ON LAPLACE'S INTEGRAL EQUATIONS*

BY

J. D. TAMARKIN

The equation with which this paper is concerned is of the form

\[ (*) \int_{(C)} e^{-\sigma F(x)} dx = f(x), \]

which is known in the literature as Laplace’s integral equation. The contour \((C)\) and the function \(f(x)\) are supposed given and \(F(x)\) is to be found.

In the case when the contour \((C)\) consists of the positive part of the axis of reals, the solution of the equation \((*)\) was given by H. Poincaré† and H. Hamburger.‡ Each of these authors considers \(F(x)\) as a function of the real variable \(x\).

When the contour \((C)\) consists of the entire axis of reals, a simple substitution reduces \((*)\) to the form studied by Riemann§ and H. Mellin.|| In the present paper we discuss the equation \((*)\) in the case of Poincaré, extending the solution \(F(x)\) to complex values of \(x\). A certain relation of reciprocity between the functions \(f(z)\) and \(F(x)\) is thereby revealed.

1. Poincaré obtained the solution of the equation

\[ (1) \quad f(x) = \int_{0}^{\infty} e^{-\sigma F(x)} dx \]

in form of a definite integral

\[ (2) \quad F(x) = \frac{1}{2\pi i} \int_{(D)} e^{\sigma f(\xi)} d\xi. \]

This solution can be easily verified assuming certain hypotheses concerning the function \(f(z)\) and the contour \((D)\).¶ Setting

\[ x = u + i v \]

---

* Presented to the Society, May 1, 1926; received by the editors in November, 1924.
† Sur la théorie des quanta, Journal de Physique, ser. 5, vol. 2 (1912).
‡ Ueber eine Riemann'sche Formel . . . , Mathematische Zeitschrift, vol. 6 (1920), pp. 6–9.
§ Ueber die Anzahl der Primzahlen . . . , Werke, 1876, p. 140.
¶ The integration over \((D)\) is taken from \(\xi = \lambda - i \infty\) to \(\xi = \lambda + i \infty\). If the integral \(\int_{(D)}\) fails to exist, but Cauchy's principal value exists, let it be denoted by the same symbol \(\int_{(D)}\), and no further modifications are necessary.
we make the following assumptions:

(A1) The function \( f(z) \) is analytic on the half-plane

\[ u > \lambda_0, \]

where \( \lambda_0 \) is a non-negative constant determined by \( f(z) \). Further, if \( \lambda \) denotes any number \( > \lambda_0 \), then \( f(z) \) approaches zero, uniformly for \( u \geq \lambda, |z| \rightarrow \infty \).

(A2) If \( (D) \) denotes the straight line \( m = \lambda \), the integrals

\[ \frac{1}{2\pi i} \int_{(D)} e^{\xi z} f(\xi) d\xi, \quad \frac{1}{2\pi i} \int_0^\infty e^{-\xi z} d\xi \int_{(D)} e^{\xi z} f(\xi) d\xi \]

exist and, in the latter, the order of integration can be interchanged.

**Theorem 1.** Under the conditions (A1) and (A2) the function \( F(x) \) given by (2) is a solution of (1) and does not depend on \( \lambda \).

We have

\[ \int_0^\infty e^{-\xi t} F(\xi) d\xi = \frac{1}{2\pi i} \int_0^\infty e^{-\xi t} d\xi \int_{(D)} e^{\xi t} f(\xi) d\xi = \frac{1}{2\pi i} \int_{(D)} f(\xi) d\xi \int_0^\infty e^{\xi (t-\lambda)} d\xi. \]

If \( z \) is any point in the region \( u > \lambda_0 \), we can always find \( \lambda > \lambda_0 \) such that \( u > \lambda > \lambda_0 \), and then

\[ \int_0^\infty e^{\xi (t-z)} d\xi = -\frac{1}{\xi - z} \]

and

\[ \int_0^\infty e^{-\xi t} F(\xi) d\xi = -\frac{1}{2\pi i} \int_{(D)} \frac{f(\xi)}{\xi - z} d\xi. \]

The point \( z \) is situated to the right of \( (D) \) and by virtue of (A1) the integral

\[ \int \frac{f(\xi)}{\xi - z} d\xi \]

taken round the right half of the circle \( |\xi - \lambda| = R \), approaches 0 as \( R \rightarrow \infty \).

