), the expression (1) may be replaced by

$$\beta(z) = \int_0^\infty \frac{e^{-zu}}{1+e^{-u}} \, du,$$

whence, integrating twice by parts,

$$\beta(z) = \frac{1}{2z} + \frac{1}{4z^2} - \frac{1}{z^2} \int_0^\infty \frac{e^{-zu} - e^{-2zu}}{(1+e^{-u})^3} \, du.$$
For \( \Re z \geq 1 \), we have
\[
\left| \int_0^\infty e^{-u} \frac{e^{-u} - e^{-2u}}{(1+e^{-u})^2} \, du \right| < \int_0^\infty e^{-u}(e^{-u} - e^{-2u}) \, du = \frac{1}{6},
\]
and (2) and (3) now give the asymptotic expression
\[
(4) \quad \beta(1-z) = \frac{\pi}{\sin \pi z} - \frac{1}{2z} - \frac{h}{z^2}, \quad |h| < \frac{5}{12} \text{ for } \Re z \geq 1.
\]

4. Let us consider first the zeros of the expression
\[
(5) \quad \frac{\pi}{\sin \pi z} - \frac{1}{2z},
\]
which constitutes the principal part of \( \beta(1-z) \). Since \( \sin \pi(x+yi) = \sin \pi x \cosh \pi y + i \cos \pi x \sinh \pi y \), a zero \( z = x+yi \) of (5) implies the two equations
\[
(6) \quad \sin \pi x \cosh \pi y = 2\pi x, \\
(7) \quad \cos \pi x \sinh \pi y = 2\pi y,
\]
where, however, the solution \( x = y = 0 \) must be discarded, since it corresponds to the pole \( z = 0 \) of (5). It is evident that when \( x+yi \) is a zero of (5), the same is true of \( x-yi, -x+yi \) and \( -x-yi \); we may therefore restrict the discussion of (6) and (7) to the case \( x \geq 0, y \geq 0 \). First assume \( x = 0 \); since \( y \neq 0 \) in this case, and \( \frac{\sinh \pi y}{\pi y} \) increases steadily from 1 to \( \infty \) as \( y \) increases from 0 to \( \infty \), (7) gives \( y = y_0 \), where \( y_0 \) is the unique positive root of
\[
(8) \quad \frac{\cosh \pi y_0}{\pi y_0} = 2.
\]
Now assume \( x > 0 \); making \( y = 0 \) in (6), we would obtain \( \frac{\sin \pi x}{\pi x} = 2 \), whereas the quotient to the left always lies between \(-1\) and \(+1\). We must consequently assume \( x > 0, y > 0 \), and (5) and (6) show that \( \sin \pi x > 0 \), \( \cos \pi x > 0 \), whence
\[
(9) \quad x = 2n + \frac{1}{2} - \xi, \\
\text{\( n \) a positive integer or zero, } 0 < \xi < \frac{1}{2}. \text{ Consider first the case } n = 0, \text{ whence } 0 < x < \frac{1}{2}; \text{ since } \pi x/\sin \pi x < \pi/2 \text{ in this interval, (6) gives } \cosh \pi y < \pi, \text{ whence } \pi y < 1.812, \text{ and we obtain from (7)}
\]
\[
\cos \pi x > \frac{2 \times 1.812}{\sin 1.812} = 1.3336 > 1,
\]
which is impossible.
Next, assume \( n > 0 \); the equations (6) and (7) become

\[
\begin{align*}
\cos \pi \xi \cosh \pi y &= (4n+1)\pi - 2\pi \xi, \\
\sin \pi \xi \sinh \pi y &= 2\pi y.
\end{align*}
\]

Since \( \pi y / \sinh \pi y \) is a decreasing function of \( y \), equation (11) represents a curve in the rectangular coordinates \( \xi, y \) such that \( y \) decreases steadily from \( +\infty \) to \( y_0 \) when \( \xi \) increases from 0 to \( \frac{1}{2} \). On the curve (10), \( y \) is an extremum when

\[
\cos^2 \pi \xi \sinh \pi y \frac{dy}{d\xi} = \left( (4n+1)\pi - 2\pi \xi \right) \sin \pi \xi - 2 \cos \pi \xi = 0,
\]

and writing this equation in the form \( (4n+1)\pi - 2\pi \xi - 2 \cot \pi \xi = 0 \), we see that it has a unique root \( \xi_0 \) in the interval \( 0 < \xi < \frac{1}{2} \), since the derivative of the left hand member is \( 2\pi \cot^2 \pi \xi > 0 \). This root evidently corresponds to a minimum, since \( y \to +\infty \) when \( \xi \to \frac{1}{2} \) by (10), and we have

\[
\sin \pi \xi_0 < \tan \pi \xi_0 = \frac{2}{(4n+1)\pi - 2\pi \xi_0} < \frac{1}{2n\pi}.
\]

For \( 0 < \xi < \xi_0 \), the value of \( y \) obtained from (10) decreases steadily from \( y_1 \) given by \( \cosh \pi y_1 = (4n+1)\pi \), to \( y_2 \), given by \( \cos \pi \xi_0 \cosh \pi y_2 = (4n+1)\pi - 2\pi \xi_0 \), while the value of \( y \) obtained from (11) decreases from \( +\infty \) to \( y_3 \), given by \( \sin \pi \xi_0 \sinh \pi y_3 = 2\pi y_3 \). We shall now show that \( y_3 > y_1 \); it is evidently sufficient to prove that

\[
\frac{\sinh \pi y_3}{\pi y_3} > \frac{\sinh \pi y_1}{\pi y_1}.
\]

From \( e^{\pi y} < e^{\pi y_1} + e^{-\pi y_1} = 2 \cosh \pi y_1 = (8n+2)\pi \) it follows that \( \pi y_1 < \log(8n+2)\pi \), and since \( (\sinh \pi y)/\pi y \) is a decreasing function,

\[
\frac{\sinh \pi y_1}{\pi y_1} < \frac{\sinh \left[ \log(8n+2)\pi \right]}{\log(8n+2)\pi} = \frac{\frac{1}{2} e^{\log(8n+2)\pi}}{\log(8n+2)\pi} = \frac{(4n+1)\pi}{\log(8n+2)\pi}.
\]

On the other hand,

\[
\frac{\sinh \pi y_3}{\pi y_3} = \frac{2}{\sin \pi \xi_0} > 4n\pi,
\]

and since we have

\[
4n\pi > \frac{(4n+1)\pi}{\log(8n+2)\pi}
\]

for \( n \geq 1 \), the inequality in question is established, and it follows that the two curves (10) and (11) do not intersect in the interval \( 0 < \xi < \xi_0 \). On the
contrary, in the interval $\xi_{0} < \xi < \frac{1}{2}$ the $y$ in (10) increases steadily from $y_{1}$ to $+\infty$, while the $y$ in (11) decreases steadily from $y_{3}$ to $y_{0}$, and since $y_{3} > y_{1} > y_{2}$, the two curves have a unique point of intersection in the interval considered.

Returning to (5), it is thus seen that the only zeros of this function are the following:

The two zeros $y_{0i}, -y_{0i}$;
In each of the strips $2n < \Re z < 2n + \frac{1}{2}$ ($n = 1, 2, 3, \ldots$), two conjugate complex zeros;
In each of the strips $-2n - \frac{1}{2} < \Re z < -2n$ ($n = 1, 2, 3, \ldots$), two conjugate complex zeros, equal to the preceding ones multiplied by $-1$.