A simple application of the Cauchy integral theorem shows then that the right hand member of (4) equals \( f(z) \). A similar argument shows that the value of the integral (2) remains unchanged under any parallel displacement of \( (D) \) in the region (3), in other words that the expression (2) for \( F(x) \) does not depend on \( \lambda \). A special case of the condition (A2) is the following:

(A3) The integral

\[ \int_{(D)} f(\xi) d\xi \]

is absolutely convergent.
In this case the integral
\[ \frac{1}{2\pi i} \int_{(D)} e^{i\zeta f(\xi)} d\zeta \quad (x \geq 0) \]
exists, and the same is true of the double integral
\[ \int_{0}^{\infty} \int_{(D)} |e^{i(\xi-\tau)} f(\xi)| d\xi \, d\tau = \int_{0}^{\infty} \int_{-\infty}^{\infty} |e^{-\xi(\omega-\lambda)} f(\lambda + i\tau)| d\xi d\tau, \]
which assures the existence of the repeated integrals of the condition (A2), and also the legitimacy of interchanging of the order of integration. It should be noted that the condition (A2) is more general than the condition (A3). For instance, the equation
\[ f e^{-\pi F(\xi)} d\xi = z^p \quad (\Re p < 0) \]
admits of a solution
\[ F(x) = \frac{1}{\Gamma(\lambda - \rho)} x^{-\rho-1} = \frac{1}{2\pi i} \int_{(D)} e^{i\zeta \rho} d\zeta, \]
whereas it is readily found that the function
\[ f(z) = z^p \quad (\Re p > -1) \]
satisfies both conditions (A1) and (A2) but not (A3). In virtue of this example it is clear that the condition (A3) may be replaced by the more general (A4) The function f(x) is of the form
\[ f(x) = \sum_{s=1}^{m} c_s x^p + \varphi(x), \]
where c_s are arbitrary constants, the exponents p_s satisfy the conditions \( \Re p_s < 0 \) (s = 1, 2, \cdots, m) and the function \( \varphi(z) \) satisfies the conditions (A1) and (A3).

2. Thus far the solution F(x) of the equation (1) which is given by the formula (2) is determined only for real positive values of x. For complex values of x the integral (2) may even become divergent. Let us now consider F(x) as the given function and define f(z) by the formula (1). We set
\[ x = \xi + i\eta \]
and suppose that
\[ (B_1) \quad \text{The function } F(x) \text{ is analytic in the strip} \]
\[ (6) \quad \xi \geq 0; \quad |\eta| < \alpha_0, \]
where \( \alpha_0 \) is a positive constant determined by \( F(x) \). Further, there exists a non-negative constant \( \lambda_0 \), depending only on \( \alpha_0 \), such that for every pair of numbers \( \lambda, \alpha \) \( (\lambda > \lambda_0; 0 < \alpha < \alpha_0) \) the product

\[
e^{-\lambda x F(x)}
\]

approaches zero uniformly as \( |x| \to \infty \) in the strip \( \xi \geq 0, |\eta| \leq \alpha \).

Consider now the function \( f(z) \) which is defined by

\[
f(z) = \int_0^\infty e^{-\xi F(\xi)} d\xi.
\]

If a positive constant \( M \) is suitably chosen, we have, for any \( \lambda_1 > \lambda_0 \),

\[
|e^{-\xi F(\xi)}| = |e^{(\lambda - \lambda_0)\xi}| \cdot |e^{-\lambda_1 \xi F(\xi)}| \leq Me^{(\lambda - u)\xi}.
\]

Therefore the integral (8) is absolutely convergent in the region \( u > \lambda_0 \) and uniformly convergent in the closed region \( u \geq \lambda > \lambda_0 \). Hence the function \( f(z) \) is analytic in the region \( u > \lambda_0 \).