5. To find the distribution of the zeros of $\beta(1-z)$, we shall use the following theorem:

Let $f(z)$ and $g(z)$ be two functions, meromorphic inside a contour $C$, and holomorphic on $C$. When the inequality

$$|f(z) - g(z)| < |g(z)|$$

is satisfied everywhere on $C$, then neither $f(z)$ nor $g(z)$ vanishes on $C$, and the difference between the number of zeros and the number of poles of $f(z)$ inside $C$ equals the corresponding difference for $g(z)$.*

Let us apply this theorem to $f(z) = \beta(1-z)$ and $g(z) = (\pi/\sin \pi z) - (1/2z)$, $C$ being the rectangle with vertices at $2n - \frac{1}{2} + bi, 2n + \frac{3}{2} + bi$, where $n$ is a positive integer, and $b$ positive and very large. On the horizontal sides of the rectangle, we have $z = 2n - \frac{1}{2} + x \pm bi, 0 \leq x \leq 2$, whence

$$\sin \pi z = -\cos \pi x \sinh \pi b \pm i \sin \pi x \sinh \pi b, \quad |\sin \pi z|^{2} = \cos^{2} \pi x + \sinh^{2} \pi b$$

and

$$\frac{\pi}{\sin \pi z} \leq \frac{\pi}{\sinh \pi b} < 2\pi e^{-\pi b}.$$

* In the case when $C$ is a circle and $f(z)$ and $g(z)$ have no poles in its interior, this theorem is due to E. Rouché, Mémoire sur la série de Lagrange, Journal de l’École Polytechnique, Cahier 39 (1862), pp. 193–224 (see Theorem III, p. 217). The theorem was rediscovered and generalized by A. Hurwitz, Ueber die Nullstellen der Bessel'schen Function, Mathematische Annalen, vol. 33 (1889), p. 246–266 (see p. 248). Incidentally, the proof is extremely simple: First, neither $f(z)$ nor $g(z)$ vanishes on $C$, since $|f(z) - g(z)| < |g(z)|$ yields the impossible inequalities $|f(z)| < 0$ for $g(z) = 0$ and $|g(z)| < |g(z)|$ for $f(z) = 0$. In the identity

$$\log f(z) = \log g(z) + \log (1 + (f(z) - g(z))g(z)),$$

we perform the analytic continuation of both members along the contour $C$, described once in the positive sense. Then $\log f(z)$ will increase by $2\pi i$ times the difference between the number of zeros and the number of poles of $f(z)$ interior to $C$, the increase in $\log g(z)$ will be the corresponding expression for $g(z)$, and the third logarithm does not change, since, on account of the inequality $|f(z) - g(z)| < |g(z)|$, the point $t = |f(z) - g(z)|g(z)$ describes a closed path interior to the circle $|t| < 1$ where $\log(1 + t)$ is holomorphic.
Moreover, for $b > 2n + 2$,

$$\frac{1}{2|z|} > \frac{1}{4n-1+2x+2b} > \frac{1}{4b}, \quad \frac{1}{|z|^2} = \frac{1}{(2n-\frac{1}{2}+x)^2+b^2} < \frac{1}{b^2},$$

whence, for $b$ sufficiently large and using (4),

$$\left| \frac{\pi}{\sin \pi z} - \frac{1}{2z} \right| \geq \frac{1}{2|z|} - \frac{1}{4b} - 2\pi e^{-zb},$$

$$\left| \beta(1-z) - \left( \frac{\pi}{\sin \pi z} - \frac{1}{2z} \right) \right| < \frac{5}{12|z|^2} < \frac{5}{12b^2} < \frac{1}{4b} - 2\pi e^{-zb}$$

On the vertical sides of the rectangle, we have $z = 2m - \frac{1}{2} + yi, \ m = n$ or $n + 1$ and $-b \leq y \leq b$, whence $\sin \pi z = -\tanh \pi y$ and

$$\frac{\pi}{\sin \pi z} - \frac{1}{2z} = - \left( \frac{\pi}{\tanh \pi y} + \frac{1}{4m-1+2yi} \right);$$

the real part of the second term to the right having the same sign as the first term, it follows that

$$\left| \frac{\pi}{\sin \pi z} - \frac{1}{2z} \right| > \left| \frac{1}{4m-1+2yi} \right| = \frac{1}{|2z|}.$$