Suppose now that the number \( \lambda > \lambda_0 \) is fixed. Since the integral (8) is uniformly convergent for

\[
u \geq \lambda,
\]

it follows that to an arbitrary positive \( \epsilon \) there corresponds a positive constant \( X \) independent of \( z \) in (9) such that

\[
\left| \int_X^\infty e^{-\xi F(\xi)} d\xi \right| < \frac{\epsilon}{2}.
\]

Further, this constant \( X \) being fixed, the integral

\[
\int_0^X e^{-\xi F(\xi)} d\xi = \frac{e^{-\xi F(\xi)}}{-z} \bigg|_0^X + \frac{1}{z} \int_0^X e^{-\xi F(\xi)} d\xi
\]

likewise is \( < \epsilon/2 \) in absolute value for \( z \) in (9) and \( |z| \) sufficiently large. Thus the condition (A_1) is satisfied by \( f(z) \). Denote now by \( \alpha_1 \) an arbitrary positive number \( < \alpha_0 \). If (B_1) is satisfied, the formula (8) may be rewritten in either of the following forms:

\[f(z) = \int_0^{\alpha_1} e^{z F(\xi)} d\xi + \int_0^\infty e^{-z (\xi + i\alpha)} F(\xi + i\alpha) d\xi = \phi_1(z) + f_1(z);\]

\[f(z) = -\int_0^{\alpha_1} e^{z F(\xi)} d\xi + \int_0^\infty e^{-z (\xi - i\alpha)} F(\xi - i\alpha) d\xi = \phi_2(z) + f_2(z).
\]

We shall proceed by associating the first form with the case \( v \leq 0 \) and the second form with the case \( v \geq 0 \). Accordingly, the occurrence of the functions
\( \varphi_k(z) \), \( f_k(z) \) in a formula shall imply the former case when \( k = 1 \), and the latter case when \( k = 2 \).

Let us suppose for the moment that \( F(0) = 0 \). Integration by parts gives

\[
\varphi_1(z) = \int_0^{a_1} e^{-z\xi} F(i\eta) i d\eta = -\frac{e^{-z\eta_1}}{z} F(i\alpha_1) + \frac{i}{z} \int_0^{a_1} e^{-z\eta} F'(i\eta) d\eta = O\left(\frac{1}{z}\right),
\]

because

\[
|e^{-z\eta}| = e^{v\eta} \leq 1 \text{ for } v \leq 0 \leq \eta.
\]

Integrating by parts once more we find easily for large \(|z|\) and \(|v|\):

\[
\varphi_1(z) = O\left(\frac{e^{-a_1|v|}}{z}\right) + O\left(\frac{1}{z^2}\right).
\]

The same is true of the function \( \varphi_2(z) \):

\[
\varphi_2(z) = O\left(\frac{e^{-a_1|v|}}{z}\right) + O\left(\frac{1}{z^2}\right).
\]

It is obvious finally that \( \varphi_1(z) \) and \( \varphi_2(z) \) are entire transcendental functions of \( z \).

The formulas (101,2) and (111,2) show that the functions \( f_1(z) \) and \( f_2(z) \) satisfy the condition (A1). Moreover, denoting by \( \lambda_1 \) any number between \( \lambda_0 \) and \( \lambda \) we can write

\[
f_1(z) = \int_0^\infty e^{-z(\xi + i\alpha_1)} F(\xi + i\alpha_1) d\xi = e^{-z\alpha_1} \int_0^\infty e^{(\lambda_0 - \lambda)\xi} e^{-\lambda_1 F(\xi + i\alpha_1)} d\xi,
\]

which gives

\[
|f_1(z)| \leq e^{-a_1|v|} \int_0^\infty e^{(\lambda_0 - \lambda)\xi} |e^{-\lambda F(\xi + i\alpha_1)}| d\xi.
\]

By virtue of (B1)

\[
|e^{-\lambda F(\xi + i\alpha_1)}| = |e^{-\lambda(\xi + i\alpha_1)}| \to 0 \text{ as } \xi \to \infty.
\]

Then (12) shows that

\[
f_1(z) = O(e^{-a_1|v|})
\]

for large values of \(|v|\). In precisely the same way we find that

\[
f_2(z) = O(e^{-a_1|v|}).
\]

Suppose now that

\[
F(0) = c \neq 0.
\]
We have
\[ f(z) = \frac{c}{z} + \int_0^\infty e^{-\xi} \left\{ F(\xi) - F(0) \right\} d\xi. \]

The integral being of the form discussed above, we see that
\[ f(z) = \frac{c}{z} + \begin{cases} \varphi_1(z) + \bar{f}_1(z) & \text{for } v \leq 0, \\ \varphi_2(z) + \bar{f}_2(z) & \text{for } v \geq 0, \end{cases} \]

where the functions \( \varphi_1, \varphi_2, f_1, f_2 \) possess the same properties as the functions \( \psi_1, \psi_2, f_1, f_2 \).