Hence we see that

$$\left| \beta(1-z) - \left( \frac{\pi}{\sin \pi z} - \frac{1}{2z} \right) \right| < \frac{5}{12|z|^2} < \frac{1}{2|z|} < \left| \frac{\pi}{\sin \pi z} - \frac{1}{2z} \right|.$$
indefinitely with \( n \), and formula (11) shows that then \( \lim \sin \pi \xi = 0 \), whence \( \lim \cos \pi \xi = 1 \). By (10), we now have

\[
\lim \frac{\operatorname{ch} \pi y}{(4n+1)\pi} = 1 = \lim \frac{e^{nu}}{2(4n+1)\pi},
\]

\( \pi y = \log \left[ \frac{(8n+2)\pi(1+\eta)}{2} \right] \), \( \lim \eta = 0 \).

Substituting this value of \( \pi y \) in (11), it is seen that

\[
\lim \frac{(4n+1)\pi}{2 \log (8n+2)\pi} \cdot \pi \xi = \lim \frac{\pi \xi}{\sin \pi \xi} \cdot \lim \frac{(4n+1)\pi}{2 \log (8n+2)\pi} \cdot \frac{1}{2 \log \left[ \frac{(8n+2)\pi(1+\eta)}{2} \right]}
\]

\[
= 1,
\]

\( \pi \xi = \frac{2 \log (8n+2)\pi}{(4n+1)\pi} (1+\xi'), \lim \xi' = 0 \).

Denoting, as usual, by \( f(x) = O(g(x)) \) the fact that two constants \( A \) and \( x_0 \) exist such that \( |f(x)| < Ag(x) \) for \( x > x_0 \), we find by substituting the above values of \( \pi y \) and \( \pi \xi \) in (10) and expanding \( \cos \pi \xi \) in a power series, so that \( \cos \pi \xi = 1 + O(\xi^3) = 1 + O((\log n)/n)^2 \),

\[
1 + O\left( \left( \frac{\log n}{n} \right)^2 \right) \left[ (4n+1)\pi(1+\eta) - \frac{1}{4(4n+1)\pi(1+\eta)} \right]
\]

\[
= (4n+1)\pi + O\left( \left( \frac{\log n}{n} \right) \right),
\]

and it follows that

\( \eta = O\left( \left( \frac{\log n}{n} \right)^2 \right) \).

Finally (11) gives, taking account of the order of magnitude of \( \eta \),

\[
\left[ \frac{2 \log (8n+2)\pi}{(4n+1)\pi} (1+\xi') + O\left( \left( \frac{\log n}{n} \right)^3 \right) \right] \left\{ (4n+1)\pi \left[ 1
\right.
\]

\[
+ O\left( \left( \frac{\log n}{n} \right)^2 \right) \right] - \frac{1}{4(4n+1)\pi} \left[ 1 + O\left( \left( \frac{\log n}{n} \right)^2 \right) \right] \}
\]

\[
= 2 \left[ \log (8n+2)\pi + O\left( \left( \frac{\log n}{n} \right)^2 \right) \right. \],
\]

\( 2 \log (8n+2)\pi \cdot \xi' = O\left( \left( \frac{\log n}{n^2} \right)^3 \right) \),
\[ \xi' = O\left( \left( \frac{\log n}{n} \right)^2 \right). \]

For the two zeros of (5) in the strip \( 2n < \Re z < 2n + \frac{1}{2} \), we consequently have the asymptotic expression

\[
(12) \quad 2n + \frac{1}{2} - \frac{2 \log (8n+2)\pi}{(4n+1)\pi^2} \pm i \frac{\log (8n+2)\pi}{\pi} + O\left( \left( \frac{\log n}{n} \right)^2 \right) .
\]

7. We shall now prove that for \( n \) sufficiently large, \( \beta(1-z) \) has one zero in the neighborhood of each of the zeros (12) of its principal part (5). Consider the circle

\[
(13) \quad z = 2n + \frac{1}{2} - \frac{2 \log (8n+2)\pi}{(4n+1)\pi^2} + i \frac{\log (8n+2)\pi}{\pi} + \frac{\xi + \eta i}{2n\pi} ,
\]

\[ \xi^2 + \eta^2 < 1; \]

by (12), this circle contains in its interior a single zero of (5) when \( n \) is sufficiently large. On the circumference \( \xi^2 + \eta^2 = 1 \), we have

\[
\sin \pi z = \frac{1}{2i} \left[ \frac{i}{(8n+2)\pi} e^{-\frac{2i \log (8n+2)\pi}{(4n+1)\pi}} + \frac{\xi - \eta}{2n} \right] e^{\frac{2i (8n+2)\pi x}{(4n+1)\pi}} + \left[ \frac{\xi - \eta}{2n} + O\left( \left( \frac{\log n}{n} \right)^2 \right) \right] ,
\]

\[
\frac{\pi}{\sin \pi z} = \frac{1}{4n+1} \left[ 1 - \frac{2i \log (8n+2)\pi}{(4n+1)\pi} + \frac{\xi - \eta}{2n} + O\left( \left( \frac{\log n}{n} \right)^2 \right) \right] ,
\]

\[
\frac{1}{2z} = \frac{1}{(4n+1)} \left[ 1 + \frac{2i \log (8n+2)\pi}{(4n+1)\pi} + O\left( \frac{\log n}{n^2} \right) \right] ,
\]

whence

\[
\frac{\pi}{\sin \pi z} - \frac{1}{2z} = \frac{\xi - \eta}{2n(4n+1)} + O\left( \frac{(\log n)^2}{n^3} \right) = \frac{\xi - \eta}{8n^2} + O\left( \frac{(\log n)^2}{n^3} \right) .
\]
and since $\xi^2 + \eta^2 = 1$,

\begin{equation}
\left| \frac{\pi}{\sin \pi z} - \frac{1}{2z} \right| = \frac{1}{8n^2} + O\left( \frac{(\log n)^2}{n^3} \right).
\end{equation}

On the other hand, we have on the same circumference, according to (4),

\begin{equation}
\left| \beta(1-z) - \left( \frac{\pi}{\sin \pi z} - \frac{1}{2z} \right) \right| < \frac{5}{12 |z|^2} = \frac{5}{48n^2} + O\left( \frac{\log n}{n^3} \right),
\end{equation}

and by comparison with (14)

\begin{equation}
\left| \beta(1-z) - \left( \frac{\pi}{\sin \pi z} - \frac{1}{2z} \right) \right| < \left| \frac{\pi}{\sin \pi z} - \frac{1}{2z} \right|
\end{equation}

on the circumference $\xi^2 + \eta^2 = 1$ for $n$ sufficiently large. Applying the theorem in paragraph 5, we see that the circle $\xi^2 + \eta^2 < 1$ contains a single zero of $\beta(1-z)$, and the same is evidently true of the circle obtained by changing the sign of $i$ in (13). We have thus obtained asymptotic expressions for the two zeros of $\beta(1-z)$ contained in the strip $2n - \frac{1}{2} < \Re z < 2n + \frac{1}{2}$, and replacing $z$ by $1-z$, we finally arrive at the result stated in paragraph 1.

It is clear that closer approximations to the zeros may be obtained by using, instead of (5), the general asymptotic expansion

\begin{equation}
\beta(z) = \frac{1}{2z} + \sum_{\nu=1}^{n} (-1)^{r-1} \frac{2^{2\nu-1}}{2\nu} B_r \cdot \frac{1}{z^{2\nu}} + \frac{1}{z^{2n}} \int_{0}^{\infty} e^{-nu} \frac{d^n}{du^n} \left( \frac{1}{1+e^{-u}} \right) du,
\end{equation}

valid for $\Re z > 0$, where $B_r$ are the Bernoulli numbers.

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