Hence the function \( f(z) \) satisfies the condition (A4) and a fortiori the condition (A2).

3. In this section we return to the point of view that the function \( F(x) \) is unknown and \( f(z) \) is given. We suppose now, however, that the function \( f(z) \) satisfies the condition

(C) The function \( f(z) \) may be represented in the form
\[ f(z) = \frac{c}{z} + \begin{cases} \psi_1(z) + \theta_1(z) & \text{for } v \leq 0, \\ \psi_2(z) + \theta_2(z) & \text{for } v \geq 0, \end{cases} \]

where \( \psi_1(z), \psi_2(z) \) are entire transcendental functions which in the corresponding half-planes are of the order

\[ \psi_k(z) = O\left(\frac{e^{-a_k|v|}}{z^2} \right) + O\left(\frac{1}{z^2} \right) \quad (k = 1,2) \]

for large \(|z|\) and \(|v|\), and where \( \theta_1(z) \) and \( \theta_2(z) \) satisfy the condition (A1) and are of the order

\[ \theta_k(z) = O(e^{-a_k|v|}) \quad (k = 1,2), \]

\( a_k \) being an arbitrary positive number < \( a_\alpha \).

Obviously \( f(z) \) satisfies the conditions (A1) and (A2). Hence the solution of the equation (1) is given by the formula
\[ F(x) = \frac{1}{2\pi i} \int_{(D)} e^{izf(\xi)} d\xi. \]

We shall show now that, because of the special properties of the function \( f(z) \), this solution can be extended to complex values of \( x \) and is analytic in \( x \).

Suppose for the sake of simplicity that \( c = 0 \) and denote by \( (D_1) \) the part of the contour \( (D) \) which is situated below the axis of reals and by
(D2) the part which is above the axis of reals. The formulas (15), in which
\( \alpha_1 \) is an arbitrary positive number \(< \alpha_0 \), show that both integrals

\[
(16) \quad F_k(x) = \frac{1}{2\pi i} \int_{(D_k)} e^{\xi} \theta_k(\xi) d\xi \quad (k = 1, 2)
\]

converge absolutely for \( |\eta| < \alpha_0 \) and uniformly in any finite part of the
strip \( |\eta| \leq \alpha, \alpha \) being an arbitrary positive number \(< \alpha_0 \). Thus \( F_k(x) \) are
analytic on the strip \( |\eta| < \alpha_0 \).

We shall prove that both functions \( F_k(x) \) satisfy the condition (B1).

Let \( \lambda_1 \) be any number \( \lambda_0, \lambda < \lambda_1 \), and then, denoting by \( M \) a suitable positive constant, we have

\[
|e^{-\lambda \xi} F_k(x)| \leq M e^{(\lambda - \lambda_1)\xi} \int_0^\infty e^{\xi(y - \alpha_1)} dy \to 0 \quad \text{as} \quad \xi \to \infty,
\]

the limit being approached uniformly in any region

\[
|\eta| \leq \alpha < \alpha_1 < \alpha_0.
\]

Thus, \( \alpha_1 \) and therefore \( \alpha \) being arbitrary numbers \(< \alpha_0 \), the property (B1) is proved.

We turn now to the functions

\[
(17) \quad \Phi_k(x) = \frac{1}{2\pi i} \int_{(D_k)} e^{\xi} \psi_k(\xi) d\xi \quad (k = 1, 2).
\]

In this form the functions \( \Phi_k(x) \) can not be extended to complex values of \( x \).
This becomes possible, however, if the contour of integration is suitably
transformed. We shall show that

\[
(18) \quad \Phi_1(x) = \frac{1}{2\pi i} \int_{-\infty}^\lambda e^{\xi} \psi_1(\xi) d\xi ; \quad \Phi_2(x) = \frac{1}{2\pi i} \int_{-\infty}^\lambda e^{\xi} \psi_2(\xi) d\xi.
\]

In order to prove this we observe that both functions \( \psi_1, \psi_2 \) are entire trans-
cendental functions and that the integral

\[
\int e^{\xi} \psi_k(\xi) d\xi
\]

taken round the quarter of the circle \( |\xi - \lambda| = R \) which lies to the left of
\( (D_k) \), approaches zero as \( R \to \infty, \) as follows from (14) using Jordan's lemma.*

The formulas (18) combined with (14) show that the $k(x)$ are analytic for $\xi > 0$, and it is readily found that, for $\lambda_1 > \lambda$,

$$|e^{-\lambda_1 \xi} \Phi_k(x)| < M e^{(\lambda - \lambda_1) \xi} \to 0 \text{ as } \xi \to \infty,$$

the convergence being uniform on any strip $|\eta| < \alpha < \alpha_0$, $\xi > 0$. The case when $c \neq 0$ does not involve any substantial change in the reasoning above. Thus the following theorem is proved:

**Theorem 2.** If the function $f(z)$ satisfies the condition (C), the solution $F(x)$ of the equation (1) satisfies the condition (B) which is identical with the condition (B1) above except that the strip $\xi \geq 0$, $|\eta| < \alpha_0$, closed at the left hand end, is replaced by the open strip

$$\xi > 0, \quad |\eta| < \alpha_0.$$

We turn now to the converse theorem:

**Theorem 3.** If the function $F(x)$ satisfies the condition (B1), the function $f(z)$ defined by

$$f(z) = \int_0^\infty e^{-z \xi} F(\xi) d\xi$$

is a solution of the equation (2) for $\xi > 0$ and satisfies the condition (C).*

This theorem establishes the reciprocity between $F(x)$ and $f(z)$, which was mentioned in the beginning of the paper.

It was proved in § 2 that the function

$$f(z) = \int_0^\infty e^{-z \xi} F(\xi) d\xi$$

satisfies the condition (C). It remains only to prove that $f(z)$ is a solution of the equation

$$F(x) = \frac{1}{2\pi i} \int_{(D)} e^{z \xi} F(\xi) d\xi.$$

We suppose again that $c = F(0) = 0$

and consider first the case when $x$ is real and positive. We have then

$$\frac{1}{2\pi i} \int_{(D)} e^{z \xi} f(\xi) d\xi = \frac{1}{2\pi i} \lim_{V \to \infty} \int_{\lambda - iv}^{\lambda + iv} e^{z \xi} d\xi \int_0^\infty e^{-t \xi} F(\xi) d\xi.$$  

* If $z$ is complex, the right hand member of (2) must be transformed as was indicated above. 
† See last foot note on p. 417.
Because of the uniform convergence of the interior integral we are justified in interchanging the order of integration, which gives

\[
\frac{1}{2\pi i} \int_{(D)} \delta^{z} f(\xi) d\xi = \lim_{V \to \infty} \int_{0}^{V} F(\xi) e^{\lambda (s-t)} \int_{-V}^{V} e^{i\xi (s-t)} d\eta \\
= \frac{1}{\pi} \lim_{V \to \infty} \int_{0}^{V} F(\xi) e^{\lambda (s-t)} \sin \frac{V(x-\xi)}{x-\xi} d\xi = F(x).
\]

This last relation can be easily proved by using the Dirichlet formula. * Thus the equation (2) is proved for real positive values of \( x \). The right hand member of this equation is an analytic function of \( x \) on the region \( \xi > 0 \), as was proved above. The same is true by hypothesis of the left hand member. Hence the equation holds true on the whole region \( \xi > 0 \).

The case \( \epsilon \neq 0 \) does not involve any substantial change in the applied reasoning; neither does the case in which finite sums of terms of the form

\[
c_{\alpha} z^{\alpha-1} \quad (\text{Re} \, \rho < 0)
\]

are present in \( f(z) \) and \( F(x) \) respectively.

Under the restrictions made we can easily prove the formulas

\[
F(x) = \frac{1}{2\pi i} \int_{(D)} \delta^{z} d\xi \int_{0}^{\infty} e^{-itF(\xi)} d\xi ;
\]

\[
f(x) = \frac{1}{2\pi i} \int_{0}^{\infty} e^{-it} d\xi \int_{(D)} \delta^{z} f(\xi) d\xi .
\]

There is no difficulty either in proving that the solutions of the equations (1), (2), obtained above, are unique under the same restrictions.

* See Hamburger, loc. cit.